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ON THE ZEROS OF FUNCTIONS IN DIRICHLET-TYPE SPACES

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ABSTRACT. We study the sequences of zeros for functions in the Dirichlet spaces $\mathcal{D}_s$. Using Carleson-Newman sequences we prove that there are great similarities for this problem in the case $0 < s < 1$ with that for the classical Dirichlet space.

1. INTRODUCTION AND MAIN RESULTS

The problem of describing the zero sets for the Dirichlet-type spaces $\mathcal{D}_s$ is an old one, and to the best of our knowledge, is still an open problem whose best results are the ones given by Carleson in [8], [10], and by Shapiro and Shields in [39]. The purpose of this paper is to give some light on this difficult problem. Since the Dirichlet-type spaces are subclasses of the Hardy space H^2, any zero sequence $\{z_n\}$ satisfies the Blaschke condition $\sum(1-|z_n|^2) < \infty$ ([18, p. 18]). However, this condition is far from being sufficient. Many examples of Blaschke sequences that are not $\mathcal{D}_s$- zero sets can be found in the literature (see [12], [29] and [39]). When $0 < s < 1$, Carleson proved in [8] that the condition

$$\sum(1-|z_n|^2)^s < \infty$$

implies that the Blaschke product B with zeros $\{z_n\}$ belongs to the space $\mathcal{D}_s$, and therefore, it is a sufficient condition for the sequence $\{z_n\}$ to be a $\mathcal{D}_s$-zero set. Concerning the Dirichlet space $\mathcal{D}$ (the case $s = 0$), since it does not contain infinite Blaschke products, one must go in a different way. In [10], by constructing a function $g \in \mathcal{D}$ with $gB \in \mathcal{D}$, Carleson found the sufficient condition $\sum \left(\log \frac{1}{1-|z_n|^2} \right)^{-1+\varepsilon} < \infty$, for a sequence $\{z_n\}$ to be a zero set for the Dirichlet space. Using Hilbert space techniques, this was improved in [39] by Shapiro and Shields, who proved that the condition

$$\sum_n \left(\log \frac{1}{1-|z_n|^2} \right)^{-1} < \infty$$

is sufficient for $\{z_n\}$ to be a Dirichlet zero set.

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Note that the spaces $\mathcal{D}_s$ are Hilbert function spaces with the norm of the corresponding reproducing kernels k_z comparable to $(\log \frac{1}{1-|z|})^{1/2}$ if $s = 0$, and to $(1-|z|^2)^{-s/2}$ if $s > 0$. So, the corresponding sufficient conditions stated before can be restated as $\sum \|k_{z_n}\|_{\mathcal{D}_s}^{-2} < \infty$. On the other hand, if $\{r_n\} \subset (0,1)$ and $\sum \|k_{r_n}\|_{\mathcal{D}_s}^{-2} = \infty$, with $0 \le s < 1$, in [29], Nagel, Rudin, and Shapiro constructed a sequence of angles $\{\theta_n\}$ such that $\{r_n e^{i\theta_n}\}$ is not the zero set of any function in $\mathcal{D}_s$. Together with the previous sufficient condition, this implies that given $\{r_n\} \subset (0,1)$, then $\{r_n e^{i\theta_n}\}$ is a zero set for $\mathcal{D}_s$ for any choice of angles $\{\theta_n\}$ if and only if

$$\sum_n \|k_{r_n}\|_{\mathcal{D}_s}^{-2} < \infty. \tag{1.1}$$

We also note that, in [7], Bogdan described the regions $\Omega \subset \mathbb{D}$ for which any Blaschke sequence of points in Ω must be a Dirichlet zero set. For example, it follows that any Blaschke sequence that lies in a region with finite order of contact with the unit circle must be a Dirichlet zero set.

What about conditions on the angles? Here we touch the notion of a *Carleson set*. Given a sequence of points $\{e^{i\theta_n}\}$, the sequence $\{r_n e^{i\theta_n}\}$ is a zero sequence of $\mathcal{D}$ for any choice of radius $\{r_n\}$, $0 < r_n < 1$ with $\sum(1-r_n) < \infty$ if and only if the closure of $\{e^{i\theta_n}\}$ is a Carleson set. Indeed, if the closure of $\{e^{i\theta_n}\}$ in the unit circle is a Carleson set, Caughran proved in [13] that there is a function f with all derivatives bounded in the unit disk vanishing at the points $\{r_n e^{i\theta_n}\}$. Conversely, if $\overline{\{e^{i\theta_n}\}}$ is not a Carleson set, by modifying the construction in [12, Theorem 1], he obtained in [13] a sequence $\{r_n\}$ for which $\{r_n e^{i\theta_n}\}$ is not contained in the zero set of any function with finite Dirichlet integral. We will see that the same holds for the spaces $\mathcal{D}_s$ when $0 < s < 1$.

In [26, Corollary 13], Marshall and Sundberg proved that the zero sets of the Dirichlet-type spaces $\mathcal{D}_s, 0 \le s \le 1$, coincide with the zero sets of its multiplier algebra (see also [2, Corollary 9.39]). From this follows the remarkable result that the union of two zero sets is also a zero set for $\mathcal{D}_s$. Note that the corresponding result for the weighted Bergman spaces (the case $s > 1$) is not true; the first example was given by Horowitz in [22]. A complete description of the zeros of functions in Bergman spaces is still open, but the gap between the necessary and sufficient known conditions is small. We refer to [19, Chapter 4], [21, Chapter 4], [23], [25], [37] and [38] for more information on this interesting problem.

1.1. Main results.

Let $\mathbb{D}$ denote the open unit disk of the complex plane, let $\mathbb{T}$ denote the unit circle and let $H(\mathbb{D})$ be the class of all analytic functions on $\mathbb{D}$. For $s \ge 0$, the weighted Dirichlet-type space $\mathcal{D}_s$ consists of those functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{\mathcal{D}_s}^2 \stackrel{\text{def}}{=} |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \, (1-|z|^2)^s \, dA(z) < \infty,$$

where $dA(z) = \frac{1}{\pi} dx\, dy$ is the normalized area measure on $\mathbb{D}$. As usual, $\mathcal{D}_0$ will be simply denoted by $\mathcal{D}$.

Given a space X of analytic functions in $\mathbb{D}$, a sequence $Z = \{z_n\} \subset \mathbb{D}$ is said to be an X-*zero set* if there exists a function in X that vanishes on Z and nowhere else.

A sequence $\{z_n\} \subset \mathbb{D}$ is said to be *separated* if $\inf_{j \ne k} \varrho(z_j, z_k) > 0$, where $\varrho(z,w) = \left| \frac{z-w}{1-\bar{z}w} \right|$ denotes the pseudohyperbolic metric in $\mathbb{D}$. This condition is equivalent to the fact that there is a positive constant $\delta < 1$ such that the pseudo-hyperbolic discs $\Delta(z_j, \delta) = \{z : \varrho(z, z_j) < \delta\}$ are pairwise disjoint.

We denote by H^p $(0 < p \leq \infty)$ the classical Hardy spaces of analytic functions on $\mathbb{D}$ (see [18]). We remind the reader that $\{z_k\} \subset \mathbb{D}$ is an *interpolating sequence* if for each bounded sequence $\{w_k\}$ of complex numbers there exists $f \in H^\infty$ such that $f(z_k) = w_k$ for all k. It is a classical result of Carleson (see e.g. [18]) that $\{z_k\} \subset \mathbb{D}$ is an interpolating sequence if and only if

$$\inf_k \prod_{j \neq k} \varrho(z_j, z_k) > 0. \tag{1.2}$$

Clearly a sequence satisfying (1.2) is separated. A finite union of interpolating sequences is usually called a *Carleson-Newman sequence.*

In this research on $\mathcal{D}_s$-zero sets, $0 < s < 1$, the additional hypothesis of being a Carleson-Newman sequence enables us to obtain better results. The key is the following one which moves the problem to a new situation on the boundary.

Theorem 1. *Suppose that $0 < s < 1$ and $\{z_k\}$ is a Carleson-Newman sequence. Then the following conditions are equivalent:*

(i) $\{z_k\}$ is a $\mathcal{D}_s$-zero set.

