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# INTERPOLATING FUNCTIONS OF MINIMAL NORM, STAR-INVARIANT SUBSPACES, AND KERNELS OF TOEPLITZ OPERATORS

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ABSTRACT. It is proved that for each inner function $\theta$ there exists an interpolating sequence $\{z_n\}$ in the disk such that $\sup_n |\theta(z_n)| < 1$, but every function $g$ in $H^\infty$ with $g(z_n) = \theta(z_n)$ $(n = 1, 2, \dots)$ satisfies $\|g\|_\infty \geq 1$. Some results are obtained concerning interpolation in the star-invariant subspace $H^2 \ominus \theta H^2$. This paper also contains a "geometric" result connected with kernels of Toeplitz operators.

Let $H^p$ denote the classical Hardy space of holomorphic functions in the open unit disk $\mathbb{D}$. The norm in $H^p$ will be denoted by $\|\cdot\|_p$. If $\{z_k\}$ is a sequence of points in $\mathbb{D}$ satisfying $\sum_k(1-|z_k|) < +\infty$, let $B = B_{\{z_k\}}$ stand for the *Blaschke product* whose zero sequence is $\{z_k\}$, i.e.,

$$B(z) = \prod_{k=1}^{\infty} \frac{\overline{z}_k}{|z_k|} \frac{z_k - z}{1 - \overline{z}_k z},$$

where $\overline{z}_k/|z_k| \overset{\text{def}}{=} -1$ for $z_k = 0$. Set

$$B_j(z) = \prod_{k\,:\,k\neq j} \frac{\overline{z}_k}{|z_k|} \frac{z_k - z}{1 - \overline{z}_k z}.$$

Recall that $\{z_k\}$ is called an *interpolating sequence* (i.s.) iff $B = B_{\{z_k\}}$ satisfies $\delta(B) \overset{\text{def}}{=} \inf_j |B_j(z_j)| > 0$. In this case $B$ itself is called an *interpolating Blaschke product* (i.B.p.). A famous theorem of Carleson (see [Gar] or [K]) says that $\{\{f(z_k)\} : f \in H^\infty\} = l^\infty$ if and only if $\{z_k\}$ is an i.s.

Recall that a function $\theta$ in $H^\infty$ is called *inner* iff $|\theta(\zeta)| = \lim_{r\to 1-0} |\theta(r\zeta)| = 1$ for $m$-almost all $\zeta \in \mathbb{T}$ (here $\mathbb{T} = \partial\mathbb{D}$ and $m$ is the normalized Lebesgue measure on $\mathbb{T}$). Theorem 1 of this paper treats inner functions as interpolating functions of minimal norm. Roughly speaking, it says that, given an inner function $\theta$, one can find an i.s. $\{z_n\}$ in $\mathbb{D}$ such that the values $|\theta(z_n)|$ are bounded away from 1, but nevertheless $\theta$ is an $H^\infty$ function of minimal norm that interpolates $\theta(z_n)$ in $z_n$.

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**Theorem 1.** *There are absolute constants* $\beta$ *and* $\delta_0$ $(0 < \beta < 1,\ 0 < \delta_0 < 1)$ *with the following property: For each inner function* $\theta$ *there exists an i.B.p.* $B = B_{\{z_k\}}$ *such that*

(i) $\sup_k |\theta(z_k)| \le \beta$;
(ii) $\delta(B) \ge \delta_0$;
(iii) $\inf_{h\in H^\infty} \|\theta + Bh\|_\infty = 1$.

*Proof*. Fix a number $\alpha$, $0 < \alpha < 1$, and let $\gamma$ be another small positive number that will be chosen later on. Once $\theta$, $\alpha$, and $\gamma$ are given, a theorem due to Marshall (see [Gar, Chapter viii, Theorem 4.1]) says there exists an i.B.p. $B = B_{\{z_k\}}$ such that

(a) $|B(z)| \le \gamma$ if $|\theta(z)| \le \alpha$ $(z \in \mathbb{D})$;
(b) $\sup_k |\theta(z_k)| \le \beta < 1$;
(c) $\delta(B) \ge \delta_0 > 0$,

where $\beta$ and $\delta_0$ are some constants depending only on $\alpha$ and $\gamma$.

