TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 364, Number 5, May 2012, Pages 2241–2260
S 0002-9947(2012)05598-X
Article electronically published on January 19, 2012

ON RESTRICTION OF MAXIMAL MULTIPLIERS IN WEIGHTED SETTINGS

MARÍA J. CARRO AND SALVADOR RODRÍGUEZ-LÓPEZ

ABSTRACT. We obtain restriction results of K. De Leeuw's type for maximal operators defined through Fourier multipliers of either strong or weak type for weighted L^p spaces with $1 \leq p < \infty$. Applications to the case of Hörmander-Mihlin multipliers, singular integral operators and Bochner-Riesz sums are given.

1. INTRODUCTION

In 1965 K. De Leeuw proved that if $\mathbf{m}$ is a continuous function on $\mathbb{R}$ such that $\mathbf{m}$ is a Fourier multiplier on $L^p(\mathbb{R})$, its restriction to the integers $\mathbf{m}|_{\mathbb{Z}}$ is a Fourier multiplier on $L^p(\mathbb{T})$. Moreover, its norm does not exceed the norm of $\mathbf{m}$ as a multiplier on $L^p(\mathbb{R})$ (see [8, Proposition 3.3] and Jodeit's article [12]).

In 1980 C. Kenig and P. Tomas extended De Leeuw's result to maximal operators associated to a family of multipliers given by the dilations of a given one. More precisely, they proved that if $\mathbf{m}$ is a continuous function and if T_r denotes the multiplier operator associated to $\mathbf{m}_r(\xi) = \mathbf{m}(\xi/r)$, whenever $T^\sharp f(x) = \sup_{r>0} |T_r f(x)|$ is a bounded operator on $L^p(\mathbb{R}^d)$ the same holds for the maximal operator on $L^p(\mathbb{T}^d)$ associated to the multipliers $\mathbf{m}_r|_{\mathbb{Z}}$. Furthermore, its norm does not exceed a constant times the norm of $T^\sharp$. They also obtained similar results for operators of weak type for $p > 1$ (see [13]).

In 2003, E. Berkson and T.A. Gillespie extended De Leeuw's restriction result for multipliers on $L^p(\mathbb{R}, w)$ with w a 1-periodic weight belonging to $A_p(\mathbb{R})$ and $1 < p < \infty$. Such weights are said to be in the class $A_p(\mathbb{T})$. Their result is the following.

Theorem 1.1 ([4, Theorem 1.2]). *Let $1 < p < \infty$ and let $w \in A_p(\mathbb{T})$. If $\mathbf{m}$ is a continuous function on $\mathbb{R}$ such that it is a Fourier multiplier for $L^p(\mathbb{R}, w)$, then $\mathbf{m}|_{\mathbb{Z}}$ is a Fourier multiplier on $L^p(\mathbb{T}, w)$. Moreover, there is a constant $\mathfrak{c}_{p,w}$ depending only on p and the A_p-constant of w, such that the norm of $\mathbf{m}|_{\mathbb{Z}}$ as a multiplier on $L^p(\mathbb{T}, w)$ does not exceed $\mathfrak{c}_{p,w}$ times the norm of $\mathbf{m}$ as a multiplier on $L^p(\mathbb{R}, w)$.*

This theorem has been recently improved by K. Andersen and P. Mohanty as follows.

Theorem 1.2 ([1, Theorem 1.1]). *Let $1 < p < \infty$ and let $w \in L^1(\mathbb{T}^d)$. If $\mathbf{m}$ is a continuous function on $\mathbb{R}^d$ such that it is a Fourier multiplier on $L^p(\mathbb{R}^d, w)$,*

Received by the editors July 13, 2009.
2010 *Mathematics Subject Classification.* Primary 42B15, 42B35.
Key words and phrases. Weighted L^p spaces, Fourier multipliers.
This work was partially supported by MTM2010-14946. The second author was also partially supported by EP/H051368/1.

then $\mathbf{m}|_{\mathbb{Z}}$ is a Fourier multiplier on $L^p(\mathbb{T}^d, w)$. Moreover, the norm of $\mathbf{m}|_{\mathbb{Z}}$ as a multiplier on $L^p(\mathbb{T}^d, w)$ does not exceed the norm of $\mathbf{m}$ as a multiplier on $L^p(\mathbb{R}^d, w)$.

The purpose of this paper is twofold:

i) To give restriction results from $\mathbb{R}^d$ to $\mathbb{T}^d$ for Fourier multipliers and for associated maximal operators of weak type (and strong type) in any dimension and for $1 \leq p < \infty$. In particular, we shall prove the following.

Theorem 1.3. *Let $1 \leq p < \infty$ and let w be a periodic weight on $\mathbb{R}^d$ satisfying $w \in L^1(\mathbb{T}^d)$. Suppose that $\{\mathbf{m}_j\}_j$ is a family of multipliers that are continuous functions satisfying that the associated maximal operator (see Definition 2.2 below) is bounded from $L^p(\mathbb{R}^d, w)$ to $L^{p,\infty}(\mathbb{R}^d, w)$ (or to $L^p(\mathbb{R}^d, w)$). Then the maximal operator associated to their restriction to the integers $\{\mathbf{m}_j|_{\mathbb{Z}^d}\}_j$ (see Definition 3.1 below) is bounded from $L^p(\mathbb{T}^d, w)$ to $L^{p,\infty}(\mathbb{T}^d, w)$ (resp. to $L^p(\mathbb{T}^d, w)$) and its operator norm does not exceed c_p times the norm of the maximal operator associated to $\{\mathbf{m}_j\}_j$, where c_p is a constant that depends only on p.*

ii) K. De Leeuw in [8] and J. Jodeit in [12] also proved some restriction results for strong Fourier multipliers on $L^p(\mathbb{R}^d)$ to a lower dimensional space. In [7] a counterpart for Fourier multipliers on $L^p(\mathbb{R}^d, w)$ with w a suitable weight in $A_p(\mathbb{R}^d)$ was given. Namely, [7, Corollary 4.13] states that if $\mathbf{m}$ is a continuous and bounded function in $\mathbb{R}^d$ that is a Fourier multiplier on $L^p(\mathbb{R}^d, w)$ where $w = u \otimes v$ with $u \in A_p(\mathbb{R}^{d_1})$, $v \in A_p(\mathbb{R}^{d_2})$, then, for any $\xi \in \mathbb{R}^{d_1}$, the function $\mathbf{m}(\xi, \cdot)$ is a Fourier multiplier on $L^p(\mathbb{R}^{d_2}, v)$. In this setting, we shall prove the following.

Theorem 1.4. *Let $d = d_1 + d_2$, $1 \leq p < \infty$, $u \in A_p(\mathbb{R}^{d_1})$, $v \in A_p(\mathbb{R}^{d_2})$ and define $w(x, y) = u(x)v(y)$. Suppose that $\{\mathbf{m}_j\}_j$ is a family of multipliers that are continuous functions satisfying that the associated maximal operator is bounded from $L^p(\mathbb{R}^d, w)$ to $L^{p,\infty}(\mathbb{R}^d, w)$ (or to $L^p(\mathbb{R}^d, w)$). Then, fixed $\xi \in \mathbb{R}^{d_1}$, the maximal operator associated to the family $\{\mathbf{m}_j(\xi, \cdot)\}_j$ is bounded from $L^p(\mathbb{R}^{d_2}, v)$ to $L^{p,\infty}(\mathbb{R}^{d_2}, v)$ (resp. to $L^p(\mathbb{R}^{d_2}, v)$) and its operator norm does not exceed $\mathfrak{c}_{p,w}$ times the norm of the maximal operator associated to $\{\mathbf{m}_j\}_j$, where $\mathfrak{c}_{p,w}$ is a constant that depends only on p, d and the A_p-constant of w.*

We want to emphasize that the techniques developed in this paper are different from those in [1, 4, 7] where duality properties of Lebesgue spaces are strongly used. Our approach allows us to also consider the case of maximal multipliers of weak type $(1, 1)$, and deal with the difficulties derived from the fact that $L^{1,\infty}$ is not a Banach space. The endpoint case $p = 1$ is the weighted analogue of the results in [2, 15].

2. Definitions and notation

In this section we present some basic definitions needed for our consideration. Let $0 < p < \infty$ and let $(\mathcal{M}, \mu)$ be a σ-finite measure space. The space $L^{p,\infty}(\mu)$ is defined by the quasinorm $\|f\|_{L^{p,\infty}} = \sup_{t>0} t\mu_f(s)^{1/p}$, where $\mu_f(s) = \mu\{x : |f(x)| > s\}$. It is known (see [10, p. 485]) that, for every $q < p$,

$$\|f\|_{L^{p,\infty}(\mu)} \leq \sup \|f\chi_E\|_{L^q(\mu)} \, \mu(E)^{1/p-1/q} \leq c_{p,q} \|f\|_{L^{p,\infty}(\mu)}, \tag{2.1}$$

where the supremum is taken on the family of sets of finite measure and $c_{p,q}^q = \frac{p}{p-q}$. The finiteness of the middle expression is called Kolmogorov's condition.

If ν is a positive measure absolutely continuous with respect to μ and w denotes the Radon-Nykodym derivative of ν with respect to μ, we shall write $L^p(w)$ for $L^p(\nu)$. If any confusion can arise, we shall write $L^p(\mathcal{M},\mu)$ and $L^{p,\infty}(\mathcal{M},\mu)$ to indicate the underlying measure space $\mathcal{M}$.

Let $\mathcal{C}_c^\infty(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)$ denote the class of infinitely differentiable functions with compact support and the Schwartz class of test functions, respectively. As usual, $\mathfrak{B}(X,Y)$ indicates the set of bounded operators on X into Y and $\mathfrak{B}(X)=\mathfrak{B}(X,X)$.

A weight on $\mathbb{R}^d$ is a locally integrable function $w:\mathbb{R}^d\to[0,\infty)$ such that $0<w<\infty$ a.e.

Definition 2.1. We say that a weight w belongs to the class $A_p(\mathbb{R}^d)$, and we write $w\in A_p(\mathbb{R}^d)$ if

$$[w]_{A_p}=\sup_Q\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{1/1-p}\,dx\right)^{p-1}<\infty,$$

for $1<p<\infty$, and

$$[w]_{A_1}=\sup_Q\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left\|w^{-1}\chi_Q\right\|_\infty<+\infty,$$

where the supremum is taken over the family of cubes Q with sides parallel to the coordinate axis. These quantities will be referred to as the A_p-constant of w.

It is well known that, for $1\le p<\infty$ and $w\in A_p(\mathbb{R}^d)$, $\mathcal{S}(\mathbb{R}^d)\subset L^p(\mathbb{R}^d,w)$ and $\mathcal{C}_c^\infty(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d,w)$. We refer the reader to [10, 11] for other properties and generalities of A_p-weights.

For any function f, we shall denote by $\widehat{f}$ ($f^\vee$) the Fourier transform (resp. the inverse Fourier transform) of f, whenever it is well defined.