(ii) There exists an outer function $g \in \mathcal{D}_s$ such that

$$\sum_{k=1}^{\infty} |g(z_k)|^2 (1 - |z_k|^2)^s < \infty. \tag{1.3}$$

(iii) There exists an outer function $g \in \mathcal{D}_s$ such that

$$\sum_{k=1}^{\infty} (1 - |z_k|^2)^{1+s} \int_{\mathbb{T}} |g(e^{it})|^2 \frac{dt}{|e^{it} - z_k|^2} < \infty.$$

We recall that a function $g \in H(\mathbb{D})$ is called an *outer function* if $\log |g|$ belongs to $L^1(\mathbb{T})$ and

$$g(z) = \exp\left(\frac{1}{2\pi} \int_{\mathbb{T}} \log |g(e^{it})| \frac{e^{it} + z}{e^{it} - z}\, dt\right).$$

Although obviously there are $\mathcal{D}_s$-zero sets that are not Carleson-Newman sequences, this additional assumption is not an obstacle in order to construct relevant examples, and to get analogous results for $\mathcal{D}_s$ to those known for $\mathcal{D}$. Combining ideas from [10], [12] and Theorem 1, the next result follows.

Corollary 1. *Suppose that $0 < s < 1$ and $\{z_k\}$ is a Carleson-Newman sequence. If $\{z_k\}$ is a $\mathcal{D}_s$-zero set, then*

$$\int_{\mathbb{T}} \log \left(\sum_{k=1}^{\infty} \frac{(1 - |z_k|^2)^{1+s}}{|e^{it} - z_k|^2} \right) dt < \infty. \tag{1.4}$$

We note that this result remains true for $s = 0$ without assuming that the sequence is Carleson-Newman (see [12]); that is, if $\{z_k\}$ is a $\mathcal{D}$-zero set, then

$$\int_{\mathbb{T}} \log \left(\sum_{k=1}^{\infty} \frac{1 - |z_k|^2}{|e^{it} - z_k|^2} \right) dt < \infty. \tag{1.5}$$

Corollary 1 allows us to extend Theorem 1 of [12] to the case $0 < s < 1$.

Theorem 2. *Let $0 < s < 1$. Then there exists a Blaschke sequence $\{z_n\}$ which is not a $\mathcal{D}_s$-zero set and with 1 as a unique accumulation point.*

Denote by $|E|$ the normalized Lebesgue measure of a subset E of the unit circle $\mathbb{T}$. A *Carleson set* is a closed subset $E \subset \mathbb{T}$ of Lebesgue measure zero for which, if the intervals $\{I_k\}$ complementary to E have lengths $|I_k|$, then $\sum_k |I_k| \log |I_k| > -\infty$. This notion was introduced in [5], and in [9] Carleson used it to describe the sets of uniqueness of some function spaces. Corollary 1 is also useful to obtain results on the angular distribution of the $\mathcal{D}_s$-zero sets.

Theorem 3. *Let $0 < s < 1$, and $\{e^{i\theta_n}\} \subset \mathbb{T}$. The following are equivalent:*

(*i*) *the sequence $\{r_n e^{i\theta_n}\}$ is a $\mathcal{D}_s$-zero set for any choice of $\{r_n\} \subset (0,1)$ with $\sum(1 - r_n) < \infty$;*
(*ii*) *the closure of $\{e^{i\theta_n}\}$ in the unit circle is a Carleson set.*

As noted before, if $0 \leq s < 1$ and $\{r_n\} \subset (0,1)$ is a Blaschke sequence that does not satisfy (1.1), then there is a sequence of angles $\{\theta_n\}$ such that $Z = \{r_n e^{i\theta_n}\}$ is not a $\mathcal{D}_s$-zero set. The sequences doing that which have been constructed in [29] (and also the examples in [39]) satisfy that every $\xi \in \mathbb{T}$ is an accumulation point of Z. Ross, Richter and Sundberg proved in [36] that this can be done in $\mathcal{D}$ with a sequence Z which accumulates to a single point in $\mathbb{T}$. We shall extend this result to the range $0 < s < 1$, which improves our Theorem 2 but whose proof is much more technical.

Theorem 4. *Let $0 < s < 1$. Suppose that $\{r_n\} \subset (0,1)$ satisfies*

$$\sum_{n=0}^{\infty} (1 - r_n)^s = \infty.$$

Then there exists a sequence $\{\theta_n\}$ such that $\overline{\{r_n e^{i\theta_n}\}} \cap \mathbb{T} = \{1\}$ and $\left\{r_n e^{i\theta_n}\right\}$ is not a $\mathcal{D}_s$-zero set.

Let X be a space of analytic functions in $\mathbb{D}$ contained in the Nevanlinna class (see [18]), so every function $f \in X$ has nontangential limits a.e. on $\mathbb{T}$. Denote also by f the function of boundary values of f (taken as a nontangential limit). A closed set $E \subset \mathbb{T}$ is called a *set of uniqueness* for X if it has the property that $f \equiv 0$ if $f \in X$ vanishes at all points $\xi \in E$. It is well known that $E \subset \mathbb{T}$ is a set of uniqueness for a Lipschitz class Λ_α if and only if E is not a Carleson set. We remind the reader that $f \in H(\mathbb{D})$ belongs to Λ_α, $0 < \alpha \leq 1$, if there is $C > 0$ such that

$$|f(z) - f(w)| \leq C|z - w|^\alpha, \quad \text{for all } z, w \in \overline{\mathbb{D}}.$$

In [9, Theorem 5], under a very weak additional assumption, the sets of uniqueness for the classical Dirichlet space are described.

If $\alpha > 0$, we denote by $C_\alpha(E)$ the α-capacity of a subset of $\mathbb{T}$ (see Section 4 for a definition). The following result is an extension of Theorem 5 in [9].

Theorem 5. *Let $0 \leq s < \alpha < 1$ and $E \subset \mathbb{T}$ with null Lebesgue measure. Suppose that there exists $m > 0$ such that for each interval $I \subset \mathbb{T}$ centered at a point of E,*

$$C_\alpha(E \cap I) \geq m|I|. \tag{1.6}$$

Then E is a set of uniqueness for $\mathcal{D}_s$ if and only if E is not a Carleson set.

The paper is organized as follows. Section 2 is devoted to the study of Carleson-Newman sequences as $\mathcal{D}_s$-zero sets proving Theorem 1, Corollary 1, Theorem 2 and Theorem 3. Theorem 4 is proved in Section 3, and Theorem 5 is proved in Section 4. In Section 5, we shall give a new proof of a result of Bogdan [7] on the description of Blaschke sets for $\mathcal{D}$. Finally, in Section 6, between other results, we prove that $\mathcal{D}_s$-zero sets and the zero sets of their generated Möbius invariant spaces coincide.

In the sequel, the notation $A \asymp B$ will mean that there exist two positive constants C_1 and C_2 which only depend on some parameters $p, \alpha, s, \dots$ such that $C_1 A \le B \le C_2 A$. Also, we remark that throughout the paper we shall be using the convention that the letter C will denote a positive constant whose value may depend on some parameters $p, \alpha, s \dots$, not necessarily the same at different occurrences.

2. Carleson-Newman $\mathcal{D}_s$-zero sets

We first recall some useful concepts and results. The *Carleson square* $S(I)$ of an interval $I \subset \mathbb{T}$ is defined as

$$S(I) = \{re^{i\theta} : e^{i\theta} \in I, \quad 1 - |I| \le r < 1\}.$$

Given $s > 0$ and a positive Borel measure μ on $\mathbb{D}$, we say that μ is an *s-Carleson measure* if there exists a positive constant C such that

$$\mu(S(I)) \le C|I|^s, \qquad \text{for every interval } I \subset \mathbb{T}.$$

If $s = 1$ we simply say that μ is a Carleson measure. We recall that a sequence $\{z_n\} \subset \mathbb{D}$ is Carleson-Newman if and only if the measure $d\mu_{z_n} = \sum (1 - |z_n|)\delta_{z_n}$ is a Carleson measure (see [27] and [28]). Here, as usual, δ_{z_n} denotes the point mass at z_n. A Blaschke product whose zero sequence is Carleson-Newman is called a Carleson-Newman Blaschke product (a CN-Blaschke product, for short).

Let $P_z(e^{it})$ denote the Poisson kernel at a point $z \in \mathbb{D}$, so that

$$P_z(e^{it}) = \frac{1 - |z|^2}{|e^{it} - z|^2}, \quad e^{it} \in \mathbb{T},$$

and let

$$\Psi(z, \phi) = \frac{1}{2\pi} \int_{\mathbb{T}} \phi(e^{it}) P_z(e^{it})\, dt - \exp\left(\frac{1}{2\pi} \int_{\mathbb{T}} \log \phi(e^{it}) P_z(e^{it})\, dt\right), \quad z \in \mathbb{D},$$

where ϕ is a positive function which belongs to $L^1(\mathbb{T})$. Observe that the arithmetic-geometric inequality implies that $\Psi(z, \phi) \ge 0$. If $\phi \in L^2(\mathbb{T}),\ \phi \ge 0,$ we set

$$\Phi(z, \phi) = \Psi(z, \phi^2).$$

We observe that for an outer function $g \in H^2$,

$$\Phi(z, |g|) = P(|g|^2)(z) - |g(z)|^2, \tag{2.1}$$

where $P(|g|^2)$ is the Poisson integral of $|g|^2$.

