WEAK FACTORIZATION AND HANKEL FORMS FOR WEIGHTED BERGMAN SPACES ON THE UNIT BALL

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ABSTRACT. We establish weak factorizations for a weighted Bergman space A_α^p, with $1 < p < \infty$, into two weighted Bergman spaces on the unit ball of $\mathbb{C}^n$. To obtain this result, we characterize bounded Hankel forms on weighted Bergman spaces on the unit ball of $\mathbb{C}^n$.

1. INTRODUCTION

A classical theorem of Riesz asserts that any function in the Hardy space H^p on the unit disk can be factored as $f = Bg$ with $\|f\|_{H^p} = \|g\|_{H^p}$, where B is a Blaschke product and g is an H^p-function with no zeros on the unit disk. An immediate consequence of that result is that any function in the Hardy space H^p admits a "strong" factorization $f = f_1 f_2$ with $f_1 \in H^{p_1}$, $f_2 \in H^{p_2}$ and $\|f_1\|_{H^{p_1}} \cdot \|f_2\|_{H^{p_2}} = \|f\|_{H^p}$, for any p_1 and p_2 determined by the condition $1/p = 1/p_1 + 1/p_2$. In [12], C. Horowitz obtained strong factorizations of functions in a weighted Bergman space on the unit disk into functions of two weighted Bergman spaces with the same weight (again using Blaschke products). These strong factorization results are no longer possible to obtain [11] in the setting of Hardy and Bergman spaces in the unit ball of the complex euclidian space $\mathbb{C}^n$ of dimension n when $n \geq 2$, but it is still possible to obtain some "weak" factorizations for functions in these spaces.

For two Banach spaces of functions, A and B, defined on the same domain, the weakly factored space $A \odot B$ is defined as the completion of finite sums

$$f = \sum_k \varphi_k \psi_k, \qquad \{\varphi_k\} \subset A, \ \{\psi_k\} \subset B,$$

with the following norm:

$$\|f\|_{A \odot B} = \inf\left\{ \sum_k \|\varphi_k\|_A \|\psi_k\|_B : f = \sum_k \varphi_k \psi_k \right\}.$$

When $0 < p \leq 1$, weak type factorizations for the Hardy spaces H^p and the weighted Bergman spaces A_α^p on the unit ball of $\mathbb{C}^n$ are well known (see [6] and [9] for Hardy spaces; and [5], [20] or [24, Corollary 2.33] for Bergman spaces). However, when $1 < p < \infty$, even for unweighted Bergman spaces the problem was still open (see, for example [4]).

In this paper we completely solve the above problem for Bergman spaces by establishing weak factorizations for a weighted Bergman space A_β^q, with $1 < q < \infty$ and $\beta > -1$, into

two weighted Bergman spaces with non necessarily the same weight, on the unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$. The following is our main result.

Theorem 1. *Let* $1 < q < \infty$ *and* $\beta > -1$. *Then*

$$A_\beta^q(\mathbb{B}_n) = A_{\alpha_1}^{p_1}(\mathbb{B}_n) \odot A_{\alpha_2}^{p_2}(\mathbb{B}_n)$$

for any $p_1, p_2 > 0$ *and* $\alpha_1, \alpha_2 > -1$ *satisfying*

(1) $$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q}, \qquad \frac{\alpha_1}{p_1} + \frac{\alpha_2}{p_2} = \frac{\beta}{q}.$$

In this context, by "=" we mean equality of the function spaces and equivalence of the norms. The inclusion $A_{\alpha_1}^{p_1} \odot A_{\alpha_2}^{p_2} \subset A_\beta^q$ with the estimate $\|f\|_{q,\beta} \lesssim \|f\|_{A_{\alpha_1}^{p_1} \odot A_{\alpha_2}^{p_2}}$ is a direct consequence of Minkowski and Hölder inequalities, so that the interesting part is the other inclusion with the corresponding estimates for the norms.

Now we are going to recall the definition of the weighted Bergman spaces. First we need some notations. For any two points $z = (z_1, ..., z_n)$ and $w = (w_1, ..., w_n)$ in $\mathbb{C}^n$, we use

$$\langle z, w \rangle = z_1 \bar{w}_1 + \cdots + z_n \bar{w}_n$$

to denote the inner product of z and w, and

$$|z| = \sqrt{\langle z, z \rangle} = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$$

to denote the norm of z in $\mathbb{C}^n$. Let $\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}$ be the unit ball in $\mathbb{C}^n$ and $\mathbb{S}_n = \{z \in \mathbb{C}^n : |z| = 1\}$ be the unit sphere in $\mathbb{C}^n$. Let $H(\mathbb{B}_n)$ be the space of all analytic functions on $\mathbb{B}_n$. We use dv to denote the normalized volume measure on $\mathbb{B}_n$ and $d\sigma$ to denote the normalized area measure on $\mathbb{S}_n$. For $-1 < \alpha < \infty$, we let $dv_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha dv(z)$ denote the normalized weighted volume measure on $\mathbb{B}_n$, where $c_\alpha = \Gamma(n + \alpha + 1)/[n!\Gamma(\alpha + 1)]$.

For $0 < p < \infty$ and $-1 < \alpha < \infty$, let $L^p(\mathbb{B}_n, dv_\alpha)$ be the weighted Lebesgue space which contains measurable functions f on $\mathbb{B}_n$ such that

$$\|f\|_{p,\alpha}^p = \int_{\mathbb{B}_n} |f(z)|^p \, dv_\alpha(z) < \infty.$$

Denote by $A_\alpha^p = L^p(\mathbb{B}_n, dv_\alpha) \cap H(\mathbb{B}_n)$, the weighted Bergman space on $\mathbb{B}_n$, with the same norm as above. If $\alpha = 0$, we simply write them as $L^p(\mathbb{B}_n, dv)$ and A^p respectively and $\|f\|_p$ for the norm of f in these spaces.

It is a well-known fact that to obtain weak factorization results is equivalent to give a "good" description of the boundedness of certain *Hankel forms*. A Hankel form is a bilinear form B on a space of analytic functions such that for any f and g, $B(f, g)$ is a linear function of fg. These forms have been extensively studied on Hardy spaces and on Bergman spaces. For the case of the Hardy space on the unit disk, a classical result by Nehari [18] says that the Hankel form

$$B_b(f, g) := \langle fg, b \rangle$$

(under the usual integral pair for Hardy spaces) with an *analytic symbol* b is bounded on $H^2 \times H^2$ if and only if $b \in BMOA$, the space of analytic functions of bounded mean oscillation. The proof used the fact that a function in H^1 can be factored into product of two functions in H^2. Unfortunately, such strong factorization is not possible (see [11]) for Hardy spaces in the unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$. However, Coifman, Rochberg and Weiss [6] were

able to generalize Nehari's result to the unit ball $\mathbb{B}_n$ by using a weak factorization of H^1. Namely, they proved that

$$H^2(\mathbb{B}_n) \odot H^2(\mathbb{B}_n) = H^1(\mathbb{B}_n).$$

Our approach to the problem for weighted Bergman spaces on the unit ball is the opposite to the one of Coifman, Rochberg and Weiss in [6]. We first characterize boundedness of the Hankel forms on weighted Bergman spaces, and with this result the weak factorization easily follows.