Definition 2.2. Let $1\le p<\infty$. A function $\mathbf{m}\in L^\infty(\mathbb{R}^d)$ is called a weak type multiplier on $L^p(\mathbb{R}^d,w)$ (in symbols, $\mathbf{m}\in M^{(\mathfrak{w})}_{p,w}(\mathbb{R}^d)$) if the mapping $f\in\mathcal{S}(\mathbb{R}^d)\mapsto(\mathbf{m}\hat{f})^\vee$ can be extended from $\mathcal{S}(\mathbb{R}^d)$ to a continuous linear mapping $S_\mathbf{m}$ from $L^p(\mathbb{R}^d,w)$ to $L^{p,\infty}(\mathbb{R}^d,w)$. In this case we write

$$\|\mathbf{m}\|_{M^{(\mathfrak{w})}_{p,w}(\mathbb{R}^d)}=\|S_\mathbf{m}\|_{\mathfrak{B}(L^p(\mathbb{R}^d,w),L^{p,\infty}(\mathbb{R}^d,w))}.$$

If $S_\mathbf{m}\in\mathfrak{B}\left(L^p(\mathbb{R}^d,w)\right)$, we say that $\mathbf{m}$ is a Fourier multiplier on $L^p(\mathbb{R}^d,w)$ and write

$$\|\mathbf{m}\|_{M_{p,w}(\mathbb{R}^d)}=\|S_\mathbf{m}\|_{\mathfrak{B}(L^p(\mathbb{R}^d,w))}.$$

If $\{\mathbf{m}_j\}_j$ is a sequence in $M^{(\mathfrak{w})}_{p,w}(\mathbb{R}^d)$, we denote by $\|\{\mathbf{m}_j\}_j\|_{M^{(\mathfrak{w})}_{p,w}(\mathbb{R}^d)}$ the norm of the operator defined for $f\in\mathcal{S}(\mathbb{R}^d)$ by

$$S^\sharp_{\{\mathbf{m}_j\}_j}f(x)=\sup_j\left|S_{\mathbf{m}_j}f(x)\right|,$$

provided it defines a continuous mapping from $L^p(\mathbb{R}^d,w)$ to $L^{p,\infty}(\mathbb{R}^d,w)$. If it extends to a bounded mapping on $L^p(\mathbb{R}^d,w)$, we write its norm by $\|\{\mathbf{m}_j\}_j\|_{M_{p,w}(\mathbb{R}^d)}$.

We shall denote by $\mathbb{T}^d$ the topological group $\mathbb{R}^d/\mathbb{Z}^d$, which can be identified with the cube $[0,1)^d$ or eventually with $[-1/2,1/2)^d$ in $\mathbb{R}^d$. Functions on $\mathbb{T}^d$ will be identified with functions on $\mathbb{R}^d$ which are 1-periodic in each variable. A function

$f : \mathbb{T}^d \to \mathbb{C}$ such that for a finitely supported sequence $\{a_k\}_{k\in\mathbb{Z}^d}$ of complex numbers written as

$$f(x) = \sum_{k\in\mathbb{Z}^d} a_k e^{2\pi i k x}$$

is called a trigonometric polynomial, and we write $f \in P(\mathbb{T}^d)$. Let us recall that $P(\mathbb{T}^d)$ is dense in $L^p(\mathbb{T}^d, \mu)$ for any Radon measure μ on $\mathbb{T}^d$.

From now on, we work in the range

$$1 \le p < \infty,$$

and w is a weight in $\mathbb{R}^d$. Observe that if in addition w is 1-periodic, then $w \in L^1(\mathbb{T}^d)$.

3. Restriction of Fourier multipliers from $\mathbb{R}^d$ to $\mathbb{T}^d$

Definition 3.1. A function $\mathbf{m} \in \ell^\infty(\mathbb{Z}^d)$ is a weak type multiplier on $L^p(\mathbb{T}^d, w)$ (in symbols, $\mathbf{m} \in M_{p,w}^{(\mathfrak{w})}(\mathbb{T}^d)$) if the mapping

$$\sum_{k\in\mathbb{Z}^d} a_k e^{2\pi i k\theta} \in P(\mathbb{T}^d) \longrightarrow \sum_{k\in\mathbb{Z}^d} \mathbf{m}(k) a_k e^{2\pi i k\theta}$$

extends to a continuous operator $T_{\mathbf{m}} \in \mathfrak{B}\left(L^p(\mathbb{T}^d, w), L^{p,\infty}(\mathbb{T}^d, w)\right)$. In this case,

$$\|\mathbf{m}\|_{M_{p,w}^{(\mathfrak{w})}(\mathbb{T}^d)} = \|T_{\mathbf{m}}\|_{\mathfrak{B}(L^p(\mathbb{T}^d,w),L^{p,\infty}(\mathbb{T}^d,w))}.$$

If $T_{\mathbf{m}} \in \mathfrak{B}\left(L^p(\mathbb{T}^d, w)\right)$, $\mathbf{m}$ is said to be a multiplier on $L^p(\mathbb{T}^d, w)$, we denote it by $\mathbf{m} \in M_{p,w}(\mathbb{T}^d)$ and

$$\|\mathbf{m}\|_{M_{p,w}(\mathbb{T}^d)} = \|T_{\mathbf{m}}\|_{\mathfrak{B}(L^p(\mathbb{T}^d,w))}.$$

If $\{\mathbf{m}_j\}_j$ is a sequence in $M_{p,w}^{(\mathfrak{w})}(\mathbb{T}^d)$ we denote by $\|\{\mathbf{m}_j\}_j\|_{M_{p,w}^{(\mathfrak{w})}(\mathbb{T}^d)}$ the norm of the operator defined for every $f \in P(\mathbb{T}^d)$ by

$$T^\sharp_{\{\mathbf{m}_j\}_j} f(x) = \sup_j \left|T_{\mathbf{m}_j} f(x)\right|,$$

provided it extends to a continuous mapping from $L^p(\mathbb{T}^d, w)$ to $L^{p,\infty}(\mathbb{T}^d, w)$. We shall write $\|\{\mathbf{m}_j\}_j\|_{M_{p,w}(\mathbb{T}^d)}$ in the case that it extends to a continuous operator on $L^p(\mathbb{T}^d, w)$.

3.1. Restriction results for weak type maximal multipliers.

Theorem 3.2. *Let w be 1-periodic and let $\{\mathbf{m}_j\}_j \in M_{p,w}^{(\mathfrak{w})}(\mathbb{R}^d)$ satisfying that, for each j, there exists $K_j \in L^1(\mathbb{R}^d)$ with compact support such that $\hat{K}_j(x) = \mathbf{m}_j(x)$ for every $x \in \mathbb{R}^d$. Then $\{\mathbf{m}_j|_{\mathbb{Z}^d}\}_j \in M_{p,w}^{(\mathfrak{w})}(\mathbb{T}^d)$ and*

$$\left\|\{\mathbf{m}_j|_{\mathbb{Z}^d}\}_j\right\|_{M_{p,w}^{(\mathfrak{w})}(\mathbb{T}^d)} \le c_p \left\|\{\mathbf{m}_j\}_j\right\|_{M_{p,w}^{(\mathfrak{w})}(\mathbb{R}^d)},$$

where c_p depends only on p.

Proof. Let $\mathfrak{N} = \left\|\{\mathbf{m}_j\}_j\right\|_{M_{p,w}^{(\mathfrak{w})}(\mathbb{R}^d)}$. Since convolution operators commute with translations, it follows that for every $\theta \in [0,1)^d$ and every $N \in \mathbb{N}$,

$$\left\|\sup_{1\le j\le N} |K_j * g|\right\|_{L^{p,\infty}(\mathbb{R}^d, w(\cdot+\theta))} \le \mathfrak{N}\, \|g\|_{L^p(\mathbb{R}^d, w(\cdot+\theta))}. \tag{3.1}$$

Now, given $f(\theta)=\sum_k a_k e^{2\pi i k\theta}\in P(\mathbb{T}^d)$, let us consider

$$\begin{aligned}\tilde T_{K_j}f(\theta)&=\int_{\mathbb{R}^d}K_j(x)f(\theta-x)\,dx=\sum_k a_k\int_{\mathbb{R}^d}K_j(x)e^{2\pi i k(\theta-x)}\,dx\\&=\sum_{k\in\mathbb{Z}^d}a_k\mathbf{m}_j(k)e^{2\pi i k\theta};\end{aligned}$$

that is, $\tilde T_{K_j}$ coincides with the multiplier operator $T_{\mathbf{m}_j|_{\mathbb{Z}^d}}$.

Let $Q_r=(-r,r)^d$ with $r>0$ such that $\operatorname{supp}K_j\subset Q_r$ for $j=1,\dots,N$. Let $q<p$, and for any measurable $E\subset[0,1)^d$, let $\tilde E=\bigcup_{k\in\mathbb{Z}^d}E+k$ be its periodic extension. Set $E_\theta=\left\{x\in\mathbb{R}^d:\ x+\theta\in\tilde E\right\}$ with $\theta\in\mathbb{T}^d$ and $R_xf(\theta)=f(\theta+x)$. Then, by translation invariance, we have that, for every $x\in\mathbb{R}^d$,

$$\left\|\sup_{1\le j\le N}\left|\tilde T_{K_j}f\right|\chi_E\right\|^q_{L^q(\mathbb{T}^d,w)}=\int_{\mathbb{T}^d}\sup_{1\le j\le N}\left|R_x\tilde T_{K_j}f(\theta)\right|^q w(x+\theta)\chi_{\tilde E}(x+\theta)\,d\theta. \tag{3.2}$$

Therefore, for every $s>0$,

$$\begin{aligned}&\left\|\sup_{1\le j\le N}\left|\tilde T_{K_j}f\right|\chi_E\right\|^q_{L^q(\mathbb{T}^d,w)}\\&=\frac{1}{(2s)^d}\int_{Q_s}\int_{\mathbb{T}^d}\sup_{1\le j\le N}\left|R_xT_{K_j}f(\theta)\right|^q w(x+\theta)\chi_{\tilde E}(x+\theta)\,d\theta\,dx.\end{aligned}$$

Now, using that $\operatorname{supp}K_j\subset Q_r$ for $j=1,\dots,N$, one can easily see that, if $x\in Q_s$,

$$R_x\tilde T_{K_j}f(\theta)=B_{K_j}\left(R_{(\cdot)}f(\theta)\chi_{Q_{r+s}}\right)(x),$$

where $B_{K_j}(h)(x)=(K_j*h)(x)$, and hence,

$$\begin{aligned}&\left\|\sup_{1\le j\le N}\left|\tilde T_{K_j}f\right|\chi_E\right\|^q_{L^q(\mathbb{T}^d,w)}\\&\le\frac{1}{(2s)^d}\int_{\mathbb{T}^d}\left\{\int_{E_\theta\cap Q_s}\sup_{1\le j\le N}\left|B_{K_j}\left(R_{(\cdot)}f(\theta)\chi_{Q_{r+s}}\right)(x)\right|^q w(x+\theta)\,dx\right\}d\theta.\end{aligned}$$

By (2.1) and (3.1), the term inside curly brackets is bounded by

$$(c_{p,q}\mathfrak{N})^q\left\{\int_{Q_{r+s}}|R_xf(\theta)|^p\,w(x+\theta)\,dx\right\}^{\frac{q}{p}}\left\{\int_{E_\theta\cap Q_s}w(x+\theta)\,dx\right\}^{1-\frac{q}{p}}.$$