The following result, Theorem 3.1 of [17] (see [6] for related results), characterizes the membership in $\mathcal{D}_s$ of an outer function in terms of its modulus on the boundary.

Theorem A. *Suppose that $0 < s < 1$ and f is an outer function. Then the following are equivalent:*

(*i*) $f \in \mathcal{D}_s$.

(*ii*) $\int_{\mathbb{D}} \Phi(z, |f|) \frac{dA(z)}{(1-|z|)^{2-s}} < \infty$.

In order to prove Theorem 1 we need some lemmas. The following result is implicit in some places (see e.g. [33, Theorem 5] or [15, Theorem 8]). For completeness we sketch a proof here.

Lemma 1. *Suppose that $0 < s < 1$, $f \in \mathcal{D}_s$ and let B be a Carleson-Newman Blaschke product with zeros $\{z_k\} \subset \mathbb{D}$. Then $fB \in \mathcal{D}_s$ if and only if*

$$\sum_{k=1}^{\infty} |f(z_k)|^2 (1-|z_k|^2)^s < \infty.$$

Moreover,

$$||fB||^2_{\mathcal{D}_s} \asymp ||f||^2_{\mathcal{D}_s} + \sum_{k=1}^{\infty} |f(z_k)|^2 (1-|z_k|^2)^s.$$

Proof. Suppose first that $fB \in \mathcal{D}_s$. By Theorem 4 of [16],

$$\|fB\|^2_{\mathcal{D}_s} \asymp \|f\|^2_{\mathcal{D}_s} + \int_{\mathbb{D}} |f(z)|^2 \left(1-|B(z)|^2\right) (1-|z|^2)^{s-2}\, dA(z). \tag{2.2}$$

Since B is a CN-Blaschke product, there is a positive constant C such that (see e.g. [16, p. 15])

$$1-|B(z)|^2 \geq C \sum_n \frac{(1-|z_n|^2)(1-|z|^2)}{|1-\bar{z}_n z|^2}.$$

Therefore, if $\Delta_n = \{\varrho(z, z_n) < 1/2\}$, the subharmonicity of $|f|^2$ gives

$$\begin{aligned}
\sum_n |f(z_n)|^2 (1-|z_n|^2)^s &\leq C \sum_n \int_{\Delta_n} |f(z)|^2 \frac{(1-|z|^2)^s}{|1-\bar{z}_n z|^2}\, dA(z) \\
&\leq C \sum_n (1-|z_n|^2) \int_{\Delta_n} |f(z)|^2 \frac{(1-|z|^2)^{s-1}}{|1-\bar{z}_n z|^2} dA(z) \\
&\leq C \sum_n (1-|z_n|^2) \int_{\mathbb{D}} |f(z)|^2 \frac{(1-|z|^2)^{s-1}}{|1-\bar{z}_n z|^2} dA(z) \\
&\leq C \int_{\mathbb{D}} |f(z)|^2 \left(1-|B(z)|^2\right) (1-|z|^2)^{s-2}\, dA(z).
\end{aligned}$$

For the converse we refer to [4, Proposition 3.2], where an elementary proof is given. □

Next, if $g \in H^2$ we shall see that the function $\Phi(z, |g|)$, although it is superharmonic, verifies a certain sub-mean-value property.

Lemma 2. *Suppose that g is an outer function which belongs to H^2. Then there is a constant $M > 1$ such that*

$$\Phi(z, |g|) \leq \frac{M}{A(D(z,r))} \int_{D(z,r)} \Phi(w, |g|)\, dA(w), \quad \text{for all } r \in \left(0, \frac{1-|z|}{2}\right),$$

where $D(z, r)$ is the Euclidean disk of center z and radius r.

Proof. Take $z \in \mathbb{D}$ and $r \in \left(0, \frac{1-|z|}{2}\right)$. Using the trivial but useful identity

$$\int_0^{2\pi} |g(e^{it}) - g(z)|^2 \, P_z(e^{it}) \, \frac{dt}{2\pi} = P(|g|^2)(z) - |g(z)|^2, \tag{2.3}$$

the subharmonicity of the function $h_t(z) = |g(e^{it}) - g(z)|^2$, Fubini's theorem and (2.1), we obtain that

$$\begin{aligned} \Phi(z, |g|) &= \int_0^{2\pi} h_t(z) \, P_z(e^{it}) \, \frac{dt}{2\pi} \\ &\leq \int_0^{2\pi} \left(\frac{1}{A(D(z,r))} \int_{D(z,r)} h_t(w) \, dA(w) \right) P_z(e^{it}) \, \frac{dt}{2\pi} \\ &= \frac{1}{A(D(z,r))} \int_{D(z,r)} \int_0^{2\pi} |g(e^{it}) - g(w)|^2 \, P_z(e^{it}) \, \frac{dt}{2\pi} \, dA(w). \end{aligned} \tag{2.4}$$

Now, by the Härnack inequality, there is a constant $M > 1$ (we can take $M = 3$) such that

$$P_z(e^{it}) \leq M \, P_w(e^{it}) \quad \text{for } w \in D(z,r),$$

which, together with (2.3) and (2.4), gives that

$$\begin{aligned} \Phi(z, |g|) &\leq \frac{M}{A(D(z,r))} \int_{D(z,r)} \int_0^{2\pi} |g(e^{it}) - g(w)|^2 \, P_w(e^{it}) \, \frac{dt}{2\pi} \, dA(w) \\ &= \frac{M}{A(D(z,r))} \int_{D(z,r)} \left(P(|g|^2)(w) - |g(w)|^2 \right) \, dA(w) \\ &= \frac{M}{A(D(z,r))} \int_{D(z,r)} \Phi(w, |g|) \, dA(w), \end{aligned}$$

which finishes the proof. □

Proof of Theorem 1. $(i) \Rightarrow (ii)$. Let B be a CN-Blaschke product with zeros $\{z_n\}$, where $\{z_n\}$ is a $\mathcal{D}_s$-zero set. Thus, there is $f \in \mathcal{D}_s$ whose zero sequence is $\{z_n\}$. Since $\mathcal{D}_s$ has the property of division by inner functions (see [16]), this implies that there is an outer function $g \in \mathcal{D}_s$ such that $g \cdot B \in \mathcal{D}_s$, which together with Lemma 1 gives that

$$\sum_n |g(z_n)|^2 (1 - |z_n|^2)^s < \infty.$$

$(ii) \Rightarrow (i)$. Since B is a CN-Blachke product, this follows immediately from Lemma 1.

$(iii) \Rightarrow (ii)$ is clear.

$(ii) \Rightarrow (iii)$. Without loss of generality we may assume that $\{z_k\}$ is separated. Therefore, there is a positive constant $\varepsilon < 1$ such that the pseudohyperbolic disks $\Delta(z_k, \varepsilon)$ are pairwise disjoint.

Suppose that there is an outer function g which satisfies (1.3). It is observed that

$$\begin{aligned} &\sum_k (1 - |z_k|^2)^{1+s} \int_{\mathbb{T}} |g(e^{it})|^2 \, \frac{dt}{|e^{it} - z_k|^2} \\ &\leq \sum_k \Phi(z_k, |g|)(1 - |z_k|^2)^s + \sum_{k=1}^{\infty} |g(z_k)|^2 (1 - |z_k|^2)^s. \end{aligned} \tag{2.5}$$

Next, bearing in mind Lemma 2, the separation of $\{z_k\}$ and Theorem A, we deduce that

$$
\begin{aligned}
\sum_k \Phi(z_k,|g|)(1-|z_k|^2)^s &\le C\sum_k (1-|z_k|^2)^{s-2}\int_{\Delta(z_k,\varepsilon)} \Phi(z,|g|)\,dA(z)\\
&\le C\sum_k \int_{\Delta(z_k,\varepsilon)} (1-|z|^2)^{s-2}\Phi(z,|g|)\,dA(z) \qquad (2.6)\\
&\le C\int_{\mathbb{D}} (1-|z|^2)^{s-2}\Phi(z,|g|)\,dA(z) < \infty.
\end{aligned}
$$

Finally, (iii) follows from (1.3), (2.6) and (2.5). □

Proof of Corollary 1. By Theorem 1 there is an outer function $g\in\mathcal{D}_s$ such that

$$\int_{\mathbb{T}} |g(e^{it})|^2 \left(\sum_k \frac{(1-|z_k|^2)^{1+s}}{|e^{it}-z_k|^2}\right) dt < \infty,$$

so bearing in mind that $\log|g|\in L^1(\mathbb{T})$ and the geometric-arithmetic inequality, the result follows. □