Given $\alpha > -1$ and a holomorphic symbol function b we define the associated Hankel type bilinear form T_b^α for polynomials f and g by

$$T_b^\alpha(f, g) = \langle fg, b \rangle_\alpha,$$

where the integral pair $\langle\,,\,\rangle_\alpha$ is defined as

$$(2) \qquad \langle \varphi, \psi \rangle_\alpha = \int_{\mathbb{B}_n} \varphi(z) \, \overline{\psi(z)} \, dv_\alpha(z).$$

Since the polynomials are dense in the weighted Bergman spaces, the Hankel form T_b^α is densely defined on $A_{\alpha_1}^{p_1} \times A_{\alpha_2}^{p_2}$ for any $p_1, p_2 > 0$ and any $\alpha_1, \alpha_2 > -1$. We say that T_b^α is bounded on $A_{\alpha_1}^{p_1} \times A_{\alpha_2}^{p_2}$ if there exists a positive constant C such that

$$|T_b^\alpha(f, g)| \leq C \|f\|_{p_1, \alpha_1} \|g\|_{p_2, \alpha_2}.$$

The norm of T_b^α is given by

$$\|T_b^\alpha\| = \|T_b^\alpha\|_{A_{\alpha_1}^{p_1} \times A_{\alpha_2}^{p_2}} := \sup\{|T_b^\alpha(f, g)| : \|f\|_{p_1, \alpha_1} = \|g\|_{p_2, \alpha_2} = 1\}.$$

The next result characterizes boundedness of the Hankel form T_b^α acting on $A_{\alpha_1}^{p_1} \times A_{\alpha_2}^{p_2}$. We will see in Section 3 that this implies the weak factorization in Theorem 1.

Theorem 2. *Let $1 < p_1, p_2 < \infty$, and $\alpha, \alpha_1, \alpha_2 > -1$ satisfy*

$$(3) \qquad \frac{1}{p_1} + \frac{1}{p_2} < 1, \qquad \frac{1 + \alpha_1}{p_1} + \frac{1 + \alpha_2}{p_2} < 1 + \alpha.$$

Then T_b^α is bounded on $A_{\alpha_1}^{p_1} \times A_{\alpha_2}^{p_2}$ if and only $b \in A_{\beta'}^{q'}$, where q and β are real numbers satisfying (1)*, and q' and β' are determined by the condition*

$$(4) \qquad \frac{1}{q} + \frac{1}{q'} = 1, \qquad \frac{\beta}{q} + \frac{\beta'}{q'} = \alpha.$$

Furthermore, we have $\|T_b^\alpha\| \asymp \|b\|_{q', \beta'}$

Remarks. Note that, condition (3) guarantees that $q > 1$ and $\beta' > -1$. When q and β satisfy condition (1), automatically we would have $\beta > -1$ (to see this, simply add two equations in (1) together). By a general duality theorem for weighted Bergman spaces (see Theorem A in Section 2), the condition $b \in A_{\beta'}^{q'}$ means that the symbol b belongs to the dual space of A_β^q under the pairing given by (2).

It turns out that boundedness of the Hankel form T_b^α is equivalent to boundedness of a (small) Hankel operator, which we are going to introduce in a moment. Let $\alpha > -1$. It is well-known that, the integral operator

$$P_\alpha f(z) = \int_{\mathbb{B}_n} \frac{f(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}} \, dv_\alpha(w)$$

is the orthogonal projection from $L^2(\mathbb{B}_n, dv_\alpha)$ onto the weighted Bergman space A_α^2. The above formula can be used to extend P_α to a linear operator from $L^1(\mathbb{B}_n, dv_\alpha)$ into $H(\mathbb{B}_n)$. For $1 < p < \infty$, P_α is a bounded operator from $L^p(\mathbb{B}_n, dv_\alpha)$ onto A_α^p.

Denote by $\overline{A_\alpha^p}$ the conjugate analytic functions f on $\mathbb{B}_n$ that are in $L^p(\mathbb{B}_n, dv_\alpha)$. Clearly,

$$\overline{A_\alpha^p} = \{\overline{f} : f \in A_\alpha^p\}.$$

Let Q_α denote the orthogonal projection from $L^2(\mathbb{B}_n, dv_\alpha)$ onto $\overline{A_\alpha^2}$. Clearly one has

$$Q_\alpha f(z) = \overline{P_\alpha \overline{f}(z)} = \int_{\mathbb{B}_n} \frac{f(w)}{(1 - \langle w, z \rangle)^{n+1+\alpha}} \, dv_\alpha(w).$$

Given $f \in L^1(\mathbb{B}_n, dv_\alpha)$ and a polynomial g, the weighted (small) *Hankel operator* is defined by

$$h_f^\alpha g = Q_\alpha(fg).$$

Due to the density of polynomials, the small Hankel operator h_f^α is densely defined on the weighted Bergman space A_α^p for $1 \le p < \infty$. We will study boundedness of the small Hankel operator with conjugate analytic symbols, that is, $h_{\overline{f}}^\alpha$ with $f \in H(\mathbb{B}_n)$, from $A_{\alpha_1}^{p_1}$ to $\overline{A_{\alpha_2}^{p_2}}$ with $0 < p_2 < p_1 < \infty$.

Theorem 3. *Let $1 < p_2 < p_1 < \infty$ and $\alpha_1, \alpha_2 > -1$ such that*

$$(5) \qquad\qquad\qquad \frac{1+\alpha_1}{p_1} < \frac{1+\alpha_2}{p_2}.$$

Let $f \in H(\mathbb{B}_n)$ and α such that

$$(6) \qquad\qquad\qquad 1 + \alpha > \frac{1+\alpha_2}{p_2}.$$

Then $h_{\overline{f}}^\alpha : A_{\alpha_1}^{p_1} \to \overline{A_{\alpha_2}^{p_2}}$ is bounded if and only if $f \in A_\beta^q$, where q and β are real numbers such that

$$\frac{1}{q} = \frac{1}{p_2} - \frac{1}{p_1}, \qquad \frac{\beta}{q} = \frac{\alpha_2}{p_2} - \frac{\alpha_1}{p_1}.$$

Moreover, we have $\|h_{\overline{f}}^\alpha\| \asymp \|f\|_{q,\beta}$.

Remarks. Condition (5) guarantees that $\beta > -1$. It is known that, when $0 < p_2 < p_1 < \infty$, $A_{\alpha_1}^{p_1} \subset A_{\alpha_2}^{p_2}$ if and only if (5) is true (see [22, Theorem 70]). Hence the above result concerns the boundedness of $h_{\overline{f}}^\alpha$ from a smaller space to a larger space. Also, by [24, Theorem 2.11], condition (6) means that the integral operator P_α is a bounded projection from $L^{p_2}(\mathbb{B}_n, dv_{\alpha_2})$ onto $A_{\alpha_2}^{p_2}$.

If one considers the operator

$$S_f^\alpha g = \overline{h_{\overline{f}}^\alpha g(z)} = P_\alpha(f\overline{g}),$$

clearly, the boundedness of $h_{\overline{f}}^\alpha$ is equivalent to the boundedness of S_f^α from $A_{\alpha_1}^{p_1}$ to $A_{\alpha_2}^{p_2}$, and the norms of $h_{\overline{f}}^\alpha$ and S_f^α are equivalent. Now, if $g \in A_{\alpha_1}^{p_1}$ and $h \in A_{\alpha_2}^{p_2}$, by Fubini's theorem we easily obtain

$$T_f^\alpha(g, h) = \langle gh, f \rangle_\alpha = \langle h, P_\alpha(f\overline{g}) \rangle_\alpha = \langle h, S_f^\alpha g \rangle_\alpha.$$

Hence, for $p_2 > 1$, by duality (see Theorem A in Section 2), the Hankel form T_f^α is bounded on $A_{\alpha_1}^{p_1} \times A_{\alpha_2}^{p_2}$ if and only if the small Hankel operator $h_{\bar{f}}^\alpha$ is bounded from $A_{\alpha_1}^{p_1}$ to $\overline{A_{\alpha_2'}^{p_2'}}$, with equivalent norms. Here, the numbers α_2' and p_2' are defined by the relation

$$\frac{1}{p_2} + \frac{1}{p_2'} = 1, \qquad \alpha = \frac{\alpha_2}{p_2} + \frac{\alpha_2'}{p_2'}.$$

Comparing Theorem 2 with Theorem 3, notice that the first inequality in (3) is equivalent to condition $1 < p_2' < p_1 < \infty$. Also, when p_2 and α_2 are replaced by p_2' and α_2', condition (6) turns out to be equivalent to $\alpha_2 > -1$, and therefore is always satisfied; and the second inequality in (3) is equivalent to condition (5). Therefore, Theorem 3 implies Theorem 2.