Also, using Hölder's inequality, it follows that

$$\begin{aligned}&\left\|\sup_{1\le j\le N}\left|\tilde T_{K_j}f\right|\chi_E\right\|^q_{L^q(\mathbb{T}^d,w)}\\&\le\frac{c^q_{p,q}\mathfrak{N}^q}{(2s)^d}\left\{\int_{\mathbb{T}^d}\int_{Q_{r+s}}|R_xf(\theta)|^p\,w(x+\theta)dtd\theta\right\}^{\frac{q}{p}}\left\{\int_{\mathbb{T}^d}\int_{Q_s\cap E_\theta}w(x+\theta)dx\,d\theta\right\}^{1-\frac{q}{p}}\\&\le\frac{c^q_{p,q}\mathfrak{N}^q}{(2s)^d}(2(r+s))^{\frac{dq}{p}}(2s)^{d(1-\frac{q}{p})}w(E)^{1-\frac{q}{p}}\|f\|^q_{L^p(\mathbb{T}^d,w)}\\&\le c^q_{p,q}\mathfrak{N}^q\left(\frac{r+s}{s}\right)^{\frac{dq}{q}}w(E)^{1-\frac{q}{p}}\|f\|^q_{L^p(\mathbb{T}^d,w)}.\end{aligned}$$

Thus, taking $s \to +\infty$, and using Kolmogorov's condition (2.1), we obtain that

$$\left\| \sup_{1\leq j\leq N} \left|\tilde{T}_{K_j} f\right| \right\|_{L^{p,\infty}(\mathbb{T}^d,w)} \leq c_{p,q}\mathfrak{N} \|f\|_{L^p(\mathbb{T}^d,w)} .$$

Now, considering $c_p = \inf_{q<p} c_{p,q}$, the result easily follows by Fatou's Lemma and the density of $P(\mathbb{T}^d)$ in $L^p(\mathbb{T}^d, w)$. □

The next step is to weaken the hypothesis assumed on $\mathbf{m}_j$ in the previous theorem as is done both in [4] and [1]. As usually happens, this is the technical part of the work.

Definition 3.3. A bounded function $\mathbf{m}$ defined in $\mathbb{R}^d$ is *normalized* if for any $x \in \mathbb{R}^d$,

$$\lim_n \widehat{\varphi_n} * \mathbf{m}(x) = \mathbf{m}(x),$$

where $\varphi_n(x) = \varphi(x/n)$, $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, $\widehat{\varphi} \geq 0$ and $\|\widehat{\varphi}\|_1 = 1$.

It is easy to see that $\lim_n \widehat{\varphi_n} * \mathbf{m}(x) = \mathbf{m}(x)$ for every Lebesgue point x of $\mathbf{m}$. In particular, any continuous and bounded function is normalized.

In order to extend Theorem 3.2 to the class of normalized multipliers, we shall need some previous lemmas. The following one is a direct consequence of the proof of [15, Lemma 2.6] for $G = \mathbb{R}^d$.

Lemma 3.4. *Let $J \in \mathbb{N}$ and let $\{\mathbf{m}_j\}_{j=1}^J$ be a family of $L^\infty(\mathbb{R}^d)$ functions. For $f \in \mathcal{S}(\mathbb{R}^d)$, $j = 1, \ldots, J$ and $x \in \mathbb{R}^d$, let*

$$F_{j,x}(\xi) = S_{\mathbf{m}_j}(e^{-2\pi i \xi \cdot} f)(x), \quad \xi \in \mathbb{R}^d.$$

Let $\mathcal{K}$ be a compact set. Then, for each $k \in \mathbb{N} \setminus \{0\}$, there exists a finite family $\{V_l^k\}_{l=1}^{I_k}$ of pairwise disjoint measurable sets in $\mathbb{R}^d$ such that

1. $\mathcal{K} \subset \biguplus_{l=1}^{I_k} V_l^k$,
2. *if $l = 1, \ldots, I_k$ and $\xi, \zeta \in V_l^k$, then*

$$|F_{j,x}(\xi) - F_{j,x}(\zeta)| \leq 1/k,$$

uniformly on $j \in \{1, \ldots, J\}$ and $x \in \mathbb{R}$.

Another key ingredient is the following version of Marcinkiewicz-Zygmund's inequality, whose proof is analogous to that given in [10, Theorem V.2.9] for $p = q = 1$ for linear operators.

Theorem 3.5. *Let $\{T_j\}_j$ be a countable family of linear operators such that*

$$\left\| \sup_j |T_j f| \right\|_{L^{1,\infty}(\mathbb{R}^d,w)} \leq \|\{T_j\}_j\| \, \|f\|_{L^1(\mathbb{R}^d,w)} .$$

Then

$$\left\| \sup_j \left(\sum_l |T_j f_l|^2 \right)^{1/2} \right\|_{L^{1,\infty}(\mathbb{R}^d,w)} \leq \mathfrak{c}_1 \|\{T_j\}_j\| \left\| \left(\sum_l |f_l|^2 \right)^{1/2} \right\|_{L^1(\mathbb{R}^d,w)},$$

where

$$\mathfrak{c}_1 := \inf_{0<r<1} \frac{\sqrt{\pi}}{2\left((1-r)\,\Gamma\left(1+\frac{r}{2}\right)\right)^{1/r}}. \tag{3.3}$$

For $p>1$, the next lemma is an immediate consequence of Minkowskii's inequality, as $L^{p,\infty}$ is normable, but for $p=1$ the convexity of the space $L^{1,\infty}$ fails. Similar results in the unweighted setting are given by [3, Lemma 2.1] and [4, Theorem 1.2].

Lemma 3.6. *Let $\varphi \in L^1(\mathbb{R}^d)$ and $\{\mathbf{m}_j\}_j \in M_{p,w}^{(\mathfrak{w})}(\mathbb{R}^d)$. Then $\{\varphi * \mathbf{m}_j\}_j \in M_{p,w}^{(\mathfrak{w})}(\mathbb{R}^d)$ and*

$$\left\| \{\varphi * \mathbf{m}_j\}_j \right\|_{M_{p,w}^{(\mathfrak{w})}(\mathbb{R}^d)} \le \mathfrak{c}_p \|\varphi\|_{L^1(\mathbb{R}^d)} \left\| \{\mathbf{m}_j\}_j \right\|_{M_{p,w}^{(\mathfrak{w})}(\mathbb{R}^d)}, \tag{3.4}$$

where $\mathfrak{c}_p = p'$ if $p>1$ and $\mathfrak{c}_1$ is the constant given in (3.3).

Proof. We shall only prove the case $p=1$. Without loss of generality, we can assume that $\{\mathbf{m}_j\}_j$ is a finite family of multipliers of cardinality, say $J \in \mathbb{N}$. For $g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$,

$$\int (\varphi * \mathbf{m}_j)(\xi) \widehat{g}(\xi) e^{2\pi i \xi x}\, d\xi = \int \varphi(y) e^{2\pi i x y} S_{\mathbf{m}_j}\left(e^{-2\pi i y \cdot} g\right)(x)\, dy.$$

Hence,

$$\left|S_{\varphi * \mathbf{m}_j} g(x)\right| \le \int |\varphi(y)| \left|S_{\mathbf{m}_j}\left(e^{-2\pi i y \cdot} g\right)(x)\right|\, dy, \tag{3.5}$$

and thus

$$\sup_{1\le j\le J} \left|S_{\varphi * \mathbf{m}_j} g(x)\right| \le \int |\varphi(y)| \sup_{1\le j \le J} \left|S_{\mathbf{m}_j}\left(e^{-2\pi i y \cdot} g\right)(x)\right|\, dy.$$

Let us first assume that $\varphi \in L^1(\mathbb{R}^d)$ is supported on a compact set $\mathcal{K}$. For each $k\ge 1$ let $\{V_l^k\}_{l=1}^{I_k}$ be the family of pairwise disjoint sets given by Lemma 3.4, and for each l, select $y_l^k \in V_l^k$. Then, for every $y \in \mathcal{K}$ and any $k \ge 1$, there exists a unique $l \in \{1,\dots,I_k\}$ such that $y \in V_l^k$, and hence

$$\left|S_{\mathbf{m}_j}\left(e^{-2\pi i y \cdot} g\right)(x) - S_{\mathbf{m}_j}\left(e^{-2\pi i y_l^k \cdot} g\right)(x)\right| \le \frac{1}{k},$$

uniformly on $j=1,\dots,J$ and $x \in \mathbb{R}^d$. It follows that for every $x \in \mathbb{R}^d$, any $j \in \{1,\dots,J\}$ and all $y \in \mathcal{K}$,

$$\lim_k \sum_{l=1}^{I_k} S_{\mathbf{m}_j}\left(e^{-2\pi i y_l^k \cdot} g\right)(x) \chi_{V_l^k}(y) = S_{\mathbf{m}_j}\left(e^{-2\pi i y \cdot} g\right)(x).$$

Then, by Fatou's Lemma on (3.5),

$$\sup_{1\le j\le J} \left|S_{\varphi * \mathbf{m}_j} g(x)\right| \le \liminf_k \sup_{1\le j\le J} \left(\sum_{l=1}^{I_k} \left|S_{\mathbf{m}_j}\left(e^{-2\pi i y_l^k \cdot} g\right)(x)\right| \lambda_l^k \right),$$

where $\lambda_l^k = \int_{V_l^k} |\varphi(y)|\, dy$. Observe that the term inside brackets is less than or equal to

$$\|\varphi\|_{L^1(\mathbb{R}^d)}^{1/2} \left(\sum_{l=1}^{I_k} \left| S_{\mathbf{m}_j}\left(\sqrt{\lambda_l^k}\, e^{-2\pi i y_l^k \cdot} g\right)(x)\right|^2 \right)^{1/2},$$

where we have used that $\sum_{l=1}^{I_k} \lambda_l^k = \int_{\biguplus_{l=1}^k V_l^k} |\varphi(y)|\ dy = \|\varphi\|_{L^1(\mathbb{R}^d)}$. Then,

$$\left\| \sup_{1\le j\le J} \left|S_{\varphi * \mathbf{m}_j} g\right| \right\|_{L^{1,\infty}(\mathbb{R}^d,w)}$$
$$\le \|\varphi\|_{L^1(\mathbb{R}^d)}^{1/2} \liminf_k \left\| \sup_{1\le j\le J} \left(\sum_{l=1}^{I_k} \left| S_{\mathbf{m}_j}\left(\sqrt{\lambda_l^k}\, e^{-2\pi i y_l^k \cdot} g\right)(x)\right|^2 \right)^{1/2} \right\|_{L^{1,\infty}(\mathbb{R}^d,w)}.$$