Proof of Theorem 2. The same sequence given in the proof of [12, Theorem 1] works. Choose a sequence $\{\varepsilon_n\}$ such that $0<\varepsilon_n<1$, $\sum_n \varepsilon_n \le 1$ and $\sum_n \varepsilon_n \log\varepsilon_n = -\infty$. Next, take disjoint open arcs of $\mathbb{T}$ with $|I_n| = \varepsilon_n$ converging to 1. Let $r_n = 1-\varepsilon_n$ and $z_n = r_n e^{i\theta_n}$, where θ_n is the center of I_n. If I is an arc of $\mathbb{T}$, then

$$\sum_{z_n\in S(I)} (1-|z_n|) \le \sum_{I_n\subset 2I} |I_n| \le 2|I|,$$

proving that the measure $\mu = \sum (1-|z_n|)\delta_{z_n}$ is a Carleson measure. So, $\{z_n\}$ is a Carleson-Newman sequence which accumulates only at $\{1\}$. Moreover, since

$$
\begin{aligned}
\int_{\mathbb{T}} \log\left(\sum_{k=1}^{\infty} \frac{(1-|z_k|^2)^{1+s}}{|e^{it}-z_k|^2}\right) dt &\ge \sum_{j=1}^{\infty} \int_{I_j} \log\left(\sum_{k=1}^{\infty} \frac{(1-|z_k|^2)^{1+s}}{|e^{it}-z_k|^2}\right) dt\\
&\ge \sum_{j=1}^{\infty} \int_{I_j} \log\left(\frac{(1-|z_j|^2)^{1+s}}{|e^{it}-z_j|^2}\right) dt\\
&\ge \sum_{j=1}^{\infty} |I_j| \log\left(4|I_j|^{s-1}\right) = \infty,
\end{aligned}
$$

it follows from Corollary 1 that $\{z_n\}$ is not a $\mathcal{D}_s$-zero set. The proof is complete. □

Proof of Theorem 3. If $\overline{\{e^{i\theta_n}\}}$ is a Carleson set and $\sum(1-r_n)<\infty$, then it follows from [13, Theorem 2] that there is a function f with all derivatives bounded that vanishes only at $\{r_n e^{i\theta_n}\}$.

Suppose now that $E = \overline{\{e^{i\theta_n}\}}$ is not a Carleson set. Let $\{I_n\}$ be the complementary intervals of E, with $I_n = (e^{i\theta_n}, e^{i(\theta_n+|I_n|)})$. Set $r_n = (1-|I_n|)e^{i\theta_n}$, which satisfies $\sum(1-r_n)<\infty$. Clearly, the sequence $\{z_n\}=\{r_ne^{i\theta_n}\}$ is Carleson-Newman, and arguing as in the proof of Theorem 2 we have

$$\int_{\mathbb{T}}\log\left(\sum_n\frac{(1-|z_n|^2)^{1+s}}{|e^{it}-z_n|^2}\right)dt\geq C\sum_n|I_n|\log\left(4|I_n|^{s-1}\right)=\infty.$$

Hence, by Corollary 1, the sequence $\{r_ne^{i\theta_n}\}$ is not a $\mathcal{D}_s$-zero set. □

3. Proof of Theorem 4

Some new concepts and preliminary results will be needed in the proof of Theorem 4. For $0<s\leq 1$, the *s-dimensional Hausdorff capacity* of $E\subset\mathbb{T}$ is determined by

$$\Lambda_s^\infty(E)=\inf\left\{\sum_j|I_j|^s:\,E\subset\bigcup_j I_j\right\},$$

where the infimum is taken over all coverings of E by countable families of open arcs $I\subset\mathbb{T}$.

Although we think that the next result is known, a proof is included here since we were not able to find any clear reference.

Lemma 3. *Let $0<s\leq 1$. Then there exists a universal constant C such that $\Lambda_s^\infty(E)\geq C|E|^s$ for all $E\subset\mathbb{T}$.*

Proof. Let $E\subset\mathbb{T}$. If $|E|=0$, the result is clear. Suppose that $|E|>0$ and take $\varepsilon\in\left(0,\frac{|E|^s}{2}\right)$. Then there exists a covering $\{I_j\}_j$ of E, such that

$$\Lambda_s^\infty(E)\geq\sum_j|I_j|^s-\varepsilon\geq\left(\sum_j|I_j|\right)^s-\varepsilon\geq|E|^s-\frac{|E|^s}{2}=\frac{|E|^s}{2}.$$

This finishes the proof. □

The *homogeneous $\mathcal{D}_s$-capacity* of a set $E\subset\mathbb{T}$ is defined by

$$\operatorname{cap}(E,\mathcal{D}_s)=\inf\left\{||f||^2_{\mathcal{D}_s}:\,f\in L^2(\mathbb{T})\text{ and }f\geq 1\text{ a.e. on }E\right\}.$$

Lemma 4. *Let $J\subset\mathbb{T}$ be an open arc with center $e^{i\theta_0}$. Suppose that $F\in\mathcal{D}_s$ with*

$$E=\left\{e^{it}\in J:\,|F(e^{it})|\geq 1\right\}.$$

If $|E|\geq\frac{|J|}{2}$, then there exists a universal constant C such that

$$\int_{S(J)}|F'(z)|^2\,(1-|z|^2)^s\,dA(z)\geq C|J|^s.$$

Proof. Let $z_0=(1-\frac{|J|}{2})e^{i\theta_0}$. Arguing as in the proof of [36, Lemma 3], we deduce that there is a universal constant C such that the harmonic measure of E with respect to $Q:=S(J)$ at z_0, $\mu_{z_0}^Q(E)$, satisfies

$$\mu_{z_0}^Q(E)\geq C.$$

Consider a conformal map $\varphi:\mathbb{D}\to Q$ with $\varphi(0)=z_0$ and take $g=F\circ\varphi$. Then $g\geq 1$ on $\varphi^{-1}(E)$ and $|\varphi^{-1}(E)|=\mu_{z_0}^Q(E)\geq C$. Thus, putting together (5.1.3) of [1] and Lemma 3, we have

$$\|g\|^2_{\mathcal{D}_s} \geq \operatorname{cap}\left(\varphi^{-1}(E), \mathcal{D}_s\right) \geq C\left(\Lambda^{\infty}_{s'}\left(\varphi^{-1}(E)\right)\right)^{\gamma} \geq C\mu^{Q}_{z_0}(E)^{s'\gamma} \geq C, \tag{3.1}$$

where $s' \in (s,1)$ and $\gamma \in (0,1)$.

Next, since φ is a conformal map (see [34, Chapter 1]),

$$(1-|z|^2)|\varphi'(z)| \asymp d\left(\varphi(z), \partial Q\right), \quad z \in \mathbb{D}. \tag{3.2}$$

Moreover, since Q is convex, reasoning as in [20, Proposition 5] and bearing in mind (3.2) we obtain that

$$|\varphi'(z)| \geq \frac{1}{4}|\varphi'(0)| \geq C\, d(z_0, \partial Q) \geq C|J|, \tag{3.3}$$

where $d(z_0,\partial Q)$ is the Euclidean distance from z_0 to ∂Q.

Taking into account (3.1), (3.2) and (3.3) we deduce that

$$\begin{aligned}
\int_Q |F'(z)|^2 \,(1-|z|^2)^s \, dA(z) &\geq \int_Q |F'(z)|^2 \, d(z,\partial Q)^s \, dA(z) \\
&\geq \int_{\mathbb{D}} |g'(z)|^2 \, d(\varphi(z),\partial Q)^s \, dA(z) \\
&\geq C \int_{\mathbb{D}} |g'(z)|^2 \, \left((1-|z|^2)|\varphi'(z)|\right)^s \, dA(z) \\
&\geq C|J|^s \int_{\mathbb{D}} |g'(z)|^2 \,(1-|z|^2)^s \, dA(z) \\
&\geq C|J|^s.
\end{aligned}$$

This finishes the proof. □

Proof of Theorem 4. Let $\{r_n\} \subset (0,1)$ be an increasing sequence such that

$$\sum_n (1-r_n)^s = \infty.$$

We can find

$$1 \leq n_1 < m_1 < n_2 < m_2 < \cdots < n_k < m_k < \cdots$$

such that

$$(1-r_n)^{1-s} < k^{-2}e^{-2k^2} \quad \text{if} \quad n \geq n_k, \quad k=1,2,\dots \tag{3.4}$$

and

$$ke^{2k^2} \leq \sum_{n=n_k}^{m_k} (1-r_n)^s < ke^{2k^2} + 1, \quad k = 1,2,\dots.$$