In the case of the same weights, that is, when $\alpha_1 = \alpha_2 = \beta = \alpha$, all the restrictions in Theorem 3 reduces to $p_2 > 1$. We isolate this case here, since it proves a conjecture in [4].

Corollary 4. *Let $\alpha > -1$, $1 < p_2 < p_1 < \infty$ and $f \in H(\mathbb{B}_n)$. Then $h_{\bar{f}}^\alpha : A_\alpha^{p_1} \to \overline{A_\alpha^{p_2}}$ is bounded if and only if $f \in A_\alpha^q$, with $q = \frac{p_1 p_2}{p_1 - p_2}$.*

The paper is organized as follows: in Section 2 we give some necessary concepts and recall some key results which are needed in our proof of the main result. In Section 3 we give in detail the connection between weak factorizations and Hankel forms. The proof of Theorem 3 is given in Section 4.

In the following, the notation $A \lesssim B$ means that there is a positive constant C such that $A \leq CB$, and the notation $A \asymp B$ means that both $A \lesssim B$ and $B \lesssim A$ hold.

2. Preliminaries

We need the following duality theorem. In this generality the result is due to Luecking [16] (see also, Theorem 2.12 in [24]).

Theorem A. *Suppose $\beta, \beta' > -1$ and $1 < q < \infty$. Then*

$$(A_\beta^q)^* = A_{\beta'}^{q'}$$

(with equivalent norms) under the integral pair $\langle \, , \, \rangle_\alpha$ given by (2), where

$$\frac{1}{q} + \frac{1}{q'} = 1, \qquad \alpha = \frac{\beta}{q} + \frac{\beta'}{q'}.$$

We need the following well known integral estimate that can be found, for example, in [24, Theorem 1.12].

Lemma B. *Let $t > -1$ and $s > 0$. There is a positive constant C such that*

$$\int_{\mathbb{B}_n} \frac{(1 - |w|^2)^t \, dv(w)}{|1 - \langle z, w \rangle|^{n+1+t+s}} \leq C \, (1 - |z|^2)^{-s}$$

for all $z \in \mathbb{B}_n$.

For any $a \in \mathbb{B}_n$ with $a \neq 0$, we denote by $\varphi_a(z)$ the Möbius transformation on $\mathbb{B}_n$ that exchanges 0 and a. It is known that, for any $z \in \mathbb{B}_n$

$$\varphi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a \rangle},$$

where $s_a = 1 - |a|^2$, P_a is the orthogonal projection from $\mathbb{C}^n$ onto the one dimensional subspace $[a]$ generated by a, and Q_a is the orthogonal projection from $\mathbb{C}^n$ onto the orthogonal complement of $[a]$. When $a = 0$, $\varphi_a(z) = -z$. φ_a has the following properties: $\varphi_a \circ \varphi_a(z) = z$, and

$$1 - |\varphi_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \langle z, a \rangle|^2}.$$

For $z, w \in \mathbb{B}_n$, the *pseudo-hyperbolic distance* between z and w is defined by

$$\rho(z, w) = |\varphi_z(w)|,$$

and the *hyperbolic distance* on $\mathbb{B}_n$ between z and w induced by the Bergman metric is given by

$$\beta(z, w) = \tanh \rho(z, w).$$

For $z \in \mathbb{B}_n$ and $r > 0$, the *Bergman metric ball* at z is given by

$$D(z, r) = \{w \in \mathbb{B}_n : \beta(z, w) < r\}.$$

It is known that, for a fixed $r > 0$, the weighted volume

$$v_\alpha(D(z, r)) \asymp (1 - |z|^2)^{n+1+\alpha}.$$

We refer to [24] for all of the above facts.

A sequence $\{a_k\}$ of points in $\mathbb{B}_n$ is a *separated sequence* (in Bergman metric) if there exists a positive constant $\delta > 0$ such that $\beta(z_i, z_j) > \delta$ for any $i \neq j$. We need a well-known result on decomposition of the unit ball $\mathbb{B}_n$. The following version is Theorem 2.23 in [24]

Lemma C. *There exists a positive integer N such that for any $0 < r < 1$ we can find a sequence $\{a_k\}$ in $\mathbb{B}_n$ with the following properties:*

  (i) $\mathbb{B}_n = \cup_k D(a_k, r)$.
  (ii) *The sets $D(a_k, r/4)$ are mutually disjoint.*
  (iii) *Each point $z \in \mathbb{B}_n$ belongs to at most N of the sets $D(a_k, 4r)$.*

Any sequence $\{a_k\}$ satisfying the conditions of the above lemma is called a *lattice* (or an *r-lattice* if one wants to stress the dependence on r) in the Bergman metric. Obviously any r-lattice is separated.

For convenience, we will denote by $D_k = D(a_k, r)$ and $\tilde{D}_k = D(a_k, 4r)$. Then Lemma C says that $\mathbb{B}_n = \cup_{k=1}^\infty D_k$ and there is an positive integer N such that every point z in $\mathbb{B}_n$ belongs to at most N of sets $\tilde{D}_k$.

We also need the following atomic decomposition theorem for weighted Bergman spaces. This turns out to be a powerful theorem in the theory of Bergman spaces. The result is basically due to Coifman and Rochberg [5], and can be found in Chapter 2 of [24].

Theorem D. *Suppose $p > 0$, $\alpha > -1$, and*

$$b > n \max\left(1, \frac{1}{p}\right) + \frac{1 + \alpha}{p}.$$

Then we have

  (i) *For any separated sequence $\{a_k\}$ in $\mathbb{B}_n$ and any sequence $\lambda = \{\lambda_k\} \in \ell^p$, the function*

$$f(z) = \sum_{k=1}^\infty \lambda_k \frac{(1 - |a_k|^2)^{b-(n+1+\alpha)/p}}{(1 - \langle z, a_k \rangle)^b}$$

belongs to A_α^p and

$$\|f\|_{p,\alpha} \lesssim \|\lambda\|_{\ell^p}.$$

(ii) *There is an r-lattice $\{a_k\}$ in $\mathbb{B}_n$ such that, for any $f \in A_\alpha^p$, there is a sequence $\lambda = \{\lambda_k\} \in \ell^p$ with*

$$f(z) = \sum_{k=1}^\infty \lambda_k \frac{(1-|a_k|^2)^{b-(n+1+\alpha)/p}}{(1-\langle z, a_k\rangle)^b}.$$

and

$$\|\lambda\|_{\ell^p} \lesssim \|f\|_{p,\alpha}.$$

In the proof given in [24], part (i) requires that the sequence $\{a_k\}$ is an r-lattice for some $r \in (0,1]$, but it is well known that only the separation of the sequence $\{a_k\}$ is needed.