Applying Theorem 3.5 with the family of operators $\left\{S_{\mathbf{m}_j}\right\}_j$ to the functions $f_l = \sqrt{\lambda_l^k}\, e^{-2\pi i y_l^k \cdot} g$, we obtain that

$$\left\| \sup_{1\le j\le J} \left(\sum_{l=1}^{I_k} \left| S_{\mathbf{m}_j}\left(\sqrt{\lambda_l^k}\, e^{-2\pi i y_l^k \cdot} g\right)\right|^2 \right)^{1/2} \right\|_{L^{1,\infty}(\mathbb{R}^d,w)}$$
$$\le \mathfrak{c}_1 \left\| \{\mathbf{m}_j\}_j \right\|_{M_{1,w}^{(\mathfrak{w})}(\mathbb{R}^d)} \left\| \left(\sum_{l=1}^{I_k} \left| \sqrt{\lambda_l^k}\, e^{-2\pi i y_l^k \cdot} g\right|^2 \right)^{1/2} \right\|_{L^1(\mathbb{R}^d,w)}$$
$$= \mathfrak{c}_1 \left\| \{\mathbf{m}_j\}_j \right\|_{M_{1,w}^{(\mathfrak{w})}(\mathbb{R}^d)} \|\varphi\|_{L^1(\mathbb{R}^d)}^{1/2} \|g\|_{L^1(\mathbb{R}^d,w)}.$$

Therefore,

$$\left\| \sup_{1\le j\le J} \left|S_{\varphi * \mathbf{m}_j} g\right| \right\|_{L^{1,\infty}(\mathbb{R}^d,w)} \le \mathfrak{c}_1 \|\varphi\|_{L^1(\mathbb{R}^d)} \left\| \{\mathbf{m}_j\}_j \right\|_{M_{1,w}^{(\mathfrak{w})}(\mathbb{R}^d)} \|g\|_{L^1(\mathbb{R}^d,w)}. \tag{3.6}$$

In the case that φ is not compactly supported, considering $\varphi_n = \varphi \chi_{B(0,n)}$, we can write

$$\sup_{1\le j\le J} \left|S_{\varphi * \mathbf{m}_j} g(x)\right| \le \lim_n \int |\varphi_n(y)| \sup_{1\le j\le J} \left| S_{\mathbf{m}_j}\left(e^{-2\pi i y \cdot} g\right)(x)\right|\ dy,$$

and using the previous argument we obtain that

$$\left\| \sup_{1\le j\le J} \left|S_{\varphi * \mathbf{m}_j} g\right| \right\|_{L^{1,\infty}(\mathbb{R}^d,w)} \le \mathfrak{c}_1 \liminf_n \|\varphi_n\|_{L^1(\mathbb{R}^d)} \left\| \{\mathbf{m}_j\}_j \right\|_{M_{1,w}^{(\mathfrak{w})}(\mathbb{R}^d)} \|g\|_{L^1(\mathbb{R}^d,w)},$$

from where it follows that (3.6) holds for any $\varphi \in L^1(\mathbb{R}^d)$. The result now follows by the density of $\mathcal{C}_c^\infty(\mathbb{R}^d)$ in $L^p(\mathbb{R}^d, w)$. □

Lemma 3.7. *Let $w \in \mathcal{C}(\mathbb{T}^d)$ such that $\inf_{x\in\mathbb{T}^d} w(x) > 0$. Consider $h \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ satisfying $0 \le h \le 1$ and $\int_{\mathbb{R}^d} h = 1$ and define $h_n(x) = n^d h(nx)$. Then,*

1. *There exists $n_0 = n_0(w) \in \mathbb{N}$ such that, for any $p \in [1,\infty)$,*
$$\sup_{n\ge n_0} ||\widehat{h_n}||_{M_{p,w}(\mathbb{R}^d)} \le 2^{1/p}.$$
2. $\sup_n ||\widehat{h_n}||_{L^\infty(\mathbb{R}^d)} \le 1$.
3. *For every $\xi \in \mathbb{R}^d$, $\lim_n \widehat{h_n}(\xi) = 1$.*

Proof. Since $\|h_n\|_{L^1} = 1$, it follows that $\|\widehat{h_n}\|_\infty \le 1$. On the other hand, for every $\xi \in \mathbb{R}^d$ and for every $\epsilon > 0$ there exists n_0 such that for all $|x| < \frac{1}{n_0}$, $|1 - e^{2\pi i x \xi}| < \epsilon$.

Hence, for every $n \geq n_0$,

$$\left|1 - \widehat{h_n}(\xi)\right| \leq \int h_n(x) \left|1 - e^{2\pi i x\xi}\right| \, dx \leq \epsilon.$$

Then, it follows that $\widehat{h_n} \to 1$ pointwise. It remains to show that $||\widehat{h_n}||_{M_{p,w}(\mathbb{R}^d)}$ are uniformly bounded on n.

Observe that $\|f\|_{L^\infty(w)} = \|f\|_{L^\infty}$ and hence, for any $n \geq 1$,

$$\|h_n * f\|_{L^\infty(\mathbb{R}^d,w)} \leq \|f\|_{L^\infty(\mathbb{R}^d,w)} .$$

Let $\delta = \inf_{x\in\mathbb{T}^d} w(x) > 0$. Since $w \in \mathcal{C}(\mathbb{T}^d)$, there exists $n_0 = n_0(\delta)$ such that, for any $n \geq n_0$, for any x and any $y \in \operatorname{supp} h_n$,

$$|w(x) - w(x - y)| \leq \delta,$$

which implies that, for any $x \in \mathbb{T}^d$,

$$h_n * w(x) \leq \delta + w(x) \leq 2w(x).$$

Then, for $n \geq n_0$,

$$\|h_n * f\|_{L^1(\mathbb{R}^d,w)} \leq \int_{\mathbb{R}^d} |f(y)| \, h_n * w(y) \, dy \leq 2 \|f\|_{L^1(\mathbb{R}^d,w)} .$$

In other words, we have seen that for $n \geq n_0$ the linear operator defined by $T_n f = h_n * f$ is uniformly bounded on $L^1(\mathbb{R}^d, w)$ and $L^\infty(\mathbb{R}^d, w)$ with norm respectively bounded by 2 and 1. Riesz-Thorin's Theorem implies the result. □

Lemma 3.8. *Let w be 1-periodic. If $g \in \mathcal{C}_c(\mathbb{R}^d)$ is nonnegative, $\int_{\mathbb{R}^d} g = 1$ and $\operatorname{supp} g \subset [-1/2, 1/2]^d$, then $\inf_{x\in\mathbb{R}^d} g * w(x) > 0$.*

Proof. Clearly $g * w \in \mathcal{C}(\mathbb{T}^d)$, and hence there exists $x_0 \in [-1/2, 1/2]^d$ such that $\inf_{x\in\mathbb{R}^d} g * w(x) = \min_{|x|_\infty \leq 1/2} g * w(x) = g * w(x_0)$. Since $g \in \mathcal{C}_c(\mathbb{R}^d)$, there exists a set of positive Lebesgue measure Q where $g(y) > 0$ for all $y \in Q$. Thus if $0 = g * w(x_0) = \int g(y)w(x_0 - y) \, dy$, then $g(y)w(x_0 - y) = 0$ a.e. $y \in Q$, which implies that $w(z) = 0$ a.e. $z \in x_0 - Q$, but this contradicts the fact that the set $\{x : w(x) = 0\}$ is null. □

Lemma 3.9. *Let T be any bounded operator from $L^p(\mathbb{R}^d, w)$ to $L^{p,\infty}(\mathbb{R}^d, w)$ that conmutes with translations. Then, for any nonnegative function $g \in \mathcal{C}_c(\mathbb{R}^d)$, T is bounded from $L^p(\mathbb{R}^d, g * w)$ to $L^{p,\infty}(\mathbb{R}^d, g * w)$ and*

$$\|T\|_{\mathcal{B}(L^p(\mathbb{R}^d,g*w),L^{p,\infty}(\mathbb{R}^d,g*w))} \leq c_p \|T\|_{\mathcal{B}(L^p(\mathbb{R}^d,w),L^{p,\infty}(\mathbb{R}^d,w))} ,$$

where $c_p = \inf_{q<p} \left(\frac{p}{p-q}\right)^{\frac{1}{q}}$.

Proof. Let E be any measurable set in $\mathbb{R}^d$ such that $0 < g * w(E) < +\infty$. Then, for any $q < p$,

$$\begin{aligned}\|Tf\chi_E\|^q_{L^q(\mathbb{R}^d,g*w)} &= \int_E |Tf(x)|^q \, g * w(x) \, dx = \int g(y) \int_E |Tf(x)|^q \, w(x - y) \, dx \, dy \\ &= \int g(y) \int_{E-y} |Tf_y(x)|^q \, w(x) \, dx \, dy,\end{aligned}$$

with $f_y(z) = f(z+y)$. Thus, by the boundedness hypothesis, Kolmogorov's condition and Hölder's inequality,

$$\|Tf\chi_E\|^q_{L^q(\mathbb{R}^d, g*w)} \leq \|T\|^q \int g(y) w(E-y)^{1-\frac{q}{p}} \left(\int |f_y(x)|^p\, w(x)\, dx\right)^{q/p} dy$$
$$\leq c^q_{p,q} \|T\|^q \left(g * w(E)\right)^{1-\frac{q}{p}} \|f\|^q_{L^p(\mathbb{R}^d, g*w)},$$

where $c^q_{p,q} = p/(p-q)$. Then, the result follows by Kolmogorov's condition and by taking the infimum for $q < p$. □

Theorem 3.10. *Let w be a 1-periodic weight on $\mathbb{R}^d$. Suppose that $\{\mathbf{m}_j\}_j$ are normalized functions and $\{\mathbf{m}_j\}_j \in M^{(\mathfrak{w})}_{p,w}(\mathbb{R}^d)$. Then $\{\mathbf{m}_j|_{\mathbb{Z}^d}\}_j \in M^{(\mathfrak{w})}_{p,w}(\mathbb{T}^d)$ and*

$$\left\| \{\mathbf{m}_j|_{\mathbb{Z}^d}\}_j \right\|_{M^{(\mathfrak{w})}_{p,w}(\mathbb{T}^d)} \leq c_p \left\| \{\mathbf{m}_j\}_j \right\|_{M^{(\mathfrak{w})}_{p,w}(\mathbb{R}^d)},$$

where c_p depends only on p.

Proof. Let $\{g_l\}_l$ be a family of nonnegative functions in $\mathcal{C}^\infty_c(\mathbb{R}^d)$, supported in $[-1/2, 1/2]^d$ such that it is an approximation of the identity in $L^1(\mathbb{T}^d)$. We can also assume that $\lim_l g_l * w(x) = w(x)$ a.e. $x \in [-1/2, 1/2]^d$.

For a fixed $l \in \mathbb{N}$, by Lemma 3.9,

$$\left\| \{\mathbf{m}_j\}_j \right\|_{M^{(\mathfrak{w})}_{p,g_l*w}(\mathbb{R}^d)} \leq c_p \left\| \{\mathbf{m}_j\}_j \right\|_{M^{(\mathfrak{w})}_{p,w}(\mathbb{R}^d)}.$$

By Lemma 3.8 it follows that for any $h \in \mathcal{C}^\infty_c(\mathbb{R}^d)$ such that $0 \leq h \leq 1$ and that $\int_{\mathbb{R}^d} h = 1$, there exists an n_l such that, for any $n \geq n_l$, the conclusions of Lemma 3.7 hold for the periodic weight $g_l * w$.