For each k, lay out arcs $J_{n_k}, J_{n_k+1}, \dots, J_{m_k}$ on the unit circle end-to-end starting at $e^{i\theta} = 1$ and such that

$$|J_n| = (1-r_n)^s k^{-2}\, e^{-2k^2}, \quad n_k \leq n \leq m_k. \tag{3.5}$$

Observe that (3.4) together with (3.5) implies that

$$|J_n| > (1-r_n). \tag{3.6}$$

Let $e^{i\theta_n}$ be the center of J_n and set $\lambda_n = (1-r_n)e^{i\theta_n}$. Suppose that there is $F \in \mathcal{D}_s$ with $F(\lambda_n) = 0$ for all $n_k \le n \le m_k$. By [6, Theorem 3.4] we may assume that $||F||_{H^\infty} \le 1$. Set

$$A_k = \left\{ n : n_k \le n \le m_k \text{ and } |F| \ge e^{-k^2} \text{ on a set } E_n \subset J_n \text{ with } |E_n| \ge \frac{|J_n|}{2} \right\},$$

$$B_k = \{ n : n_k \le n \le m_k, \quad n \notin A_k \}.$$

Using Lemma 4 and (3.6) with $S(J_n)$, $n \in A_k$, we deduce that

$$\int_{S(J_n)} |F'(z)|^2 (1-|z|^2)^s \, dA(z) \ge Ce^{-2k^2}|J_n|^s \ge Ce^{-2k^2}(1-r_n)^s.$$

Moreover if $n \in B_k$,

$$\int_{J_n} \log \frac{1}{|F(\xi)|} \, d\xi \ge \frac{1}{2}k^2|J_n| = \frac{1}{2}(1-r_n)^s e^{-2k^2}.$$

So, bearing in mind (3),

$$\sum_{n\in A_k} \int_{S(J_n)} |F'(z)|^2 (1-|z|^2)^s \, dA(z) + \sum_{n \in B_k} \int_{J_n} \log \frac{1}{|F(\xi)|} \, d\xi$$

$$\ge Ce^{-2k^2} \sum_{n=n_k}^{m_k} (1-r_n)^s \ge Ck,$$

which together with the integrability of $\log|F|$ on the boundary (see Theorem 2.2 of [18]), implies that F must be the zero function. Finally, arguing as in the proof of Theorem 2 of [36], the proof can be finished. □

4. Zeros on the boundary. Sets of uniqueness

In order to prove Theorem 5, the notion of α-capacity must be introduced. We shall recall some definitions (see [41] and [8]). Given $E \subset [0, 2\pi)$, let $\mathcal{P}(E)$ be the set of all probability measures supported on E. If $\alpha > 0$ and $\sigma \in \mathcal{P}(E)$, the α-potential associated to σ is

$$U_\alpha \sigma(\tau) = \int_E \frac{d\sigma(\theta)}{|\theta - \tau|^\alpha}.$$

Let

$$V_{E,\alpha} = \inf \int_E U_\alpha \sigma(\tau) \, d\sigma(\tau),$$

where the infimum is taken over all $\sigma \in \mathcal{P}(E)$. If $V_{E,\alpha} < \infty$, there is $\mu \in \mathcal{P}(E)$ where the value $V_{E,\alpha}$ is attained, and that measure μ is called the equilibrium distribution for the α-potentials of E. It is known that $U_\alpha\mu(\tau) = V_{E,\alpha}$ for a.e. (μ). The α-capacity of E is determined by

$$C_\alpha(E) = (V_{E,\alpha})^{-1}.$$

Proof of Theorem 5. Suppose that E is a set of uniqueness for $\mathcal{D}_s$. Then E is also a set of uniqueness for any Lipschitz class Λ_β with $\beta > \frac{1-s}{2}$, due to $\Lambda_\beta \subset D_s$. So, by Theorem 1 of [9], E is not a Carleson set.

For the converse, we shall follow the argument in the proof of Theorem 5 in [9]. Let μ be the equilibrium distribution for the α-potentials of E. Then, if $\{\gamma_n\}$ are the Fourier-Stieltjes coefficients of μ, there is a constant C which only depends on α such that

$$\sum_n n^{\alpha-1}|\gamma_n|^2 \leq CV_{E,\alpha}. \tag{4.1}$$

Suppose that there is a bounded function $f \in \mathcal{D}_s$, $f \neq 0$, that vanishes on E. We shall see that this leads to a contradiction. The function $h(\theta) = |f(e^{i\theta})|$ can be written as

$$h(\theta) = \sum_n c_n e^{in\theta},$$

where

$$\sum_n n^{1-s}|c_n|^2 < \infty. \tag{4.2}$$

For each $t \in (0, \pi)$, let us consider $h_t(\theta) = \frac{1}{2t}\int_{\theta-t}^{\theta+t} h(s)\, ds$. Integrating the Fourier series of h, it follows that the Fourier coefficients of h_t are $\frac{\sin(nt)}{nt}c_n$. Then by (4.1) and Schwarz's inequality,

$$\begin{aligned} \int_E h_t(\theta)\, d\mu(\theta) &= \left| \int_E (h_t(\theta) - h(\theta))\, d\mu(\theta) \right| \\ &= \left| \sum_n \left(1 - \frac{\sin(nt)}{nt}\right) c_n \int_E e^{in\theta}\, d\mu(\theta) \right| \\ &\leq C \sum_n \left(1 - \frac{\sin(nt)}{nt}\right) |c_n||\gamma_n| \\ &\leq C \left(\sum_n \left(1 - \frac{\sin(nt)}{nt}\right)^2 |c_n|^2 n^{1-\alpha} \right)^{\frac{1}{2}} \left(\sum_n n^{\alpha-1}|\gamma_n|^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{4.3}$$

We claim that there is $C > 0$ such that

$$n^{s-\alpha}\left(1 - \frac{\sin(nt)}{nt}\right)^2 \leq Ct^{\alpha-s}, \quad t > 0, \quad n = 1, 2, \ldots. \tag{4.4}$$

If $nt \leq 1$, there is a positive constant C which does not depend on n or t, such that $1 - \frac{\sin(nt)}{nt} \leq C(nt)^2$, so

$$n^{s-\alpha}\left(1 - \frac{\sin(nt)}{nt}\right)^2 \leq C^2 n^{s-\alpha}(nt)^4 \leq C^2 n^{s-\alpha}(nt)^{\alpha-s} \leq C^2 t^{\alpha-s}. \tag{4.5}$$

On the other hand, if $nt \geq 1$, bearing in mind that $1 - \frac{\sin(\theta)}{\theta}$ is a bounded function of θ, we deduce that

$$n^{s-\alpha}\left(1 - \frac{\sin(nt)}{nt}\right)^2 \leq Cn^{s-\alpha} \leq Ct^{\alpha-s},$$

which together with (4.5) gives (4.4).

Therefore, using (4.3), (4.4), (4.1) and (4.2), it follows that

$$\begin{aligned} \int_E h_t(\theta)\, d\mu(\theta) &\leq Ct^{\frac{\alpha-s}{2}} \left(\sum_n n^{1-s}|c_n|^2 \right)^{\frac{1}{2}} \left(\sum_n n^{\alpha-1}|\gamma_n|^2 \right)^{\frac{1}{2}} \\ &\leq Ct^{\frac{\alpha-s}{2}} ||f||_{D_s} V_{E,\alpha}^{1/2}. \end{aligned} \tag{4.6}$$

Now, let k_n be the number of complementary intervals of E whose lengths are in $[2^{-n}, 2^{-n+1})$. Since E is not a Carleson set,

$$\sum \frac{nk_n}{2^n} = \infty. \tag{4.7}$$

Let $\{\omega_i\}_{i=1}^{k_n}$ be those intervals, and let $\{\theta_i\}_{i=1}^{2k_n}$ be the endpoints of $\{\omega_i\}_{i=1}^{k_n}$. We consider the open intervals $\{\delta_i\}_{i=1}^{2k_n}$ of length 2^{-n} with midpoints $\{\theta_i\}_{i=1}^{2k_n}$. Take $\gamma \in \left(0, \frac{\alpha-s}{2}\right)$ and let S be the set of those δ_i such that

$$h_\tau(\theta_i) > 2^{-\gamma n}, \quad \tau = 2^{-n}. \tag{4.8}$$

Observe that (4.8) implies that $h_{2\tau}(\theta) > 2^{-\gamma n-1}$ holds for $\theta \in \delta_i$ whenever $\delta_i \in S$, which, together with the general relation (4.6), gives that for $\mu^\star$ the equilibrium distribution for the α-potentials of $E \cap S$,

$$2^{-\gamma n-1} \le \int_{E\cap S} h_\tau(\theta)\, d\mu^\star(\theta) \le C V_{E\cap S}^{1/2} 2^{-n\frac{(\alpha-s)}{2}},$$