Thus $B$ satisfies (i) and (ii), so it suffices to show that for some choice of $\alpha$ and $\gamma$ we have

(iii) $\operatorname{dist}(\theta, BH^\infty) = \inf_{h\in H^\infty} \|\theta + Bh\|_\infty = 1$.

(From now on $\operatorname{dist}(\cdot, \cdot)$ denotes the distance measured in $L^\infty$.)

We now make the following

*Claim*. For each $\alpha$, $0 < \alpha < 1$, there is a positive number $\gamma = \gamma(\alpha)$ such that the property (a) above implies

$$\operatorname{dist}(\overline{\theta}^2 B, H^\infty) < 1. \tag{1}$$

Assuming that the Claim is already established, we complete the proof. Let $\alpha$ and $\gamma = \gamma(\alpha)$ be such that (a) implies (1), and let $B$ be the i.B.p. given by Marshall's construction when applied to $\theta$, $\alpha$, and $\gamma$. Suppose (iii) fails, then we have $\operatorname{dist}(\theta\overline{B}, H^\infty) < 1$ and $\operatorname{dist}(\overline{\theta}B, \theta H^\infty) < 1$. (The former inequality is contrary to (iii) and the latter coincides with (1).) However, such a situation is incompatible with the following well-known fact (e.g., see [Do]): if $u$ is a unimodular function on $\mathbb{T}$ such that $\operatorname{dist}(u, H^\infty) < 1$ and $\operatorname{dist}(\overline{u}, H^\infty) < 1$, then every function $h$ in $H^\infty$ satisfying $\|u - h\|_\infty < 1$ is outer (in fact, it is even invertible in $H^\infty$). In our case set $u = \overline{\theta}B$. Both distances are $< 1$, but there is a function $h$ in $\theta H^\infty$ (hence not outer) with $\|u - h\|_\infty < 1$. This is impossible, and the contradiction proves (iii).

Since $\alpha$ was arbitrary throughout, one can take $\alpha = \frac{1}{2}$. This done, all the constants in the above construction become numerical. The proof is complete, except for the Claim.

To prove the Claim, fix $\alpha \in (0, 1)$ and consider the system $\Gamma_\alpha$ of the so-called *Carleson curves* associated with $\theta$ and $\alpha$. More precisely, let $\Gamma_\alpha$ be a countable union of simple closed rectifiable curves in $\operatorname{Clos}\mathbb{D}$ with the following properties.

1. The curves in the system $\Gamma_\alpha$ have pairwise disjoint interiors.

2. For $z \in \Gamma_\alpha \cap \mathbb{D}$ we have $\eta(\alpha) \le |\theta(z)| \le \alpha$, where $\eta(\alpha)$ is some positive number depending only on $\alpha$.

3. Arc length $|dz|$ on $\Gamma_\alpha \cap \mathbb{D}$ is a *Carleson measure*, i.e., $H^1 \subset L^1(\Gamma_\alpha, |dz|)$ and the norm $N(\alpha)$ of the arising embedding operator depends only on $\alpha$.

4. For any $\Phi$, $\Phi \in H^1$, we have

$$\int_{\mathbb{T}} \frac{\Phi(z)}{\theta(z)}\, dz = \int_{\Gamma_\alpha} \frac{\Phi(z)}{\theta(z)}\, dz,$$

provided the curves in $\Gamma_\alpha$ are oriented in the appropriate way. A similar equality holds if $\theta$ is replaced by any positive power of itself.

Carleson's construction is described in [Gar, Chapter viii].

Now let $B$ satisfy (a). Then $\sup_{\Gamma_\alpha \cap \mathbb{D}} |B| \le \gamma$ and a standard duality argument yields

$$\begin{aligned} \operatorname{dist}(\overline{\theta}^2 B, H^\infty) &= \sup_{k \in H^1, \|k\|_1 = 1} \left| \int \overline{\theta}^2 B z k\, dm \right| \\ &= \sup_k \left| \frac{1}{2\pi} \int \frac{Bk}{\theta^2}\, dz \right| = \sup_k \left| \frac{1}{2\pi} \int_{\Gamma_\alpha} \frac{Bk}{\theta^2}\, dz \right| \\ &\le \frac{1}{2\pi} \frac{\gamma}{\eta(\alpha)^2} \sup_k \int_{\Gamma_\alpha} |k||dz| \le \frac{1}{2\pi} \frac{\gamma N(\alpha)}{\eta(\alpha)^2}, \end{aligned}$$

where we have used the properties of $\Gamma_\alpha$ listed above. Setting $\gamma = \eta(\alpha)^2 / N(\alpha)$ we get $\operatorname{dist}(\overline{\theta}^2 B, H^\infty) < 1$ as desired. □

*Remark.* This theorem should be compared to the following result of Stray and Øyma [SØ].