3. WEAK FACTORIZATIONS AND HANKEL FORMS

It is well known to specialists, but difficult to find in the literature, that the obtention of weak factorizations is equivalent to estimates for small Hankel operators or Hankel forms (in our case, estimates with loss). The equivalence between boundedness of the bilinear Hankel form T_b^α and weak factorization can be formulated as the following result. Since this is the basis for our obtention of the weak factorization for Bergman spaces, for completeness, we offer the proof here of the implication that gives the factorization.

Proposition 5. *Let $1 < q < \infty$ and $\alpha, \beta > -1$. Let p_1, p_2 and α_1, α_2 satisfy (3) and (1), and let q' and β' satisfy (4). The following are equivalent:*

(i) $A_\beta^q \subset A_{\alpha_1}^{p_1} \odot A_{\alpha_2}^{p_2}$ *with* $\|f\|_{A_{\alpha_1}^{p_1} \odot A_{\alpha_2}^{p_2}} \lesssim \|f\|_{q,\beta}$ *for $f \in A_\beta^q$.*

(ii) *For any analytic function b, if T_b^α is bounded on $A_{\alpha_1}^{p_1} \times A_{\alpha_2}^{p_2}$, then $b \in A_{\beta'}^{q'}$ with* $\|b\|_{q',\beta'} \lesssim \|T_b^\alpha\|$.

Proof. We will prove (ii) implies (i). The other implication is easier, and the interested reader can follow the argument in [1, Corollary 1.2] for a proof. By the atomic decomposition in Theorem D, we have the inclusion $A_\beta^q \subset A_{\alpha_1}^{p_1} \odot A_{\alpha_2}^{p_2}$. In order to have the corresponding estimate for the norms, we will show that, for any bounded linear functional F on $A_{\alpha_1}^{p_1} \odot A_{\alpha_2}^{p_2}$, there is a unique function $b_F \in A_{\beta'}^{q'}$ with $\|b_F\|_{q',\beta'} \lesssim \|F\|$ such that $F(f) = \langle f, b_F\rangle_\alpha$ for $f \in A_\beta^q$. This would give

$$\|f\|_{A_{\alpha_1}^{p_1} \odot A_{\alpha_2}^{p_2}} = \sup_{\|F\|=1} |F(f)| \leq \sup_{\|F\|=1} \|b_F\|_{q',\beta'} \cdot \|f\|_{q,\beta} \lesssim \|f\|_{q,\beta}.$$

Thus, suppose $F \in (A_{\alpha_1}^{p_1} \odot A_{\alpha_2}^{p_2})^*$ with norm $\|F\|$. Then for all $\varphi \in A_{\alpha_2}^{p_2}$ we have

$$|F(\varphi)| = |F(1 \cdot \varphi)| \leq \|F\| \cdot \|1\|_{p_1,\alpha_1} \cdot \|\varphi\|_{p_2,\alpha_2} = \|F\| \cdot \|\varphi\|_{p_2,\alpha_2}.$$

Hence $F \in (A_{\alpha_2}^{p_2})^*$, and so, by Theorem A, there is an unique function $b = b_F \in A_{\alpha_2'}^{p_2'}$ such that $F(\varphi) = \langle \varphi, b\rangle_\alpha$ for all $\varphi \in A_{\alpha_2}^{p_2}$, where p_2' and α_2' satisfy

$$\frac{1}{p_2} + \frac{1}{p_2'} = 1, \qquad \frac{\alpha_2}{p_2} + \frac{\alpha_2'}{p_2'} = \alpha.$$

Now, for polynomials g and h we have

$$\begin{aligned}
|T_b^\alpha(g,h)| = |\langle gh, b\rangle_\alpha| &= |F(gh)| \\
&\leq \|F\| \cdot \|gh\|_{A_{\alpha_1}^{p_1} \odot A_{\alpha_2}^{p_2}} \\
&\leq \|F\| \cdot \|g\|_{p_1,\alpha_1} \cdot \|h\|_{p_2,\alpha_2},
\end{aligned}$$

which shows that T_b^α is bounded on $A_{\alpha_1}^{p_1} \times A_{\alpha_2}^{p_2}$ with $\|T_b^\alpha\| \le \|F\|$. Therefore, we know that $b \in A_{\beta'}^{q'}$ with $\|b\|_{q',\beta'} \lesssim \|T_b^\alpha\| \lesssim \|F\|$. Hence $\Lambda(f) = \langle f, b \rangle_\alpha$ defines a bounded linear form on A_β^q, and coincides with F on polynomials. Thus, for $f \in A_\beta^q$, we have $F(f) = \Lambda(f) = \langle f, b \rangle_\alpha$. The proof is complete. $\qquad\square$

4. PROOF OF THEOREM 3

In this section we prove Theorem 3, from which Theorem 2 follows. Before that, for $s \ge 0$ and $\alpha > -1$, let $R^{\alpha,s}$ denote the unique continuous linear operator on $H(\mathbb{B}_n)$ satisfying

$$R^{\alpha,s}\left(\frac{1}{(1 - \langle z, w\rangle)^{n+1+\alpha}}\right) = \frac{1}{(1 - \langle z, w\rangle)^{n+1+\alpha+s}}$$

for all $w \in \mathbb{B}_n$. If $f \in A_\alpha^1$, then $R^{\alpha,s} f$ is given by the following integral expression

$$(7) \qquad R^{\alpha,s} f(z) = \int_{\mathbb{B}_n} \frac{f(w)}{(1 - \langle z, w\rangle)^{n+1+\alpha+s}} \, dv_\alpha(w).$$

More properties of the "differential type" operators $R^{\alpha,s}$ can be found in [24, Section 1.4]. Now we are ready to prove Theorem 3.

Proof of Theorem 3. As we noticed before, we just need to prove that $S_f^\alpha : A_{\alpha_1}^{p_1} \to A_{\alpha_2}^{p_2}$ is bounded if and only if $f \in A_\beta^q$.

Suppose first that $f \in A_\beta^q$. We need to show $S_f^\alpha : A_{\alpha_1}^{p_1} \to A_{\alpha_2}^{p_2}$ is bounded. Let $g \in A_{\alpha_1}^{p_1}$. If $p_2 > 1$ then $P_\alpha : L^{p_2}(\mathbb{B}_n, dv_{\alpha_2}) \to A_{\alpha_2}^{p_2}$ is bounded, and then from Hölder's inequality the result follows. Indeed,

$$\|S_f^\alpha g\|_{p_2,\alpha_2} = \|P_\alpha(f\bar{g})\|_{p_2,\alpha_2} \le C\|fg\|_{p_2,\alpha_2} \le C\|f\|_{q,\beta} \cdot \|g\|_{p_1,\alpha_1}$$

which shows that $S_f^\alpha : A_{\alpha_1}^{p_1} \to A_{\alpha_2}^{p_2}$ is bounded with

$$\|S_f^\alpha\| \lesssim \|f\|_{q,\beta}.$$

Conversely, suppose $S_f^\alpha : A_{\alpha_1}^{p_1} \to A_{\alpha_2}^{p_2}$ is bounded, we are going to show that $f \in A_\beta^q$. We begin with using an argument of Luecking (see, e.g., [17]). Let $r_k(t)$ be a sequence of Rademacher functions (see [8, Appendix A]). Let b be large enough so that

$$(8) \qquad b > n + \frac{1 + \alpha_1}{p_1}.$$

Fix any $r > 0$, and let $\{a_k\}$ be an r-lattice and $\{D_k\}$ be the associated sets in Lemma C. By Theorem D, we know that, for any sequence of real numbers $\lambda = \{\lambda_k\} \in \ell^{p_1}$, the function

$$g_t(z) = \sum_{k=1}^\infty \lambda_k r_k(t) \frac{(1 - |a_k|^2)^{b-(n+1+\alpha_1)/p_1}}{(1 - \langle z, a_k\rangle)^b}$$

belongs to $A_{\alpha_1}^{p_1}$ with $\|g_t\|_{p_1,\alpha_1} \lesssim \|\lambda\|_{\ell^{p_1}}$ for almost every t in $(0, 1)$. Denote by

$$g_k(z) = \frac{(1 - |a_k|^2)^{b-(n+1+\alpha_1)/p_1}}{(1 - \langle z, a_k\rangle)^b}.$$