so

$$C_\alpha(E \cap S) \le C 2^{(2\gamma-(\alpha-s))n}. \tag{4.9}$$

Let N be the number of intervals δ_i which belong to S. We shall estimate N using condition (1.6). Take μ_i to be the equilibrium distribution for the α-potentials of $E \cap \delta_i$. Let us consider $\sigma = N^{-1} \sum_{\delta_i \subset S} \mu_i$ and u the corresponding α-potential. Suppose that $\tau \in \delta_k$, where $\delta_k \in S$, and let δ_{k-1} and δ_{k+1} be the intervals in S which are on the left and on the right of δ_k. We shall define $\mathcal{F} = \{k-1, k, k+1\}$. Then bearing in mind that the intervals $\{\delta_j\}$ are disjoint, the distance between the intervals $\{\delta_j\}$, and condition (1.6) we deduce that

$$\begin{aligned}
u(\tau) &= \int_{E\cap S} \frac{d\sigma(\theta)}{|\theta-\tau|^\alpha} \\
&\le \sum_{j\in\mathcal{F}} \int_{\delta_j\cap S} \frac{d\sigma(\theta)}{|\theta-\tau|^\alpha} + \sum_{j=1,\, j\notin\mathcal{F}}^{N} \int_{\delta_j\cap S} \frac{d\sigma(\theta)}{|\theta-\tau|^\alpha} \\
&\le N^{-1} \left(\sum_{j\in\mathcal{F}} \int_{\delta_j\cap S} \frac{d\mu_j(\theta)}{|\theta-\tau|^\alpha} + \sum_{j=1,\, j\notin\mathcal{F}}^{N} \int_{\delta_j\cap S} \frac{d\mu_j(\theta)}{|\theta-\tau|^\alpha} \right) \\
&\le CN^{-1} \left(2^n + \sum_{j=1}^{N} \frac{1}{(j2^{-n})^\alpha} \right) \\
&\le CN^{-1} 2^n,
\end{aligned}$$

which together with (4.9) gives

$$N^{-1}2^n \ge Cu \ge \frac{C}{C_\alpha(E\cap S)} \ge C 2^{(-2\gamma+(\alpha-s))n},$$

so due to $\gamma < \frac{\alpha-s}{2}$, one obtains

$$N \le C 2^{pn}, \quad \text{for some } p \in (0,1). \tag{4.10}$$

If $\omega_\nu = (\theta_{2\nu-1}, \theta_{2\nu})$ and (4.8) does not hold for $\theta_{2\nu-1}$ and $\theta_{2\nu}$, then by the arithmetic-geometric inequality,

$$\begin{aligned}\frac{1}{|\omega_\nu|}\int_{\omega_\nu} \log h(\theta)\,d\theta &\le \log\left(\frac{1}{|\omega_\nu|}\int_{\omega_\nu} h(\theta)\,d\theta\right)\\ &\le \log\left[\frac{1}{|\omega_\nu|}\left(\int_{\theta_{2\nu-1}-2^{-n}}^{\theta_{2\nu-1}+2^{-n}} h(\theta)\,d\theta + \int_{\theta_{2\nu}-2^{-n}}^{\theta_{2\nu}+2^{-n}} h(\theta)\,d\theta\right)\right]\\ &= \log\left[\frac{2^{-(n+1)}}{|\omega_\nu|}\left(h_\tau(\theta_{2\nu-1}) + h_\tau(\theta_{2\nu})\right)\right]\\ &\le -\gamma n + C.\end{aligned}$$

By (4.10), the number of indices n for which the above inequality is true is greater than $k_n - 2N \ge k_n - C2^{pn}$. Hence

$$\sum_{\nu=1}^{k_n}\int_{\omega_\nu} \log h(\theta)\,d\theta \le -\gamma n 2^{-n}(k_n - C2^{pn}) + C\sum_{\nu=1}^{k_n}|\omega_\nu|,$$

which, joined to the fact that $p < 1$, gives

$$\int_0^{2\pi} \log h(\theta)\,d\theta \le -\gamma \sum_n n 2^{-n} k_n + C.$$

Consequently, bearing in mind that $\gamma > 0$ and (4.7), this implies a contradiction. □

5. Blaschke sets

A subset A of the unit disc $\mathbb{D}$ is called a *Blaschke set* for $\mathcal{D}$ if any Blaschke sequence with elements in A is a zero set of $\mathcal{D}$. These sets were characterized by Bogdan in [7]. Here we shall give a new proof of that result.

Theorem 6. *$A \subset \mathbb{D}$ is a Blaschke set for $\mathcal{D}$ if and only if*

$$\int_{\mathbb{T}} \log \operatorname{dist}(e^{it}, A)\,dt > -\infty. \tag{5.1}$$

Some definitions and results will be introduced. A *tent* is an open subset T of $\mathbb{D}$ bounded by an arc $I \subset \mathbb{T}$ with $|I| < \frac{1}{4}$ and two straight lines through the endpoints of I forming with I an angle of $\frac{\pi}{4}$. The closed arc $\overline{I}$ will be called the base of the tent $T = T_I$. A tent T is said to support A if $T \cap A = \emptyset$ but $\overline{T} \cap \overline{A} \ne \emptyset$. A finite or countable collection of tents $\{T_n\}$ is an *A-belt* if $\{T_n\}$ are pairwise disjoint, A-supporting and $\mathbb{T} \setminus \overline{A} \subset \bigcup_n \overline{T_n}$. The following result can be found in [24, Lemma 1].

Lemma B. *Let $A \subset \mathbb{D}$ such that $\mathbb{T} \setminus \overline{A} \ne \emptyset$. Let $\{T_{I_n}\}$ be an A-belt. Then* (5.1) *holds if and only if $\overline{A} \cap \mathbb{T}$ has zero Lebesgue measure, and*

$$\sum_n |I_n| \log\left(\frac{e}{|I_n|}\right) < \infty.$$

Lemma 5. *Let $\{z_n\}$ be a $\mathcal{D}$-zero set. If $\{\lambda_n\} \subset \mathbb{D}$ satisfies that $\varrho(z_n, \lambda_n) < \delta < 1$ for each n, then $\{\lambda_n\}$ is a $\mathcal{D}$-zero set.*

Proof. Since $Z=\{z_n\}$ is a $\mathcal{D}$-zero set, there is a function g in $\mathcal{D}$ such that $gB_Z \in \mathcal{D}$, where B_Z is the Blaschke product with zeros $\{z_n\}$. By Carleson's formula for the Dirichlet integral (see [11] and also [35]), we have

$$\begin{aligned}\|gB_\Lambda\|^2_{\mathcal{D}} &= \|g\|^2_{\mathcal{D}} + \int_{\mathbb{T}} \sum_n P_{\lambda_n}(e^{it})\, |g(e^{it})|^2\, dt \\ &\le \|g\|^2_{\mathcal{D}} + C \int_{\mathbb{T}} \sum_n P_{z_n}(e^{it})\, |g(e^{it})|^2\, dt \\ &\le C\, \|gB_Z\|^2_{\mathcal{D}} < \infty.\end{aligned}$$

Hence, $\{\alpha_n\}$ is a $\mathcal{D}$-zero set, and the proof is complete. □

Remark 1. Note that this result implies that, if A is a Blaschke set for $\mathcal{D}$ and $\{w_k\}$ is a sequence such that $\varrho(\{w_k\}, A) \le C < 1$, then $A \cup \{w_k\}$ is also a a Blaschke set for $\mathcal{D}$.

Proof of Theorem 6. Suppose that (5.1) holds, and let Z be a Blaschke sequence of points in A. Then

$$\int_{\mathbb{T}} \log \operatorname{dist}(e^{it}, Z)\, dt > -\infty,$$

and by a result of Taylor and Williams in [40], Z is a Λ_α-zero set for any α. Since $\Lambda_\alpha \subset \mathcal{D}$ for $\alpha > \frac{1}{2}$, it follows that A is a Blaschke set for $\mathcal{D}$.

Suppose that A is a Blaschke set for $\mathcal{D}$. We shall use Lemma B to see that (5.1) holds. Suppose that $|\overline{A} \cap \mathbb{T}| > 0$. Then we can choose a sequence $\{\varepsilon_n\}$ of positive numbers satisfying

$$\sum_n \varepsilon_n \le |\overline{A} \cap \mathbb{T}|, \qquad \sum_n \varepsilon_n \log \frac{1}{\varepsilon_n} = \infty,$$

and a collection of disjoint arcs $\{I_n\}$ in $\mathbb{T}$ such that

$$|I_n| = \varepsilon_n, \qquad I_n \cap \overline{A} \ne \emptyset, \qquad n \ge 1.$$

In order to construct this sequence of subsets $\{I_n\}$, take I_1 with $|I_1| = \varepsilon_1$ and $I_1 \cap \overline{A} \ne \emptyset$, and once I_n has been taken, choose I_{n+1} such that $I_{n+1} \cap \left(\overline{A} \setminus \bigcup_{j=1}^n I_j\right) \ne \emptyset$ with $|I_{n+1}| = \varepsilon_{n+1}$.