Consider, for $j, n \in \mathbb{N}$,

$$\mathbf{m}_{j,n}(\xi) = \widehat{K_{j,n}}(\xi) = (\widehat{\varphi_n} * \mathbf{m}_j)(\xi)\widehat{h_n}(\xi),$$

where φ_n are the functions given by the normalized condition. First observe that $\widehat{K_{j,n}} \in \mathcal{S}(\mathbb{R}^d)$ and hence $K_{j,n} \in \mathcal{S}(\mathbb{R}^d)$. Moreover, since

$$K_{j,n}(x) = (\varphi_n\, \mathbf{m}_j{}^\vee)(h_n(x-\cdot)) = \mathbf{m}_j{}^\vee(\varphi_n(\cdot)\, h_n(x-\cdot)),$$

and φ_n, h_n are compactly supported, it follows that $K_{j,n} \in \mathcal{C}^\infty_c(\mathbb{R}^d)$. On the other hand, since $\mathbf{m}_j$ is normalized and $\widehat{h_n} \to 1$, it holds that for every $\xi \in \mathbb{R}^d$,

$$\lim_n \widehat{K_{j,n}}(\xi) = \mathbf{m}_j(\xi).$$

Since $\left\|\widehat{h_n}\right\|_{L^\infty(\mathbb{R}^d)} \leq 1$ and $\|\widehat{\varphi_n}\|_{L^1(\mathbb{R}^d)} \leq 1$, then $\|\mathbf{m}_{j,n}\|_{L^\infty(\mathbb{R}^d)} \leq \|\mathbf{m}_j\|_{L^\infty(\mathbb{R}^d)}$.

Let us fix $J \in \mathbb{N}$. Since for any $f \in \mathcal{C}^\infty_c(\mathbb{R})$

$$K_{j,n} * f = T_{\widehat{\varphi_n} * \mathbf{m}_j}(h_n * f),$$

it follows that for every $n \geq n_l$,

$$\left\| \sup_{1 \leq j \leq J} |K_{j,n} * f| \right\|_{L^{p,\infty}(\mathbb{R}^d, g_l*w)} \leq \left\| \{\widehat{\varphi_n} * \mathbf{m}_j\}_j \right\|_{M^{(\mathfrak{w})}_{p,g_l*w}(\mathbb{R}^d)} \|h_n * f\|_{L^p(\mathbb{R}^d, g_l*w)}$$
$$\leq c_p 2^{\frac{1}{p}} \left\| \{\mathbf{m}_j\}_j \right\|_{M^{(\mathfrak{w})}_{p,w}(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d, g_l*w)},$$

where we have used that, by Lemma 3.6,

$$\left\|\{\widehat{\varphi_n} * \mathbf{m}_j\}_j\right\|_{M^{(\mathfrak{w})}_{p,g_l*w}(\mathbb{R}^d)} \leq c_p \left\|\{\mathbf{m}_j\}_j\right\|_{M^{(\mathfrak{w})}_{p,g_l*w}(\mathbb{R}^d)} \leq c_p^2 \left\|\{\mathbf{m}_j\}_j\right\|_{M^{(\mathfrak{w})}_{p,w}(\mathbb{R}^d)}.$$

We can now apply Theorem 3.2 to deduce that for any $n \geq n_l$,

$$\left\|\{\mathbf{m}_{j,n}|_{\mathbb{Z}^d}\}_{j=1}^J\right\|_{M^{(\mathfrak{w})}_{p,g_l*w}(\mathbb{T}^d)} \leq 2^{\frac{1}{p}} c_p^2 \left\|\{\mathbf{m}_j\}_j\right\|_{M^{(\mathfrak{w})}_{p,w}(\mathbb{R}^d)}.$$

Since, for any $f \in P(\mathbb{T}^d)$,

$$\lim_n S_{\mathbf{m}_{j,n}} f(s) = \lim_n \sum_{k\in\mathbb{Z}^d} \mathbf{m}_{j,n}(k)\widehat{f}(k)e^{2\pi iks} = \sum_{k\in\mathbb{Z}^d} \mathbf{m}_j(k)\widehat{f}(k)e^{2\pi iks} = S_{\mathbf{m}_j} f(s),$$

by Fatou's Lemma, the following inequality holds:

$$\begin{aligned}\left\|\sup_{1\leq j\leq J} \left|S_{\mathbf{m}_j} f\right|\right\|_{L^{p,\infty}(\mathbb{T}^d,g_l*w)} &\leq \liminf_n \left\|\sup_{1\leq j\leq J} \left|S_{\mathbf{m}_{j,n}} f\right|\right\|_{L^{p,\infty}(\mathbb{T}^d,g_l*w)} \\ &\leq 2^{\frac{1}{p}} c_p^2 \left\|\{\mathbf{m}_j\}_j\right\|_{M^{(\mathfrak{w})}_{p,w}(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{T}^d,g_l*w)},\end{aligned}$$

which implies

$$\left\|\sup_{1\leq j} \left|S_{\mathbf{m}_j} f\right|\right\|_{L^{p,\infty}(\mathbb{T}^d,w)} \leq 2^{\frac{1}{p}} c_p^2 \left\|\{\mathbf{m}_j\}_j\right\|_{M^{(\mathfrak{w})}_{p,w}(\mathbb{R}^d)} \liminf_{l\to\infty} \|f\|_{L^p(\mathbb{T}^d,g_l*w)}.$$

Observe that

$$\|f\|_{L^p(\mathbb{T}^d,g_l*w)} \leq \|f\|_{L^\infty(\mathbb{T}^d)} \|g_l * w - w\|_{L^1(\mathbb{T}^d)} + \|f\|_{L^p(\mathbb{T}^d,w)},$$

and since $\lim_l \|g_l * w - w\|_{L^1(\mathbb{T}^d)} = 0$, it follows that

$$\left\|\sup_{1\leq j} \left|S_{\mathbf{m}_j} f\right|\right\|_{L^{p,\infty}(\mathbb{T}^d,w)} \leq 2^{\frac{1}{p}} c_p^2 \|f\|_{L^p(\mathbb{T}^d,w)}.$$

The result follows by the density of $P(\mathbb{T}^d)$ in $L^p(\mathbb{T}^d, w)$. □

With minor modifications in the proofs, the analogous result for operators of strong type can be proved. In the particular case of a single multiplier, we recover K. Andersen and P. Mohanty's [1, Theorem 1.1].

Theorem 3.11. *Let w be 1-periodic. Suppose that $\{\mathbf{m}_j\}_j \subset M_{p,w}(\mathbb{R}^d)$ and that they are normalized functions. Then $\{\mathbf{m}_j|_{\mathbb{Z}^d}\}_j \subset M_{p,w}(\mathbb{T}^d)$ and*

$$\left\|\{\mathbf{m}_j|_{\mathbb{Z}^d}\}_j\right\|_{M_{p,w}(\mathbb{T}^d)} \leq 2^{1/p} \left\|\{\mathbf{m}_j\}_j\right\|_{M_{p,w}(\mathbb{R}^d)}.$$

3.2. **An improvement for nonperiodic weights.** A similar approach to that in the previous section allows us to obtain a more general version of Theorem 3.10 (and also of Theorem 3.11) for a class of nonnecessarily periodic weights which includes those in $A_p(\mathbb{T}^d)$.

Definition 3.12. We say that a weight $v \in W(\mathbb{R}^d)$ if it satisfies the following conditions:

i) For every $x \in \mathbb{R}^d$, $\theta \in [0,1)^d$,

$$\frac{1}{\zeta} \leq \frac{v(x)}{v(x+\theta)} \leq \zeta.$$

ii)

$$\lim_{s\to\infty} \frac{v(Q_{r+s})}{v(Q_s)} = 1.$$

Theorem 3.13. *Let u be a periodic weight in $\mathbb{R}^d$, let $v \in W$ and set $w = uv$. Assume that $\{\mathbf{m}_j\}_j \in M_{p,w}^{(\mathfrak{w})}(\mathbb{R}^d)$ (respectively $M_{p,w}(\mathbb{R}^d)$) and that they are normalized functions in $\mathbb{R}^d$. Then $\{\mathbf{m}_j|_{\mathbb{Z}^d}\}_j \in M_{p,w}^{(\mathfrak{w})}(\mathbb{T}^d)$ (respectively $M_{p,w}(\mathbb{T}^d)$) and*

$$\left\| \{\mathbf{m}_j|_{\mathbb{Z}^d}\}_j \right\|_{M_{p,u}^{(\mathfrak{w})}(\mathbb{T}^d)} \leq \mathfrak{c}_{p,w} c_{p,v} \left\| \{\mathbf{m}_j\}_j \right\|_{M_{p,u}^{(\mathfrak{w})}(\mathbb{R}^d)}$$

(respectively replacing $M_{p,u}^{(\mathfrak{w})}$ with $M_{p,u}$ in the previous inequality), where $\mathfrak{c}_{p,w}$ and $c_{p,v}$ depend only on p.

Proof. We shall prove the weak case. The proof for the strong case is similar and we leave the details to the reader. Assume first that $\{\mathbf{m}_j\}_j$ is a finite sequence. The argument is similar to that for Theorem 3.2, and we shall sketch the major changes to be done in the proof.

Let $\mathfrak{N} = \left\| \{\mathbf{m}_j\}_j \right\|_{M_{p,w}^{(\mathfrak{w})}(\mathbb{R}^d)}$. Since $v \in W$,

$$\frac{1}{\zeta} w(x+\theta) \leq u(x+\theta)v(x) \leq \zeta w(x+\theta). \tag{3.7}$$

By (3.2), for every $f \in P(\mathbb{T})$ and every measurable set $E \subset \mathbb{T}$,

$$\begin{aligned}
&\left\| \sup_{1\leq j\leq N} \left| T_{K_j} f \right| \chi_E \right\|_{L^q(\mathbb{T}^d,u)}^q \\
&= \frac{1}{v(Q_s)} \int_{Q_s} \int_{\mathbb{T}^d} \sup_{1\leq j\leq N} \left| R_x T_{K_j} f(\theta) \right|^q u(x+\theta) v(x) \chi_{\tilde{E}}(x+\theta)\, d\theta\, dx \\
&\leq \frac{\zeta}{v(Q_s)} \int_{\mathbb{T}^d} \left\{ \int_{E_\theta \cap Q_s} \sup_{1\leq j\leq N} \left| B_{K_j}\left(R_{(\cdot)} f(\theta) \chi_{Q_{r+s}} \right)(x) \right|^q w(x+\theta)\, dx \right\} d\theta,
\end{aligned}$$

where $E_\theta = \left\{ x \in \mathbb{R}^d :\ x+\theta \in \tilde{E} \right\}$ and $\tilde{E}$ is the periodic extension of E. By (2.1) and (3.1), the term inside curly brackets is bounded by

$$(c_{p,q}\mathfrak{N})^q \left\{ \int_{Q_{r+s}} |R_x f(\theta)|^p w(x+\theta)\, dt \right\}^{\frac{q}{p}} \left\{ \int_{E_\theta \cap Q_s} w(x+\theta)\, dt \right\}^{1-\frac{q}{p}}.$$