Since $S_f^\alpha : A_{\alpha_1}^{p_1} \to A_{\alpha_2}^{p_2}$ is bounded, we get that

$$\begin{aligned}
\|S_f^\alpha g_t\|_{p_2,\alpha_2}^{p_2} &= \int_{\mathbb{B}_n} \left|\sum_{k=1}^\infty \lambda_k r_k(t) S_f^\alpha g_k(z)\right|^{p_2} dv_{\alpha_2}(z) \\
&\lesssim \|S_f^\alpha\|^{p_2} \cdot \|g_t\|_{p_1,\alpha_1}^{p_2} \lesssim \|S_f^\alpha\|^{p_2} \cdot \|\lambda\|_{\ell^{p_1}}^{p_2}
\end{aligned}$$

for almost every t in $(0, 1)$. Integrating both sides with respect to t from 0 to 1, and using Fubini's Theorem and Khinchine's inequality (see [21, p.12]), we get

$$(9) \qquad \int_{\mathbb{B}_n} \left(\sum_{k=1}^{\infty} |\lambda_k|^2 |S_f^{\alpha} g_k(z)|^2 \right)^{p_2/2} dv_{\alpha_2}(z) \lesssim \|S_f^{\alpha}\|^{p_2} \cdot \|\lambda\|_{\ell^{p_1}}^{p_2}.$$

Now we estimate

$$(10) \quad \sum_{k=1}^{\infty} |\lambda_k|^{p_2} \int_{\tilde{D}_k} |S_f^{\alpha} g_k(z)|^{p_2} dv_{\alpha_2}(z) = \int_{\mathbb{B}_n} \left(\sum_{k=1}^{\infty} |\lambda_k|^{p_2} |S_f^{\alpha} g_k(z)|^{p_2} \chi_{\tilde{D}_k}(z) \right)^{\frac{2}{p_2} \cdot \frac{p_2}{2}} dv_{\alpha_2}(z).$$

If $p_2 \geq 2$, then $2/p_2 \leq 1$, and from (10) we have

$$\sum_{k=1}^{\infty} |\lambda_k|^{p_2} \int_{\tilde{D}_k} |S_f^{\alpha} g_k(z)|^{p_2} dv_{\alpha_2}(z)$$

$$\leq \int_{\mathbb{B}_n} \left(\sum_{k=1}^{\infty} |\lambda_k|^2 |S_f^{\alpha} g_k(z)|^2 \chi_{\tilde{D}_k}(z) \right)^{p_2/2} dv_{\alpha_2}(z)$$

$$\leq \int_{\mathbb{B}_n} \left(\sum_{k=1}^{\infty} |\lambda_k|^2 |S_f^{\alpha} g_k(z)|^2 \right)^{p_2/2} dv_{\alpha_2}(z).$$

If $1 < p_2 < 2$, then $2/p_2 > 1$, from (10), by Hölder's inequality we get

$$\sum_{k=1}^{\infty} |\lambda_k|^{p_2} \int_{\tilde{D}_k} |S_f^{\alpha} g_k(z)|^{p_2} dv_{\alpha_2}(z)$$

$$\leq \int_{\mathbb{B}_n} \left(\sum_{k=1}^{\infty} |\lambda_k|^2 |S_f^{\alpha} g_k(z)|^2 \right)^{p_2/2} \left(\sum_{k=1}^{\infty} \chi_{\tilde{D}_k}(z) \right)^{1-p_2/2} dv_{\alpha_2}(z)$$

$$\leq N^{1-p_2/2} \int_{\mathbb{B}_n} \left(\sum_{k=1}^{\infty} |\lambda_k|^2 |S_f^{\alpha} g_k(z)|^2 \right)^{p_2/2} dv_{\alpha_2}(z),$$

since each $z \in \mathbb{B}_n$ belongs to at most N of the sets $\tilde{D}_k$. Combining the above two inequalities, and applying (9) we have

$$\sum_{k=1}^{\infty} |\lambda_k|^{p_2} \int_{\tilde{D}_k} |S_f^{\alpha} g_k(z)|^{p_2} dv_{\alpha_2}(z)$$

$$\leq \min\{1, N^{1-p_2/2}\} \int_{\mathbb{B}_n} \left(\sum_{k=1}^{\infty} |\lambda_k|^2 |S_f^{\alpha} g_k(z)|^2 \right)^{p_2/2} dv_{\alpha_2}(z)$$

$$\lesssim \|S_f^{\alpha}\|^{p_2} \cdot \|\lambda\|_{\ell^{p_1}}^{p_2}.$$

By subharmonicity we know that,

$$|S_f^{\alpha} g_k(a_k)|^{p_2} \lesssim \frac{1}{(1 - |a_k|^2)^{n+1+\alpha_2}} \int_{\tilde{D}_k} |S_f^{\alpha} g_k(z)|^{p_2} dv_{\alpha_2}(z).$$

From this we obtain

$$(11) \qquad \sum_{k=1}^{\infty} |\lambda_k|^{p_2} (1 - |a_k|^2)^{n+1+\alpha_2} \left| S_f^{\alpha} g_k(a_k) \right|^{p_2} \lesssim \|S_f^{\alpha}\|^{p_2} \cdot \|\lambda\|_{\ell^{p_1}}^{p_2}.$$

Let $R^{\alpha,b}$ be the integral operator defined in (7). Then

$$
\begin{aligned}
S_f^\alpha g_k(a_k) &= \int_{\mathbb{B}_n} \frac{f(w)\overline{g_k(w)}}{(1-\langle a_k, w\rangle)^{n+1+\alpha}}\, dv_\alpha(w) \\
&= \int_{\mathbb{B}_n} \frac{f(w)(1-|a_k|^2)^{b-(n+1+\alpha_1)/p_1}}{(1-\langle a_k, w\rangle)^{n+1+\alpha}(1-\langle a_k, w\rangle)^b}\, dv_\alpha(w) \\
&= (1-|a_k|^2)^{b-(n+1+\alpha_1)/p_1} \int_{\mathbb{B}_n} \frac{f(w)}{(1-\langle a_k, w\rangle)^{n+1+\alpha+b}}\, dv_\alpha(w) \\
&= (1-|a_k|^2)^{b-(n+1+\alpha_1)/p_1} R^{\alpha,b} f(a_k).
\end{aligned}
$$

Thus (11) becomes

$$
(12) \quad \sum_{k=1}^\infty |\lambda_k|^{p_2}(1-|a_k|^2)^{(n+1+\alpha_2)+[b-(n+1+\alpha_1)/p_1]p_2}|R^{\alpha,b}f(a_k)|^{p_2} \lesssim \|S_f^\alpha\|^{p_2}\cdot\|\lambda\|_{\ell^{p_1}}^{p_2}.
$$

Since

$$
(n+1+\alpha_2)+\left(b-\frac{n+1+\alpha_1}{p_1}\right)p_2 = \left(b+\frac{n+1+\beta}{q}\right)p_2,
$$

the equation (12) is the same as

$$
(13) \quad \sum_{k=1}^\infty |\lambda_k|^{p_2}\left[(1-|a_k|^2)^{b+(n+1+\beta)/q}|R^{\alpha,b}f(a_k)|\right]^{p_2} \lesssim \|S_f^\alpha\|^{p_2}\cdot\|\lambda\|_{\ell^{p_1}}^{p_2}.
$$