Next, take a sequence $\{w_n\} \subset A$ such that $\operatorname{dist}(w_n, I_n \cap \overline{A}) \le \varepsilon_n$ and let p_n be the integer part of $\varepsilon_n/(1-|w_n|)$. Let Z be the sequence of points in A that consists of p_n repetitions of each point w_n. Observe that Z is a Blaschke sequence,

$$\sum_{z\in Z} (1-|z|) = \sum_n p_n(1-|w_n|) \le \sum_n \varepsilon_n < \infty,$$

so that Z must be a sequence of zeros of $\mathcal{D}$. We also have

$$\begin{aligned}\int_{\mathbb{T}} \log\left(\sum_{z\in Z} \frac{1-|z|^2}{|e^{it}-z|^2}\right) dt &= \int_{\mathbb{T}} \log\left(\sum_n p_n \frac{1-|w_n|^2}{|e^{it}-w_n|^2}\right) dt \\ &\geq \sum_k \int_{I_k} \log\left(p_k \frac{1-|w_k|^2}{|e^{it}-w_k|^2}\right) dt \\ &\geq \sum_k |I_k| \log\left(p_k \frac{1-|w_k|^2}{4\varepsilon_k^2}\right) \\ &\geq \sum_k \varepsilon_k \log\left(\frac{1}{8\varepsilon_k}\right) = \infty,\end{aligned}$$

which gives a contradiction with condition (1.5). Therefore, $\overline{A}\cap\mathbb{T}$ has zero Lebesgue measure.

Next, let $\{T_n\}$ be an A-belt. Then for each n there is $w_n \in \overline{A}\cap\partial T_n$. We may assume that w_n belongs to A. Indeed, if w_n is an endpoint of the arc I_n, there is a point $\alpha_n \in A$ which is in the Stolz angle with vertex w_n and aperture $\frac{\pi}{2}$. Consequently, if $\tilde{\alpha}_n$ is the closest point in ∂T_n with the same modulus as α_n, then $\varrho(\alpha_n, \tilde{\alpha}_n) \leq C < 1$, where C is independent of n, and now we can use the remark after Lemma 5.

Let v_n be the vertex of the tent T_n. Since $\{I_n\}$ is a sequence of disjoint arcs, $\{v_n\}$ is a Blaschke sequence. We denote by q_n the integer part of $(1-|v_n|)/(1-|w_n|)$ and we consider Z to be the sequence of points in A that consists of q_n repetitions of each point w_n. Arguing as before, it follows that Z is a Blaschke sequence, and moreover there is $C>0$ such that

$$|w_n - e^{it}|^2 \leq C|v_n - e^{it}|^2, \quad \text{for each } n \text{ and } e^{it}\in\mathbb{T}. \tag{5.2}$$

So, bearing in mind that A is a Blaschke set for $\mathcal{D}$, (1.5) and (5.2), we have that

$$\begin{aligned}\infty > \int_{\mathbb{T}} \log\left(\sum_{z\in Z} \frac{1-|z|^2}{|e^{it}-z|^2}\right) dt &= \int_{\mathbb{T}} \log\left(\sum_n q_n \frac{1-|w_n|^2}{|e^{it}-w_n|^2}\right) dt \\ &\geq \int_{\mathbb{T}} \log\left(C\sum_n q_n \frac{1-|w_n|^2}{1-|v_n|^2}\frac{1-|v_n|^2}{|e^{it}-v_n|^2}\right) dt \\ &\geq \int_{\mathbb{T}} \log\left(\sum_n C \frac{1-|v_n|^2}{|e^{it}-v_n|^2}\right) dt \\ &\geq \sum_k \int_{I_k} \log\left(C \frac{1-|v_k|^2}{|e^{it}-v_k|^2}\right) dt \\ &\geq \sum_k |I_k| \log\left(\frac{C}{|I_k|}\right).\end{aligned}$$

This finishes the proof. □

6. Other results

6.1. **Other necessary angular conditions on $\mathcal{D}_s$-zero sets.** First we shall prove the following result of its own interest.

Lemma 6. *Suppose that $0 < s < 1$, B is a Blaschke product with ordered sequence of zeros $\{z_k\}_{k=1}^{\infty}$ and $f \in \mathcal{D}_s$. Then*

$$||fB||^2_{\mathcal{D}_s} \asymp ||f||^2_{\mathcal{D}_s} + \sum_{k=1}^{\infty}(1-|z_k|^2)\int_{\mathbb{D}} \frac{|f(z)|^2|B_k(z)|^2}{|1-\overline{z_k}z|^2}\frac{dA(z)}{(1-|z|^2)^{1-s}},$$

where $B_k(z)$ is the Blaschke product of the first $k-1$ zeros.

Proof. Bearing in mind (2.2), the result follows from the identity (see [3, p. 191])

$$\frac{1-|B(z)|^2}{1-|z|^2} = \sum_k |B_k(z)|^2 \frac{1-|z_k|^2}{|1-\overline{z_k}z|^2}, \quad z \in \mathbb{D}.$$

□

We also obtain different conditions from (1.4) (which can work for any Blaschke sequence) on the angular distribution of a Blaschke sequence $\{z_k\}$ to be a $\mathcal{D}_s$-zero set, $0 < s < 1$.

Proposition 1. *Suppose that $0 < s < 1$ and $\{z_k\} \subset \mathbb{D}$. If there exists $r_0 \in (0,1)$ such that*

$$M(\{z_k\}) \overset{\text{def}}{=} \inf_{r_0 \le |z| < 1} \sum_k \frac{(1-|z_k|^2)(1-|z|^2)^s}{|1-\overline{z_k}z|^2} > 0, \tag{6.1}$$

then $\{z_k\}$ is not a $\mathcal{D}_s$-zero set.

Proof. Suppose that $\{z_k\}$ is a $\mathcal{D}_s$-zero set and satisfies (6.1). Then, there exists $F \in \mathcal{D}_s$ which vanishes uniquely on $\{z_k\}$, so $F = f \cdot B$, where $f \in \mathcal{D}_s$ and B is the Blaschke product with zeros $\{z_k\}$. Thus, Lemma 6 and (6.1) imply that

$$\begin{aligned}\infty &> \sum_k (1-|z_k|^2)\int_{\mathbb{D}} \frac{|f(z)|^2|B_k(z)|^2}{|1-\overline{z_k}z|^2}\frac{dA(z)}{(1-|z|^2)^{1-s}} \\ &\ge \int_{\mathbb{D}} |f(z)|^2|B(z)|^2 \left(\sum_k \frac{(1-|z_k|^2)(1-|z|^2)^s}{|1-\overline{z_k}z|^2}\right)\frac{dA(z)}{(1-|z|^2)} \\ &\ge M(\{z_k\})\int_{\mathbb{D}} |F(z)|^2 \frac{dA(z)}{(1-|z|^2)};\end{aligned}$$

consequently $F \equiv 0$. This finishes the proof. □

This result allows us to make constructions of Blaschke sequences which are not $\mathcal{D}_s$-zero sets.

Corollary 2. *For $0 < s < 1$, set*

$$z^{(s)}_{k,j} \overset{\text{def}}{=} \left(1 - 2^{-\frac{2}{1+s}k}\right)\exp\left(\frac{2\pi j}{2^k} i\right), \quad k = 0,1,2,\dots, \quad j = 0,1,\dots,2^k - 1.$$

The sequence $\{z^{(s)}_{k,j}\}$ is not a $\mathcal{D}_s$-zero set.

Proof. There is $\beta = \beta(s) > 0$ such that for each $z \in \mathbb{D}$ we can find a pair $(k(z), j(z))$ with $1-|z| \asymp 1 - |z_{k(z),j(z)}|$, and

$$|1 - \overline{z_{k(z),j(z)}}z|^2 \le \beta(1-|z|^2)^{1+s}.$$

Therefore

$$\sum_{k=0}^{\infty}\sum_{j=0}^{2^k-1}\frac{(1-|z_{k,j}|^2)(1-|z|^2)^s}{|1-\overline{z_{k,j}}z|^2}\geq\frac{(1-|z_{k(z),j(z)}|^2)(1-|z|^2)^s}{|1-\overline{z_{k,j}}z|^2}\geq C\beta^{-1},$$

so, by Proposition 2, $\{z_{k,j}^{(s)}\}$ is not a $\mathcal{D}_s$-zero set. □

6.2. **Möbius invariant spaces generated by $\mathcal{D}_s$.** The *space* Q_s, $0\leq s<\infty$, is the Möbius invariant space generated by $\mathcal{D}_s$, that is, $f\in Q_s$ if

$$\sup_{a\in\mathbb{D}}\|f\circ\varphi_a-f(a)\|^2_{\mathcal{D}_s}<\infty.$$

It is known that Q_1 coincides with $BMOA$. However, if $0<s<1$, Q_s is a proper subspace of $BMOA$ and has many interesting properties (see the detailed monograph [42]).