Hence, by Hölder's inequality, it follows that

$$\begin{aligned}
&\left\| \sup_{1\leq j\leq N} \left| T_{K_j} f \right| \chi_E \right\|_{L^q(\mathbb{T}^d,u)}^q \\
&\leq \frac{c_{p,q}^q \mathfrak{N}^q \zeta}{v(Q_s)} \left[\int_{\mathbb{T}^d} \int_{Q_{r+s}} |R_x f(\theta)|^p w(x+\theta) dt d\theta \right]^{\frac{q}{p}} \left[\int_{\mathbb{T}^d} \int_{Q_s \cap E_\theta} w(x+\theta) dx\, d\theta \right]^{1-\frac{q}{p}}.
\end{aligned}$$

By (3.7), the first term is bounded by $[\zeta v(Q_{r+s})]^{\frac{q}{p}} \|f\|_{L^p(\mathbb{T},u)}^q$, and the second one by $[\zeta v(Q_s) u(E)]^{1-\frac{q}{p}}$. Hence,

$$u(E)^{\frac{q}{p}-1} \left\| \sup_{1\leq j\leq N} \left| T_{K_j} f \right| \chi_E \right\|_{L^q(\mathbb{T}^d,u)}^q \leq c_{p,q}^q \mathfrak{N}^q \zeta^2 \left(\frac{v(Q_{r+s})}{v(Q_s)} \right)^{\frac{q}{p}} \|f\|_{L^p(\mathbb{T}^d,u)}^q.$$

Letting $s \to \infty$ and using (2.1) we obtain that

$$\left\| \sup_{1\le j\le N} \left|T_{K_j} f\right| \right\|_{L^{p,\infty}(\mathbb{T}^d,u)} \le c_{p,q}\zeta^{\frac{2}{q}}\mathfrak{N}\,\|f\|_{L^p(\mathbb{T}^d,u)}\,.$$

Considering $c_{p,v} = \inf_{q<p} \zeta^{2/q} c_{p,q}$, the result easily follows by Fatou's Lemma and the density of $P(\mathbb{T}^d)$ in $L^p(\mathbb{T}^d, u)$. □

4. Restriction of Fourier multipliers to lower dimension

Restriction of Fourier multipliers of strong type to a lower dimensional space was studied in [7, Corollary 4.13]. Here we shall give a weak counterpart to that result.

We have to mention here that in this section we work with $A_p(\mathbb{R}^d)$ weights mainly because, under this condition, we can prove the analogue to Lemma 3.7 (see Lemma 4.2 below). Other conditions that we can assume in w in order to have an approximation lemma are, for example, that w is uniformly continuous and $\inf_{x\in\mathbb{R}^d} w(x) > 0$. In this case the proof is a simple modification of the proof of Lemma 3.7.

Lemma 4.1. *If $w \in \bigcup_{1\le p<\infty} A_p(\mathbb{R}^d)$, then, for any $s > 0$, $\lim_{s\to\infty} \frac{w(Q_{r+s})}{w(Q_s)} = 1$.*

Proof. By the A_∞-condition [10, Theorem IV.2.9], there exist $\delta, C > 0$ such that

$$0 \le 1 - \frac{w\left(Q_s\right)}{w(Q_{r+s})} = \frac{w\left(Q_{r+s} \setminus Q_s\right)}{w(Q_{r+s})} \le C\left(1 - \frac{s^d}{(r+s)^d}\right)^{\delta},$$

from where the result easily follows. □

Lemma 4.2. *If $w \in A_p(\mathbb{R}^d)$, there exists $\{h_n\}_n \subset \mathcal{C}_c^\infty(\mathbb{R}^d)$, such that*

1. $\mathfrak{s}_{p,w} := \sup_n ||\widehat{h_n}||_{M_{p,w}(\mathbb{R}^d)} < \infty$,
2. $\sup_n ||\widehat{h_n}||_{L^\infty(\mathbb{R}^d)} \le 1$,
3. *for every $\xi \in \mathbb{R}^d$, $\lim_n \widehat{h_n}(\xi) = 1$.*

Proof. Properties (2) and (3) are proved as in Lemma 3.7. To prove (1), we first observe that clearly

$$\sup_n |h_n * f|(x) \lesssim Mf(x)$$

and hence the case $p > 1$ is trivial.

To prove the case $p = 1$, fix $\beta \in \mathbb{N}$, $\beta > d$. Then,

$$\|h_n * f\|_{L^1(w)} \le \int |f(y)| \int h_n(x-y) w(x)\, dx dy.$$

For a fixed $y \in \mathbb{R}^d$ and $n > 0$, the inner integral can be split into

$$\int_{|x-y|<n^{-1}} + \sum_{j\ge 0} \int_{2^j n^{-1} < |x-y| \le 2^{j+1} n^{-1}} h\left(n(x-y)\right) n^d w(x)\, dx.$$

The first term can be bounded by

$$\|h\|_\infty\, n^d \int_{|x-y|<n^{-1}} w(x)\, dx \le [w]_{A_1} 2^d w(y),$$

as the ball $|x-y| < 1/n$ is included in $y+[-1/n,1/n]^d$. On the other hand, if $p_{0,\beta}(h)=\sup_{x\in\mathbb{R}^d}|h(x)|\,|x|^{\beta}$, each term on the sum can be bounded from above by

$$p_{0,\beta}(h)n^{d-\beta}\int_{2^j<n|x-y|\le 2^{j+1}}|x-y|^{-\beta}\,w(x)\,dx \le p_{0,\beta}(h)4^d2^{j(d-\beta)}[w]_{A_1}w(y).$$

Thus, the sum is bounded from above by $\frac{p_{0,\beta}(h)4^d}{1-2^{d-\beta}}[w]_{A_1}w(y)$. Hence

$$\|h_n * f\|_{L^1(w)} \le [w]_{A_1}c_{d,h}\,\|f\|_{L^1(w)}\,,$$

where $c_{d,h}=2^d\left(1+\inf_{\beta>d}p_{0,\beta}(h)\frac{2^d}{1-2^{d-\beta}}\right)$. □

The following result is the weighted version of [6, Lemma 2] and the weak type maximal counterpart of [7, Proposition 4.10].

Proposition 4.3. *Let $w\in A_p(\mathbb{R}^d)$ and let $\{\mathbf{m}_j\}_j\subset M^{(\mathfrak{w})}_{p,w}(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ that are normalized functions. Then, there exist $\{\mathbf{m}_{j,n}\}_{j,n}\subset L^\infty(\mathbb{R}^d)$ satisfying:*

(1) *For any j and every $\xi\in\mathbb{R}^d$,*

$$\mathbf{m}_j(\xi)=\lim_n \mathbf{m}_{j,n}(\xi). \tag{4.1}$$

(2) *$K_{j,n}={\mathbf{m}_{j,n}}^\vee\in L^1(\mathbb{R}^d)$, and it is compactly supported.*
(3) *$\sup_n\|\mathbf{m}_{j,n}\|_{L^\infty(\mathbb{R}^d)}\le\|\mathbf{m}_j\|_{L^\infty(\mathbb{R}^d)}$.*
(4) *$\sup_n\|\{\mathbf{m}_{j,n}\}_j\|_{M^{(\mathfrak{w})}_{p,w}(\mathbb{R}^d)}\le\mathfrak{d}_{p,w}\,\|\{\mathbf{m}_j\}\|_{M^{(\mathfrak{w})}_{p,w}(\mathbb{R}^d)}$, where $\mathfrak{d}_{p,w}$ depends only on p, d and the A_p-constant of w.*

Proof. Let $\{h_n\}$ be the functions given by Lemma 4.2 and $\varphi_n(x)$ as in Definition 3.3. Consider, for $j,n\in\mathbb{N}$,

$$m_{j,n}(\xi)=\widehat{K_{j,n}}(\xi)=(\widehat{\varphi_n}*\mathbf{m}_j)(\xi)\widehat{h_n}(\xi),$$

and proceed as in the proof of Theorem 3.10. □

Theorem 4.4. *Let $d=d_1+d_2$, $u\in A_p(\mathbb{R}^{d_1})$, $v\in A_p(\mathbb{R}^{d_2})$ and define $w(x,y)=u(x)v(y)$. Suppose that $\{\mathbf{m}_j\}_j\in M^{(\mathfrak{w})}_{p,w}(\mathbb{R}^d)$ and are normalized functions. Then, for a fixed $\xi\in\mathbb{R}^{d_1}$, $\{\mathbf{m}_j(\xi,\cdot)\}_j\in M^{(\mathfrak{w})}_{p,w}(\mathbb{R}^{d_2})$ and*

$$\sup_{\xi\in\mathbb{R}^{d_1}}\left\|\{\mathbf{m}_j(\xi,\cdot)\}_j\right\|_{M^{(\mathfrak{w})}_{p,u}(\mathbb{R}^{d_2})}\le\mathfrak{c}_{p,w}\left\|\{\mathbf{m}_j\}_j\right\|_{\mathbf{m}\in M^{(\mathfrak{w})}_{p,u}(\mathbb{R}^d)},$$

where $\mathfrak{c}_{p,w}$ depends only on p, d and the A_p-constant of w.

Proof. Since $u\in A_p(\mathbb{R}^{d_1})$ and $v\in A_p(\mathbb{R}^{d_2})$ we have that $w\in A_p(\mathbb{R}^d)$ and $[w]_{A_p(\mathbb{R}^d)}\le[v]_{A_p(\mathbb{R}^{d_1})}[u]_{A_p(\mathbb{R}^{d_2})}$. Then, by Proposition 4.3, we can assume that $\{\mathbf{m}_j\}_{j=1}^J$ is a finite family such that $K_j=\mathbf{m}_j^\vee\in L^1$ with compact support.

Let $\mathfrak{N}=\left\|\{\mathbf{m}_j\}_j\right\|_{M^{(\mathfrak{w})}_{p,w}(\mathbb{R}^d)}$. Since translations and convolution commute, it follows that for every $z\in\mathbb{R}^{d_2}$,

$$\left\|\sup_{1\le j\le J}\left|B_{K_j}g\right|\right\|_{L^{p,\infty}(\mathbb{R}^d,u(\cdot)v(\cdot+z))}\le\mathfrak{N}\,\|g\|_{L^p(\mathbb{R}^d,u(\cdot)v(\cdot+z))}\,. \tag{4.2}$$

Fix $\xi\in\mathbb{R}^{d_1}$. For any $f\in\mathcal{C}_c^\infty(\mathbb{R}^{d_2})$, write

$$R_{(x,y)}f(z)=e^{2\pi ix\xi}f(z+y),\quad (x,y)\in\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}. \tag{4.3}$$

Observe that in this way,

$$\begin{aligned}\tilde{T}_{K_j}f(z) &= \int_{\mathbb{R}^d} K_j(x,y)R_{-(x,y)}f(z)\,dxdy \\ &= \int_{\mathbb{R}^{d_2}}\left(\int_{\mathbb{R}^{d_1}} K_j(x,y)e^{-2\pi i\xi x}\,dx\right)f(z-y)\,dy \\ &= \int_{\mathbb{R}^{d_2}} \mathbf{m}_j(\xi,\eta)\widehat{f}(\eta)e^{-2\pi iz\eta}\,d\eta.\end{aligned}$$

Fix $q < p$ and fix $E \subset \mathbb{R}^{d_1}$ a set of finite measure. For any $z \in \mathbb{R}^{d_2}$, let $A_z = \{(x,y) \in \mathbb{R}^d : \, y+z \in E\}$. Let $r > 0$ such that $\operatorname{supp} K_j \subset (-r,r)^d = Q_r$ for $j = 1,\ldots,J$.