Since $\{\lambda_k\}$ was an arbitrary sequence in ℓ^{p_1}, we know that $\{\lambda_k^{p_2}\}$ is an arbitrary sequence in ℓ^{p_1/p_2}. Since the conjugate exponent of p_1/p_2 is $(p_1/p_2)' = p_1/(p_1-p_2)$, by duality we obtain that

$$
\left\{(1-|a_k|^2)^{b+(n+1+\beta)/q}|R^{\alpha,b}f(a_k)|\right\} \in \ell^{p_1 p_2/(p_1-p_2)} = \ell^q,
$$

and

$$
(14) \quad \sum_{k=1}^\infty (1-|a_k|^2)^{bq+(n+1+\beta)}|R^{\alpha,b}f(a_k)|^q \lesssim \|S_f^\alpha\|^q.
$$

This is the discrete version of what we want. Now, we will deduce that $f \in A_\beta^q$ with $\|f\|_{q,\beta} \lesssim \|S_f^\alpha\|$ using duality and the atomic decomposition for Bergman spaces. Indeed, choose $\beta' = q'(\alpha - \beta/q)$ (which means $\alpha = \beta/q + \beta'/q'$). Note that condition (6) implies that $1+\alpha > (1+\beta)/q$, and this guarantees that $\beta' > -1$. Hence, by the duality result in Theorem A,

$$
(15) \quad \|f\|_{q,\beta} \asymp \sup_{\|h\|_{q',\beta'}=1} |\langle h, f\rangle_\alpha|.
$$

Observe that

$$
n+1+\alpha+b > n+\frac{1+\beta'}{q'}.
$$

Then, we can apply (14) with the r-lattice $\{a_k\}$ for which the atomic decomposition in Theorem D for $A_{\beta'}^{q'}$ holds. That is, for any $h \in A_{\beta'}^{q'}$, there exists a sequence $\mu = \{\mu_k\} \in \ell^{q'}$ with $\|\mu\|_{\ell^{q'}} \lesssim \|h\|_{q',\beta'}$ such that

$$
h(z) = \sum_{k=1}^\infty \mu_k \frac{(1-|a_k|^2)^{n+1+\alpha+b-(n+1+\beta')/q'}}{(1-\langle z, a_k\rangle)^{n+1+\alpha+b}}.
$$

Since
$$n + 1 + \alpha + b - (n+1+\beta')/q' = b + (n+1+\beta)/q,$$
then (15), Hölder's inequality and (14) gives

$$
\begin{aligned}
\|f\|_{q,\beta} &\asymp \sup_{\|h\|_{q',\beta'}=1} \left| \sum_{k=1}^{\infty} \mu_k (1-|a_k|^2)^{b+(n+1+\beta)/q}\, \overline{R^{\alpha,b} f(a_k)} \right| \\
&\leq \sup_{\|h\|_{q',\beta'}=1} \|\mu\|_{\ell^{q'}} \left[\sum_{k=1}^{\infty} (1-|a_k|^2)^{bq+(n+1+\beta)} |R^{\alpha,b} f(a_k)|^q \right]^{1/q} \\
&\lesssim \|S_f^{\alpha}\|.
\end{aligned}
$$

Hence $f \in A_\beta^q$ with
$$\|f\|_{q,\beta} \lesssim \|S_f^{\alpha}\|.$$
This finishes the proof.

$\square$

5. FURTHER RESULTS

5.1. **Compactness.** Under the assumptions of Theorem 3, actually one has that the small Hankel operator $h_f^{\frac{\alpha}{f}} : A_{\alpha_1}^{p_1} \to \overline{A_{\alpha_2}^{p_2}}$ is bounded if and only if it is compact. This is from a general result of Banach space theory. It is known that, for $0 < p_2 < p_1 < \infty$, every bounded operator from ℓ^{p_1} to ℓ^{p_2} is compact (see, for example Theorem I.2.7, p.31 in [15]). Since the weighted Bergman space A_α^p is isomorphic to ℓ^p (see, Theorem 11, p.89 in [21], note that the same proof there works for weighted Bergman spaces on the unit ball $\mathbb{B}_n$), we get directly the above result.

5.2. **Small Hankel operators with the same weights.** Concerning the boundedness of the small Hankel operator $h_f^{\frac{\alpha}{f}} : A_\alpha^{p_1} \to \overline{A_\alpha^{p_2}}$ (the case when all the weights are the same) for all possible choices of $0 < p_1, p_2 < \infty$, we mention here that the case $p_1 = p_2 > 1$ is by now classical (see [13], [23] and [4]), and in this case the boundedness is equivalent to the symbol f being in the Bloch space $\mathcal{B}$, that consists of those holomorphic functions f on $\mathbb{B}_n$ with
$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbb{B}_n} (1-|z|^2)|Rf(z)| < \infty.$$
Here, Rf denotes the radial derivative of f, that is,
$$Rf(z) = \sum_{k=1}^{n} z_k \frac{\partial f}{\partial z_k}(z), \qquad z = (z_1, \ldots, z_n) \in \mathbb{B}_n.$$

The Bloch space also admits an equivalent norm in terms of the invariant gradient $\widetilde{\nabla} f(z) := \nabla(f \circ \varphi_z)(0)$ as follows
$$\|f\|_{\mathcal{B}} \asymp |f(0)| + \sup_{z\in\mathbb{B}_n} |\widetilde{\nabla} f(z)|.$$

The case $0 < p_1 \leq p_2$ is completely settled in [4] (actually the results are stated for the unweighted Bergman spaces A^p, but the proofs works also for the weighted case). The description for the case $p_1 = p_2 = 1$ is that f must belong to the so called logarithmic Bloch space, a result that goes back to the one dimensional result of Attele [2]. Concerning estimates with loss, in [4] Bonami and Luo obtained a description for the case $0 < p_2 < p_1$ with $p_2 < 1$ (again the result in [4] is stated for the unweighted Bergman spaces). Thus, in view of Corollary 4, to complete the picture it remains to deal with the case $p_1 > p_2 = 1$

(this problem is also open for the unit disk). In that case, also in [4], some partial results are obtained (again for the unweighted case). Mainly, they provide a pointwise estimate that is necessary for the small Hankel operator to be bounded, and they show that the condition

$$(16) \qquad\qquad f(z) \log \frac{2}{1 - |z|} \in L^{p_1'}(\mathbb{B}_n, dv_\alpha)$$

is sufficient. Moreover, they conjecture that the previous condition is also necessary. We have not been able to prove the conjecture, but we are going to shed some light on that problem.

Theorem 6. *Let $f \in H(\mathbb{B}_n)$, $\alpha > -1$ and $p_1 > 1$. Let p_1' be the conjugate exponent of p_1. Then $h_{\bar{f}}^\alpha : A_\alpha^{p_1} \to \overline{A_\alpha^1}$ is bounded if and only if the multiplication operator $M_f : \mathcal{B} \to A_\alpha^{p_1'}$ is bounded.*

Before going to the proof we need first some preparation. First of all, recall that the Bloch space is the dual of A_α^1 under the integral pairing $\langle \, , \, \rangle_\alpha$ (see [24, Theorem 3.17]). We also need the following lemma, whose one dimensional analogue is essentially proved in [3].