As usual, for a space of analytic functions X, we shall write $M(X)$ for the algebra of (pointwise) multipliers of X, that is,

$$M(X)\overset{\text{def}}{=}\{g\in H(\mathbb{D}):gf\in X\text{ for all }f\in X\}.$$

Theorem 7. *Suppose that $0<s\leq 1$. Then $\mathcal{D}_s$, Q_s, $Q_s\cap H^\infty$ and $M(\mathcal{D}_s)$ have the same zero sets.*

Proof. If $s=1$, the result is well known because $\mathcal{D}_1=H^2$, $M(H^2)=H^\infty$ and $Q_1=BMOA$. If $0<s<1$, by [26, Corollary 13] the zeros sets of $\mathcal{D}_s$ and $M(\mathcal{D}_s)$ coincide, so the result follows from the chain of embeddings (see [4, Lemma 5.1])

$$M(\mathcal{D}_s)\subset Q_s\cap H^\infty\subset Q_s\subset\mathcal{D}_s.$$

This finishes the proof. □

Since from different values of $s\in(0,1)$, the D_s-zero sets are not the same, we obtain directly the following result.

Corollary 3. *Suppose that $0\leq s<p<1$. Then there exists $Z\subset\mathbb{D}$, which is a Q_p-zero set but not a Q_s-zero set.*

A stronger result, in the following sense, can be proved. A sequence $\{z_n\}$ is interpolating for $Q_p\cap H^\infty$, $0<p<1$, if for each bounded sequence $\{w_k\}$ of complex numbers, there exists $f\in Q_p\cap H^\infty$ such that $f(z_k)=w_k$ for all k. A characterization of these sequences in terms of p-Carleson measures is given in [30]. It is clear that each interpolating sequence for $Q_p\cap H^\infty$ is a $\mathcal{D}_p$-zero set.

Theorem 8. *Suppose that $0<s<p<1$. Then, there exists $Z=\{z_n\}_{n=0}^\infty\subset\mathbb{D}$ which is an interpolating sequence for $Q_p\cap H^\infty$ and such that it is not a $\mathcal{D}_s$-zero set.*

Proof. Set

$$z_n=\left(1-\frac{1}{n^{1/s}}\right)e^{i\theta_n},\quad n=2,3,\dots,$$

where

$$\theta_n=\sum_{k=1}^{n-1}\frac{1}{k}+\frac{1}{2n},\quad n=2,3,\dots.$$

The proof of [29, Theorem 5.10] gives that $\{z_n\}$ is not a $\mathcal{D}_s$-zero set. Moreover, borrowing the argument of the proof of [32, Theorem 2], we have that $\{z_n\}$ is separated and $\mu_{z_n,p} = \sum_n (1-|z_n|)^p \delta_{z_n}$ is a p-Carleson measure. So [30, Theorem 1.3] gives that $\{z_n\}$ is an interpolating sequence for $Q_p \cap H^\infty$. This finishes the proof. $\square$

Finally, we note that in a recent paper [31], the algebra of (pointwise) multipliers of Q_s, $0 < s < 1$, has been characterized in terms of α-logarithmic s-Carleson measures. Using Corollary 3 as a main tool we shall prove the following result.

Corollary 4. *Suppose that* $0 < s < p < 1$. *Then*

$$M(Q_p, Q_s) \stackrel{\text{def}}{=} \{g \in H(\mathbb{D}) : gf \in Q_s \textit{ for all } f \in Q_p\} = \{0\}\,.$$

Proof. Suppose that $M(Q_p, Q_s) \neq \{0\}$. Let $g \in M(Q_p, Q_s)$, $g \neq 0$ and denote by W its zero set. By Corollary 3 there exists $f \in Q_p$, $f \neq 0$, whose sequence of zeros Z is not a Q_s-zero set. It is clear that $Z \cup W$ is the zero set of $fg \in Q_s$, and since $g \in Q_s$, W satisfies the Blaschke condition. Now, taking B to be the Blaschke product with zeros W and bearing in mind that Q_s has the f-property (see Corollary 1 of [14] or Corollary 5.4.1 of [42]), we obtain that $\frac{fg}{B} \in Q_s$, whose zero set is Z. This finishes the proof. $\square$

7. Further remarks

We would like to emphasize that conditions (*ii*) and (*iii*) of Theorem 1 are equivalent when $\{z_n\}$ is a finite union of separated Blaschke sequences. So, it seems natural to ask whether or not for finite unions of separated Blaschke sequences, condition (*ii*) implies that $\{z_n\}$ is a $\mathcal{D}_s$-zero set. Although we are not able to answer this question, if the function g has some additional regularity properties, one can prove that condition (*ii*) implies that $\{z_n\}$ is a $\mathcal{D}_s$-zero set, as the following result shows.

Proposition 2. *Let* $\{z_n\} \subset \mathbb{D}$ *be a Blaschke sequence,* $0 < s < 1$ *and* $\alpha > \frac{1-s}{2}$. *If there exists a function* $g \in \Lambda_\alpha$ *such that*

$$\sum_n |g(z_n)|^2 \, (1-|z_n|^2)^s < \infty,$$

then $\{z_n\}$ *is a* $\mathcal{D}_s$*-zero set.*

Proof. Let B be the Blaschke product with zeros $\{z_n\}$. We shall prove that $gB \in \mathcal{D}_s$. Using the fact that $g \in \Lambda_\alpha$, and [43, Lemma 4.2.2], one has

$$\begin{aligned}
&\sum_n (1-|z_n|^2) \int_{\mathbb{D}} |g(z)-g(z_n)|^2 \, \frac{(1-|z|^2)^{s-1}}{|1-\bar{z}_n z|^2} \, dA(z) \\
&\qquad \leq C \sum_n (1-|z_n|^2) \int_{\mathbb{D}} \frac{(1-|z|^2)^{s-1}}{|1-\bar{z}_n z|^{2-2\alpha}} \, dA(z) \\
&\qquad \leq C \sum_n (1-|z_n|^2) < \infty.
\end{aligned} \tag{7.1}$$

Also, by our assumption and [43, Lemma 4.2.2],

$$\begin{aligned}
&\sum_n (1-|z_n|^2)|g(z_n)|^2 \int_{\mathbb{D}} \frac{(1-|z|^2)^{s-1}}{|1-\bar{z}_n z|^2} \, dA(z) \\
&\qquad \leq C \sum_n |g(z_n)|^2 \, (1-|z_n|^2)^s < \infty.
\end{aligned} \tag{7.2}$$

Now, since $\Lambda_\alpha \subset \mathcal{D}_s$ for $\alpha > \frac{1-s}{2}$, it follows easily from (7.1) and (7.2) that

$$\|gB\|^2_{\mathcal{D}_s} \leq C\|g\|^2_{\mathcal{D}_s} + C\int_{\mathbb{D}} |(gB')(z)|^2\,(1-|z|^2)^s\,dA(z) < \infty.$$

□

In view of all this, we state the following related problem.

Problem. For $0 < s < 1$, describe those separated Blaschke sequences $\{z_n\} \subset \mathbb{D}$ such that there is $g \in \mathcal{D}_s,\ g \neq 0$, with

$$\sum_n |g(z_n)|^2\,(1-|z_n|^2)^s < \infty.$$

Another interesting problem is to find sufficient conditions in order for a sequence $\{z_n\}$ to be a zero set for the analytic Besov space B_p, $1 < p < \infty$ (see [43, Chapter 5]). Since the point evaluations are bounded linear functionals in B_p, there are reproducing kernels $k_z \in B_{p'}$, where p' is the conjugate exponent of p. Also, it is well known that

$$\|k_z\|^{-p}_{B_{p'}} \asymp \left(\log\frac{1}{1-|z|}\right)^{-(p-1)}.$$

So, bearing in mind (1.1), it seems natural to ask the following.

Question. Let $1 < p < \infty$, and let $\{z_n\} \subset \mathbb{D}$ such that

$$\sum_n \left(\log\frac{1}{1-|z_n|^2}\right)^{-(p-1)} < \infty.$$

Is the sequence $\{z_n\}$ a B_p-zero set?

In order to answer that question, it seems that a more constructive proof of the case $p = 2$ (the Shapiro-Shields result [39]) must be given, not relying so heavily on Hilbert space techniques.

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