Let $s > 0$. For any $(x,y) \in Q_s = (-s,s)^d$,

$$\left\|\sup_{1\le j\le J}\left|\tilde{T}_{K_j}f\right|\chi_E\right\|^q_{L^q(\mathbb{R}^{d_2},w)} = \int_{\mathbb{R}^{d_2}}\sup_{1\le j\le J}\left|R_{(x,y)}T_{K_j}f(z)\right|^q v(y+z)\chi_E(y+z)\,dz.$$

If we consider the weight $\omega = u\otimes 1$ on $\mathbb{R}^d$, it follows that

$$\begin{aligned}&\left\|\sup_{1\le j\le J}\left|\tilde{T}_{K_j}f\right|\chi_E\right\|^q_{L^q(\mathbb{T}^d,w)} \\ &= \frac{1}{\omega(Q_s)}\int_{Q_s}\int_{\mathbb{R}^{d_2}}\sup_{1\le j\le J}\left|R_{(x,y)}\tilde{T}_{K_j}f(z)\right|^q u(x)v(y+z)\chi_E(y+z)\,dxdy\,dz \\ &\le \frac{1}{\omega(Q_s)}\int_{\mathbb{R}^{d_2}}\left\{\int_{A_z\cap Q_s}\sup_{1\le j\le J}\left|B_{K_j}\left(R_{(\cdot)}f(z)\chi_{Q_{r+s}}\right)(x,y)\right|^q u(x)v(y+z)\,dxdy\right\}dz.\end{aligned}$$

By Kolmogorov's condition (2.1) and (4.2), the term inside curly brackets is bounded by

$$(c_{p,q}\mathfrak{N})^q\left\{\int_{Q_{r+s}}\left|R_{(x,y)}f(z)\right|^p u(x)v(y+z)\,dt\right\}^{\frac{q}{p}}\left\{\int_{A_z\cap Q_s}u(x)v(y+z)\,dt\right\}^{1-\frac{q}{p}}.$$

Then, by Hölder's inequality, it follows that

$$\begin{aligned}\left\|\sup_{1\le j\le J}\left|T_{K_j}f\right|\chi_E\right\|^q_{L^q(\mathbb{R}^{d_2},v)} &\le \frac{c^q_{p,q}\mathfrak{N}^q}{\omega(Q_s)}\left\{\int_{\mathbb{R}^{d_2}}\int_{Q_s\cap A_z}u(x)v(y+z)dt\,dz\right\}^{1-\frac{q}{p}} \\ &\quad\times\left\{\int_{\mathbb{R}^{d_2}}\int_{Q_{r+s}}\left|R_{(x,y)}f(z)\right|^p u(x)v(y+z)\,dxdy\,dz\right\}^{\frac{q}{p}} \\ &\le c^q_{p,q}\mathfrak{N}^q\left(\frac{\omega(Q_{r+s})}{\omega(Q_s)}\right)^{\frac{q}{p}} v(E)^{1-\frac{q}{p}}\,\|f\|^q_{L^p(\mathbb{R}^{d_2},w)}.\end{aligned}$$

Since $u \in A_p(\mathbb{R}^{d_1})$, $\omega \in A_p(\mathbb{R}^d)$. Then by Lemma 4.1 and Kolmogorov's condition (2.1), it follows that

$$\left\|\sup_{1\le j\le J}\left|T_{K_j}f\right|\right\|_{L^{p,\infty}(\mathbb{R}^{d_2},w)} \le c_{p,q}\mathfrak{N}\,\|f\|_{L^p(\mathbb{R}^{d_2},w)}.$$

Finally, considering $c_p = \inf_{q<p} c_{p,q}$, the result easily follows by Fatou's Lemma and the density of $\mathcal{C}^\infty_c(\mathbb{R}^{d_2})$ in $L^p(\mathbb{R}^{d_2},w)$. □

5. Consequences and applications

5.1. **Hörmander-Mihlin type multipliers.** The first application involves multipliers satisfying a Hörmander-Mihlin type condition.

Definition 5.1 (see [14]). Let $\mathbf{m} \in L^\infty(\mathbb{R}^d) \cap \mathcal{C}^d\left(\mathbb{R}^d \setminus \{0\}\right)$, $l \in \mathbb{N}$ and $s \geq 1$. We say $\mathbf{m} \in M(s,l)$ if it satisfies

$$(5.1)\quad c_{\mathbf{m},s,l} = \sup_{\substack{|\alpha|\leq l \\ \alpha=(\alpha_1,\ldots,\alpha_d)}} \sup_{r>0} \left(r^{s|\alpha|-d} \int_{r<|x|<2r} \left| \frac{\partial^{|\alpha|}\mathbf{m}}{\partial x_1^{\alpha_1}\ldots\partial x_d^{\alpha_d}}(x)\right|^s dx \right)^{1/s} < \infty.$$

In 1979, D. Kurtz and R. Wheeden proved the following result.

Theorem 5.2 ([14, Theorem 1]). *Let $1 < s \leq 2$, $\frac{d}{s} < l \leq d$ and $\mathbf{m} \in M(s,l)$. If*

(1) $d/l < p < \infty$ *and* $w \in A_{pl/d}(\mathbb{R}^d)$ *or*
(2) $1 < p < (d/l)'$ *and* $w^{-1/(p-1)} \in A_{p'l/d}(\mathbb{R}^d)$,

then $\mathbf{m} \in M_{p,w}(\mathbb{R}^d)$. When $l < d$ it can be taken $p = d/l$ in (1) or $p = (d/l)'$ in (2).

Moreover, if $w^{d/l} \in A_1(\mathbb{R}^d)$, then $\mathbf{m} \in M_{1,w}^{(\mathfrak{w})}(\mathbb{R}^d)$.

Corollary 5.3. *Under the hypothesis of Theorem 5.2 and assuming that $\mathbf{m}$ is a normalized function, the following holds: If*

(1) $d/l < p < \infty$ *and* $w \in A_{pl/d}(\mathbb{T}^d)$ *or*
(2) $1 < p < (d/l)'$ *and* $w^{-1/(p-1)} \in A_{p'l/d}(\mathbb{T}^d)$,

then $\mathbf{m}|_{\mathbb{Z}^d} \in M_{p,w}(\mathbb{T}^d)$. When $l < d$ it can be taken $p = d/l$ in (1)or $p = (d/l)'$ in (2).

Moreover, if $w^{d/l} \in A_1(\mathbb{T}^d)$, then $\mathbf{m}|_{\mathbb{Z}^d} \in M_{1,w}^{(\mathfrak{w})}(\mathbb{T}^d)$.

Proof. The result follows by applying Theorems 3.10 and 3.11 to $\mathbf{m}$. □

5.2. **Singular integral operators.** Our second example involves the classical theory of Calderón-Zygmund singular integrals.

Definition 5.4 ([10, Definition II.5.17]). A function $K \in L^1_{\text{loc}}(\mathbb{R}^d \setminus \{0\})$ is said to be a regular kernel if $\widehat{K} \in L^\infty(\mathbb{R}^d)$ and it satisfies

$$(5.2)\qquad |K(x)| \leq C\,|x|^{-d}, \quad \forall x \in \mathbb{R}^d \setminus \{0\},$$

$$(5.3)\qquad |K(x-y) - K(x)| \leq C\,|y|\,|x|^{-d-1}, \quad |x| > 2\,|y|\,.$$

Corollary 5.5. *Let K be a regular kernel and consider for any $0 < r < s < \infty$, $K_{r,s} = K\chi_{r<|x|<s}$ and $\mathbf{m}_{r,s} = \widehat{K_{r,s}}$. If $1 < p < \infty$ and $w \in A_p(\mathbb{T}^d)$ there exists a constant c such that*

$$\left\|\{\mathbf{m}_{r,s}|_{\mathbb{Z}^d}\}_{r<s}\right\|_{M_{p,w}(\mathbb{T}^d)} \leq c.$$

If $w \in A_1(\mathbb{T}^d)$, then there exists a constant c such that

$$\left\|\{\mathbf{m}_{r,s}|_{\mathbb{Z}^d}\}_{r<s}\right\|_{M_{1,w}^{(\mathfrak{w})}(\mathbb{T}^d)} \leq c.$$

Proof. It is easy to see that $T^\sharp_{\{\mathbf{m}_{r,s}|_{\mathbb{Z}^d}\}_{r<s}} f(x) = T^\sharp_{\{\mathbf{m}_{r,s}|_{\mathbb{Z}^d}\}_{r,s\in\mathbb{Q}_+,\ r<s}} f(x)$ for every $f \in P(\mathbb{T}^d)$. Then, the result follows by the known corresponding result for functions in $\mathbb{R}^d$ (see [10, Theorem IV.3.6 and V.4.11]) by applying Theorem 3.2 and its corresponding strong version. □

5.3. **Bochner-Riesz partial sums.** Our third application involves Bochner-Riesz partial sums. Let us recall that the Bochner-Riesz operators in $\mathbb{R}^d$ are defined as

$$(B^r_\lambda f)\widehat{\ }(\xi) = \mathbf{m}_r(\xi)\widehat{f}(\xi), \quad \text{where} \quad \mathbf{m}_r(x) = \left(1 - \frac{|x|^2}{r^2}\right)^{\lambda}_{+},$$

$t_+ = \max(t, 0)$, and the associated maximal operator is defined by

$$B^\sharp_\lambda f(x) = \sup_{r>0} |B^r_\lambda f(x)|$$

for $\lambda > 0$. It is known that for $\lambda > \frac{d-1}{2}$, $B^\sharp_\lambda f$ is pointwise majorized by the Hardy-Littlewood maximal operator; then it inherits its boundedness properties. For the critical index the following is known (S. Shi and Q. Sun [17, Theorem 1] and A. Vargas [16, Theorem 1]).

Theorem 5.6. *Let $\lambda = \frac{d-1}{2}$. If $1 < p < \infty$ and $w \in A_p(\mathbb{R}^d)$, then*

$$\left\| B^\sharp_\lambda f \right\|_{L^p(\mathbb{R}^d,w)} \le C \left\| f \right\|_{L^p(\mathbb{R}^d,w)},$$

and if $w \in A_1(\mathbb{R}^d)$, there is a constant C such that for each $r > 0$,

$$\left\| B^r_\lambda f \right\|_{L^{1,\infty}(\mathbb{R}^d,w)} \le C \left\| f \right\|_{L^1(\mathbb{R}^d,w)},$$

where the constants depend only on the A_p-constant of w and the dimension d.

Let us observe that for $\lambda = (d-1)/2$, the kernel of the operator B^r_λ, say K, satisfies the size condition $|K(x)| \lesssim |x|^{-d}$, but it does not satisfy any Hörmander type condition such as (5.3) above. Then we can't apply the result obtained in the previous example.