Lemma 7. *Let $1 < p < \infty$, $\sigma > -1$, and $n + 1 + \sigma < b$. Then*

$$\int_{\mathbb{B}_n} \frac{|f(z) - f(a)|^p}{|1 - \langle a, z \rangle|^b} \, dv_\sigma(z) \lesssim \int_{\mathbb{B}_n} |\widetilde{\nabla} f(z)|^p \frac{dv_\sigma(z)}{|1 - \langle a, z \rangle|^b}$$

for any $f \in H(\mathbb{B}_n)$ and $a \in \mathbb{B}_n$.

Proof. We are going to prove first that, for $0 \le t < n + 1 + \sigma$,

$$(17) \qquad \int_{\mathbb{B}_n} \frac{|f(z) - f(0)|^p}{|1 - \langle a, z \rangle|^t} \, dv_\sigma(z) \lesssim \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^p |Rf(w)|^p}{|1 - \langle a, w \rangle|^t} \, dv_\sigma(w).$$

From [24, p.51], for β big enough, say $\beta \ge 1 + \sigma$, we have

$$|f(z) - f(0)| \le C \int_{\mathbb{B}_n} \frac{(1 - |w|^2) |Rf(w)| \, dv_{\beta-1}(w)}{|1 - \langle z, w \rangle|^{n+\beta}}.$$

Take a small number $\varepsilon > 0$ with $\sigma - \varepsilon \max(p, p') > -1$, where p' denotes the conjugate exponent of p, and $t < n + 1 + \sigma - \varepsilon p$. An application of Hölder's inequality and Lemma B yields

$$|f(z) - f(0)|^p \lesssim (1 - |z|^2)^{-\varepsilon p} \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^p |Rf(w)|^p \, dv_{\beta-1+\varepsilon p}(w)}{|1 - \langle z, w \rangle|^{n+\beta}}.$$

This together with Fubini's theorem and [19, Lemma 2.5] gives

$$\int_{\mathbb{B}_n} \frac{|f(z) - f(0)|^p}{|1 - \langle a, z \rangle|^t} \, dv_\sigma(z)$$

$$\lesssim \int_{\mathbb{B}_n} (1 - |w|^2)^p |Rf(w)|^p \left(\int_{\mathbb{B}_n} \frac{dv_{\sigma - \varepsilon p}(z)}{|1 - \langle a, z \rangle|^t \, |1 - \langle z, w \rangle|^{n+\beta}} \right) dv_{\beta-1+\varepsilon p}(w)$$

$$\lesssim \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^p |Rf(w)|^p}{|1 - \langle a, w \rangle|^t} \, dv_\sigma(w)$$

proving (17). Now, a change of variables $z = \varphi_a(\zeta)$ gives (see [24, Proposition 1.13])

$$\int_{\mathbb{B}_n} \frac{|f(z) - f(a)|^p}{|1 - \langle a, z \rangle|^b} \, dv_\sigma(z) = \int_{\mathbb{B}_n} \frac{|(f \circ \varphi_a)(\zeta) - (f \circ \varphi_a)(0)|^p}{|1 - \langle a, \varphi_a(\zeta) \rangle|^b} \frac{(1 - |a|^2)^{n+1+\sigma}}{|1 - \langle a, \zeta \rangle|^{2(n+1+\sigma)}} \, dv_\sigma(\zeta).$$

From [24, Lemma 1.3] we have

$$1 - \langle a, \varphi_a(\zeta) \rangle = 1 - \langle \varphi_a(0), \varphi_a(\zeta) \rangle = \frac{1 - |a|^2}{1 - \langle a, \zeta \rangle}.$$

Therefore we obtain

$$\int_{\mathbb{B}_n} \frac{|f(z) - f(a)|^p}{|1 - \langle a, z \rangle|^b} \, dv_\sigma(z) = (1 - |a|^2)^{n+1+\sigma-b} \int_{\mathbb{B}_n} \frac{|(f \circ \varphi_a)(\zeta) - (f \circ \varphi_a)(0)|^p}{|1 - \langle a, \zeta \rangle|^{2(n+1+\sigma)-b}} \, dv_\sigma(\zeta).$$

Due to our condition $b > n + 1 + \sigma$, we have

$$t = 2(n + 1 + \sigma) - b < n + 1 + \sigma$$

and we can apply (17) to get

$$\int_{\mathbb{B}_n} \frac{|f(z) - f(a)|^p}{|1 - \langle a, z \rangle|^b} \, dv_\sigma(z) \lesssim (1 - |a|^2)^{n+1+\sigma-b} \int_{\mathbb{B}_n} \frac{(1 - |\zeta|^2)^p \, |R(f \circ \varphi_a)(\zeta)|^p}{|1 - \langle a, \zeta \rangle|^{2(n+1+\sigma)-b}} \, dv_\sigma(\zeta).$$

Since

$$(1 - |\zeta|^2) \, |R(f \circ \varphi_a)(\zeta)| \leq |\widetilde{\nabla}(f \circ \varphi_a)(\zeta)| = |\widetilde{\nabla} f(\varphi_a(\zeta))|,$$

another change of variables $w = \varphi_a(\zeta)$ finally gives

$$\int_{\mathbb{B}_n} \frac{|f(z) - f(a)|^p}{|1 - \langle a, z \rangle|^b} \, dv_\sigma(z) \lesssim \int_{\mathbb{B}_n} \frac{|\widetilde{\nabla} f(w)|^p}{|1 - \langle a, w \rangle|^b} \, dv_\sigma(w).$$

completing the proof of the lemma. $\square$

After these preparations, we are now ready for the proof of Theorem 6.

Proof of Theorem 6. Assume first that the small Hankel operator $h_{\bar{f}}^\alpha : A_\alpha^{p_1} \to \overline{A_\alpha^1}$ is bounded. Let $g \in A_\alpha^{p_1}$. From the pointwise estimate for Bergman spaces, we get

$$|\langle g, f \rangle_\alpha| = |h_{\bar{f}}^\alpha g(0)| \leq C \|h_{\bar{f}}^\alpha g\|_{1,\alpha} \leq C \|h_{\bar{f}}^\alpha\| \cdot \|g\|_{p_1,\alpha}.$$

Therefore, by duality, we have that $f \in A_\alpha^{p_1'}$ with

$$(18) \qquad \|f\|_{p_1',\alpha} \leq C \|h_{\bar{f}}^\alpha\|.$$

Recall that $h_{\bar{f}}^\alpha : A_\alpha^{p_1} \to \overline{A_\alpha^1}$ is bounded, if and only if, $S_f^\alpha : A_\alpha^{p_1} \to A_\alpha^1$ is bounded, with $\|S_f^\alpha\| \asymp \|h_{\bar{f}}^\alpha\|$. Also, since for any $g \in A_\alpha^{p_1}$ and $h \in \mathcal{B}$,

$$(19) \qquad \langle S_f^\alpha g, h \rangle_\alpha = \langle f, gh \rangle_\alpha = \langle S_f^\alpha h, g \rangle_\alpha$$

we know that $S_f^\alpha : \mathcal{B} \to A_\alpha^{p_1'}$ is bounded, and moreover, we have

$$\|S_f^\alpha\|_{\mathcal{B} \to A_\alpha^{p_1'}} \lesssim \|h_{\bar{f}}^\alpha\|.$$

For g in the Bloch space $\mathcal{B}$, one has

$$(20) \qquad \begin{aligned} \|M_f g\|_{p_1',\alpha}^{p_1'} &= \int_{\mathbb{B}_n} |f(z) \overline{g(z)}|^{p_1'} \, dv_\alpha(z) \\ &\lesssim \int_{\mathbb{B}_n} |S_f^\alpha g(z)|^{p_1'} \, dv_\alpha(z) + \int_{\mathbb{B}_n} |f(z) \overline{g(z)} - S_f^\alpha g(z)|^{p_1'} \, dv_\alpha(z). \end{aligned}$$