In the periodic case, for $r > 0$, the Bochner-Riesz partial sum of order $\lambda > 0$ is defined for every $f \in P(\mathbb{T}^d)$ by

$$S^r_\lambda f(\theta) = \sum_{|n| \le r} \left(1 - \frac{|n|^2}{r^2}\right)^{\lambda}_{+} \widehat{f}(n) e^{2\pi i n\theta},$$

and we denote by $S^\sharp_\lambda$ the associated maximal operator. Observe that since the function $(1 - |x|^2)^\lambda_+$ is continuous, for every $f \in P(\mathbb{T}^d)$,

$$S^\sharp_\lambda f(x) = \sup_{r \in \mathbb{Q}_+} |S^r_\lambda f(x)|.$$

Then, as a consequence of our results, the following counterpart to Theorem 5.6 is obtained.

Corollary 5.7. *Let $\lambda \ge \frac{d-1}{2}$. If $1 < p < \infty$ and $w \in A_p(\mathbb{T}^d)$, then there exists $C > 0$ such that*

$$\left\| S^\sharp_\lambda f \right\|_{L^p(\mathbb{T}^d,w)} \le C \left\| f \right\|_{L^p(\mathbb{T}^d,w)}.$$

If $w \in A_1(\mathbb{T}^d)$ and $\lambda = \frac{d-1}{2}$ there exists $C > 0$ such that for any $r > 0$,

$$\left\| S^r_\lambda f \right\|_{L^{1,\infty}(\mathbb{T}^d,w)} \le C \left\| f \right\|_{L^1(\mathbb{T}^d,w)},$$

and if $\lambda > \frac{d-1}{2}$, there exists $C > 0$ such that

$$\left\| S^\sharp_\lambda f \right\|_{L^{1,\infty}(\mathbb{T}^d,w)} \le C \left\| f \right\|_{L^1(\mathbb{T}^d,w)}.$$

By standard arguments the theorem implies:

Corollary 5.8. *Let $\lambda \geq \frac{d-1}{2}$, $1 \leq p < \infty$ and $w \in A_p(\mathbb{T}^d)$. For any $f \in L^p(\mathbb{T}^d, w)$,*

$$\lim_{r \to 0^+} S_\lambda^r f = f,$$

where the convergence is considered in measure for $p = 1$ and $\lambda = \frac{d-1}{2}$ and pointwise almost everywhere in the other cases.

For λ below the critical index the study of the boundedness properties of the Bochner-Riesz operators constitutes an active area of research (see [9] for instance and the references therein). In this setting, the following is a direct consequence of [9, Theorem 5.1] (taking $u_0 = 1$ with the notation therein).

Theorem 5.9. *Let $0 < \lambda < (d-1)/2$. If $w(x) = v(x)^{2\lambda/d-1}$ with $v \in A_2(\mathbb{R}^d)$, then, for any $r > 0$, B_λ^r is bounded in $L^2(\mathbb{R}^d, w)$ uniformly on r.*

Theorem 3.11 leads to obtain the following periodic counterpart result.

Corollary 5.10. *Let $0 < \lambda < (d-1)/2$. If $w(x) = v(x)^{2\lambda/d-1}$ with $v \in A_2(\mathbb{T}^d)$, then, for any $r > 0$, S_λ^r is bounded in $L^2(\mathbb{T}^d, w)$ uniformly on r.*

5.4. **Extension of multipliers from $L^p(\mathbb{T})$ to $L^p(\mathbb{R}, w)$.** In this section we are going to show how Theorem 3.13 allows us to see the strong ties between $M_p(\mathbb{T})$ and a subspace of $M_{p,w}(\mathbb{R})$ for a subclass of weights in $A_p(\mathbb{R})$ (see Corollary 5.12 below).

Following M. Jodeit's ideas in [12], E. Berkson, M. Paluszyński and G. Weiss in [5] gave a way to extend multipliers from $L^p(\mathbb{T})$ to $L^p(\mathbb{R}, w)$ with $w \in A_p(\mathbb{R})$ satisfying that there exists a constant $\rho \geq 1$ such that for each $k \in \mathbb{Z}$

$$\rho^{-1} w(k) \leq w(x) \leq \rho w(k), \quad \text{for all} x \in [k, k+1). \tag{5.4}$$

These weights are said to be in W_p.

In this framework, E. Berkson, M. Paluszyński and G. Weiss proved the following result.

Theorem 5.11 ([5, Theorem 4.21])**.** *Let $1 < p < \infty$, $w \in W_p$, $\Psi \in M_{p,w}(\mathbb{R})$ and the support of Ψ is contained in $[-1/2, 1/2]$. Then, if $\{\phi_n\}_n \in M_p(\mathbb{T})$, we have that*

$$\mathcal{W}_{\phi,\Psi}(t) = \sum_{m \in \mathbb{Z}} \phi(m) \Psi(t-m) \in M_{p,w}(\mathbb{R})$$

and

$$\left\| \{\mathcal{W}_{\phi_n,\Psi}\}_n \right\|_{M_{p,w}(\mathbb{R})} \leq K_{p,w} \left\| \Psi \right\|_{M_{p,w}(\mathbb{R})} \left\| \{\phi_n\}_n \right\|_{M_p(\mathbb{T})}.$$

Since $\mathcal{W}_{\phi_n,\Psi}|_{\mathbb{Z}} = \Psi(0)\phi_n|_{\mathbb{Z}}$, a direct consequence of Theorem 3.13 with $u = 1$ and $v = w$ is that the converse of Theorem 5.11 also holds:

Corollary 5.12. *Let $\{\phi_n\}_n \subset \ell^\infty(\mathbb{Z})$. Then, under the hypothesis of Theorem 5.11, we have that if $\Psi(0) \neq 0$,*

$$\left\| \{\mathcal{W}_{\phi_n,\Psi}\}_n \right\|_{M_{p,w}(\mathbb{R})} < +\infty \quad \text{if and only if} \quad \left\| \{\phi_n\}_n \right\|_{M_p(\mathbb{T})} < +\infty.$$

Moreover,

(5.5)
$$\frac{C_{p,w}\left\|\{\phi_n\}_n\right\|_{M_p(\mathbb{T})}}{|\Psi(0)|} \leq \left\|\{\mathcal{W}_{\phi_n,\Psi}\}_n\right\|_{M_{p,w}(\mathbb{R})} \leq K_{p,w}\left\|\Psi\right\|_{M_{p,w}(\mathbb{R})}\left\|\{\phi_n\}_n\right\|_{M_p(\mathbb{T})}.$$

Observation 5.13. In the particular case of a single multiplier, inequality (5.5) yields that, for any $w \in W_p$, the map $\phi \mapsto \mathcal{W}_{\phi,\Psi}$ induces an isomorphism between $M_p(\mathbb{T})$ and a subspace of $M_{p,w}(\mathbb{R})$. This result is a one dimensional weighted generalization of the unweighted result in [12, p. 225] for Ψ the characteristic function of the interval $[-1/2, 1/2)$.

Acknowledgements

We would like to thank the referee for some valuable comments which have improved the final version of this work.

References

[1] K. Andersen and P. Mohanty, *Restriction and extension of Fourier multipliers between weighted L^p spaces on $\mathbb{R}^n$ and $\mathbb{T}^n$*, Proc. Amer. Math. Soc. **137** (2009), no. 5, 1689–1697. MR2470827 (2009m:42012)

[2] N. Asmar, E. Berkson, and J. Bourgain, *Restrictions from $\mathbb{R}^n$ to $\mathbb{Z}^n$ of weak type $(1,1)$ multipliers*, Studia Math. **108** (1994), no. 3, 291–299. MR1259281 (95b:42015)

[3] N. Asmar, E. Berkson, and T. A. Gillespie, *Maximal estimates on measure spaces for weak-type multipliers*, J. Geom. Anal. **5** (1995), no. 2, 167–179. MR1341028 (96m:43006)

[4] E. Berkson and T. A. Gillespie, *On restrictions of multipliers in weighted settings*, Indiana Univ. Math. J. **52** (2003), no. 4, 927–961. MR2001939 (2005i:43006)

[5] E. Berkson, M. Paluszyński, and G. Weiss, *Transference couples and weighted maximal estimates*, Harmonic analysis and partial differential equations (Chicago, IL, 1996), 1999, pp. 49–67. MR1743855 (2001d:47045)

[6] M. J. Carro and S. Rodríguez, *New results on restriction of Fourier multipliers*, Mathematische Zeitschrift **265** (2010), no. 2, 417–435. MR2609318 (2011c:42029)

[7] ———, *Transference results on weighted Lebesgue spaces*, Proc. Roy. Soc. Edinburgh Sect. A **138** (2008), no. 2, 239–263. MR2406690 (2009d:43009)

[8] K. De Leeuw, *On L_p multipliers*, Ann. of Math. (2) **81** (1965), 364–379. MR0174937 (30:5127)

[9] J. Duoandikoetxea, A. Moyua, O. Oruetxebarria, and E. Seijo, *Radial A_p weights with applications to the disc multiplier and the Bochner-Riesz operators*, Indiana Univ. Math. J. **57** (2008), no. 3, 1261–1281. MR2429092 (2009d:42034)

[10] J. García-Cuerva and J. L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland Mathematics Studies, vol. 116, North-Holland Publishing Co., Amsterdam, 1985. Notas de Matemática [Mathematical Notes], 104. MR807149 (87d:42023)

[11] R. Hunt, B. Muckenhoupt, and R. Wheeden, *Weighted norm inequalities for the conjugate function and Hilbert transform*, Trans. Amer. Math. Soc. **176** (1973), 227–251. MR0312139 (47:701)

[12] M. Jodeit, Jr., *Restrictions and extensions of Fourier multipliers*, Studia Math. **34** (1970), 215–226. MR0262771 (41:7376)

[13] C. E. Kenig and P. A. Tomas, *Maximal operators defined by Fourier multipliers*, Studia Math. **68** (1980), no. 1, 79–83. MR583403 (82c:42016)

[14] D. S. Kurtz and R. L. Wheeden, *Results on weighted norm inequalities for multipliers*, Trans. Amer. Math. Soc. **255** (1979), 343–362. MR542885 (81j:42021)

[15] J. A. Raposo, *Weak type $(1,1)$ multipliers on LCA groups*, Studia Math. **122** (1997), no. 2, 123–130. MR1432163 (98b:43011)

[16] X. L. Shi and Q. Y. Sun, *Weighted norm inequalities for Bochner-Riesz operators and singular integral operators*, Proc. Amer. Math. Soc. **116** (1992), no. 3, 665–673. MR1136237 (93a:42009)

[17] A. M. Vargas, *Weighted weak type* $(1,1)$ *bounds for rough operators*, J. London Math. Soc. (2) **54** (1996), no. 2, 297–310. MR1405057 (98g:42027)

Departament de Matemàtica Aplicada i Anàlisi, Universitat de Barcelona, 08007 Barcelona, Spain
E-mail address: carro@ub.edu
URL: http://garf.ub.edu

Department of Mathematics, Uppsala University, Upssala, SE 75106, Sweden
E-mail address: salvador@math.uu.se
URL: www.math.uu.se/~salvador