Due to the boundedness of $S_f^\alpha : \mathcal{B} \to A_\alpha^{p_1'}$,

(21) $$\int_{\mathbb{B}_n} |S_f^\alpha g(z)|^{p_1'} \, dv_\alpha(z) \le \|S_f^\alpha\|_{\mathcal{B} \to A_\alpha^{p_1'}}^{p_1'} \cdot \|g\|_{\mathcal{B}}^{p_1'} \lesssim \|h_{\overline{f}}^\alpha\|^{p_1'} \cdot \|g\|_{\mathcal{B}}^{p_1'}.$$

On the other hand, by the reproducing formula for Bergman spaces and Hölder's inequality,

$$|f(z)\,\overline{g(z)} - S_f^\alpha g(z)|^{p_1'} = \left| \int_{\mathbb{B}_n} \frac{f(w)\,\overline{(g(z) - g(w))}}{(1 - \langle z, w \rangle)^{n+1+\alpha}} \, dv_\alpha(w) \right|^{p_1'}$$

$$\le \left(\int_{\mathbb{B}_n} \frac{|f(w)|^{p_1'}}{|1 - \langle w, z \rangle|^{n+1+\alpha}} \, dv_{\alpha + \varepsilon p_1'}(w) \right) \left(\int_{\mathbb{B}_n} \frac{|g(z) - g(w)|^{p_1}}{|1 - \langle w, z \rangle|^{n+1+\alpha}} \, dv_{\alpha - \varepsilon p_1}(w) \right)^{\frac{p_1'}{p_1}},$$

where $\varepsilon > 0$ satisfies $\alpha - \varepsilon \max(p_1, p_1') > -1$. Using Lemma 7 and Lemma B we get

$$\int_{\mathbb{B}_n} \frac{|g(z) - g(w)|^{p_1}}{|1 - \langle w, z \rangle|^{n+1+\alpha}} \, dv_{\alpha - \varepsilon p_1}(w) \lesssim \int_{\mathbb{B}_n} \frac{|\widetilde{\nabla} g(w)|^{p_1}}{|1 - \langle w, z \rangle|^{n+1+\alpha}} \, dv_{\alpha - \varepsilon p_1}(w)$$

$$\lesssim \|g\|_{\mathcal{B}}^{p_1} (1 - |z|^2)^{-\varepsilon p_1}.$$

Therefore, this together with Fubini's theorem, Lemma B and the estimate (18) gives

$$\int_{\mathbb{B}_n} |f(z)\,\overline{g(z)} - S_f^\alpha g(z)|^{p_1'} \, dv_\alpha(z)$$

(22) $$\le C\|g\|_{\mathcal{B}}^{p_1'} \int_{\mathbb{B}_n} |f(w)|^{p_1'} \left(\int_{\mathbb{B}_n} \frac{dv_{\alpha - \varepsilon p_1'}(z)}{|1 - \langle w, z \rangle|^{n+1+\alpha}} \right) dv_{\alpha + \varepsilon p_1'}(w)$$

$$\le C\|g\|_{\mathcal{B}}^{p_1'} \cdot \|f\|_{p_1',\alpha}^{p_1'} \le C\|h_{\overline{f}}^\alpha\|^{p_1'} \cdot \|g\|_{\mathcal{B}}^{p_1'}.$$

Putting together the estimates (20), (21) and (22) it follows that $M_f : \mathcal{B} \to A_\alpha^{p_1'}$ is bounded with $\|M_f\|_{\mathcal{B} \to A_\alpha^{p_1'}} \lesssim \|h_{\overline{f}}^\alpha\|$.

Conversely, suppose that $M_f : \mathcal{B} \to A_\alpha^{p_1'}$ is bounded. By the boundedness of the projection $P_\alpha : L^{p_1'}(\mathbb{B}_n, dv_\alpha) \to A_\alpha^{p_1'}$ one deduces that $S_f^\alpha : \mathcal{B} \to A_\alpha^{p_1'}$ is also bounded, and so obviously, $S_f^\alpha : \mathcal{B}_0 \to A_\alpha^{p_1'}$ is bounded, where $\mathcal{B}_0$ is the little Bloch space, and it is well-known that the dual space of $\mathcal{B}_0$ is A_α^1 under the integral pair $\langle \, , \, \rangle_\alpha$ (see, for example, Chapter 3 of [24]), from (19) we know that $S_f^\alpha : A_\alpha^{p_1} \to A_\alpha^1$ is bounded. $\qquad\square$

As a consequence of Theorem 6 we can easily obtain the sufficient and necessary conditions given in [4] as well as another relevant necessary condition for the boundedness of $h_{\overline{f}}^\alpha : A_\alpha^{p_1} \to \overline{A_\alpha^1}$.

Corollary 8. *Let $f \in H(\mathbb{B}_n)$, $\alpha > -1$ and $p_1 > 1$.*

(i) *If (16) holds, then $h_{\overline{f}}^\alpha : A_\alpha^{p_1} \to \overline{A_\alpha^1}$ is bounded.*

(ii) *If $h_{\overline{f}}^\alpha : A_\alpha^{p_1} \to \overline{A_\alpha^1}$ is bounded, then*

(23) $$\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{(n+1+\alpha)/p_1'} |f(z)| \left(\log \frac{2}{1 - |z|^2} \right) < \infty$$

and

$$(24) \qquad \int_{\mathbb{B}_n} |f(z)|^{p_1'} \left(\log \frac{2}{1-|z|^2} \right)^{\frac{p_1'}{2}} dv_\alpha(z) < \infty.$$

Proof. Part (i) follows directly from Theorem 6 and the pointwise estimate for Bloch functions

$$|g(z)| \le \|g\|_{\mathcal{B}} \, \log \frac{2}{1-|z|^2}.$$

To prove part (ii), for each $z \in \mathbb{B}_n$, the function

$$g_z(w) = \log \frac{2}{1 - \langle w, z \rangle}$$

is in the Bloch space with $\|g_z\|_{\mathcal{B}} \le C$ with the constant C independent of the point z. Therefore, from the pointwise estimate for functions in Bergman spaces, we get

$$
\begin{aligned}
(1-|z|^2)^{n+1+\alpha} \Big(|f(z)| \log \frac{2}{1-|z|^2} \Big)^{p_1'} &= (1-|z|^2)^{n+1+\alpha} |f(z) \, g_z(z)|^{p_1'} \\
&\lesssim \|f g_z\|_{p_1',\alpha}^{p_1'} = \|M_f g_z\|_{p_1',\alpha}^{p_1'} \\
&\le \|M_f\|_{\mathcal{B} \to A_\alpha^{p_1'}} \cdot \|g_z\|_{\mathcal{B}}^{p_1'} \lesssim \|M_f\|_{\mathcal{B} \to A_\alpha^{p_1'}},
\end{aligned}
$$

and (23) follows due to Theorem 6. The necessity of (24) is also a consequence of Theorem 6. Indeed, clearly $M_f : \mathcal{B} \to A_\alpha^{p_1'}$ is bounded if and only if the measure $d\mu_f(z) = |f(z)|^{p_1'} dv_\alpha(z)$ is a p_1'- Carleson measure for the Bloch space (see [7, 10] for the definition); and by Proposition 1.4 in [7] (the one dimensional case appears in [10] and [14]) this implies (24), finishing the proof. $\qquad \square$

The established connection between Hankel operators on Bergman spaces and Carleson measures for the Bloch space makes even more interesting the problem (as far as we know, still open) of describing those measures.

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