**Theorem A.** *If $h$ is an extreme point in the unit ball of $H^\infty$ (i.e., $\|h\|_\infty = 1$ and $\int \log(1 - |h|)\, dm = -\infty$), there exists an interpolating sequence $\{z_n\}$ tending to one point such that $h$ is the unique $H^\infty$ function of norm $\le 1$ that interpolates $h(z_n)$ in $z_n$.*

Of course, each inner function is an extreme point, and so Theorem A deals with a larger class of functions than Theorem 1. However, the construction in [SØ] gives $\lim_{n\to\infty} |h(z_n)| = 1$. Hence the equality $\operatorname{dist}(h, B_{\{z_n\}} H^\infty) = 1$ is immediate, and it is the uniqueness of $h$ that really matters in Theorem A.

**Theorem 2.** *There is an absolute constant $N$, $N \in \mathbb{N}$, with the following property: for each inner function $\theta$ there exists an i.B.p. $B$ such that*

$$\operatorname{dist}(B, \theta H^\infty) < 1, \tag{2}$$

$$\operatorname{dist}(\theta^N, B H^\infty) < 1. \tag{3}$$

*Proof.* Let $\alpha = \frac{1}{2}$, $\gamma$ be small enough, and $B = B_{\{z_k\}}$ be the i.B.p. possessing the properties (a), (b), (c) given by Marshall's theorem (see the proof of Theorem 1). It has been actually proved (see the proof of the Claim above) that for a suitable $\gamma$ the inequality (2) holds.

Clearly, $\operatorname{dist}(\theta^N, BH^\infty) \overset{\text{def}}{=} d(N)$ equals $\inf\{\|f\|_\infty : f \in H^\infty,\ f(z_n) = \theta^N(z_n)\ (n = 1, 2, \dots)\}$. Since $\sup_n |\theta(z_n)| \le \beta$ and $\delta(B) \ge \delta_0$ (conditions (b) and (c)), Carleson's interpolation theorem [Gar, Chapter vii] yields

$$d(N) \le \frac{c}{\delta_0} \left(1 + \log \frac{1}{\delta_0}\right) \beta^N,$$

where the constant $c$ is numerical.

Choosing $N$ to be large enough, so that

$$\frac{c}{\delta_0}\left(1+\log\frac{1}{\delta_0}\right)\beta^N < 1,$$

one arrives at (3). Since the constants $\alpha$, $\beta$, $\gamma$, $\delta_0$ were numerical, the same is true of $N$. $\square$

Here is an interesting open question, posed by Nikolskiĭ [N]. Given an arbitrary inner function $\theta$, does there necessarily exist an i.B.p. $B$ such that (2) and (3) hold with $N = 1$? An affirmative answer (which seems quite probable in view of Theorem 2) could be restated to say that for any $\theta$ the corresponding *star-invariant subspace* $K_\theta^2 \stackrel{\text{def}}{=} H^2 \ominus \theta H^2$ possesses an unconditional basis of reproducing kernels.

Now let $\{z_n\}$ be an i.s.; by $T = T_{\{z_n\}}$ we denote the operator given by

$$Tf = \{(1-|z_n|)^{1/2} f(z_n)\}_{n=1}^\infty.$$

It is well known [Gar, K] that $TH^2 = l^2$.

Once again, let $\theta$ be an inner function and let $K_\theta^2 = H^2 \ominus \theta H^2$ be the star-invariant (i.e., invariant under the backward shift operator) subspace generated by $\theta$. Consider the following interpolation problem: Describe the conditions on $\theta$ and i.s. $\{z_n\}$ under which

$$T_{\{z_n\}} K_\theta^2 = l^2. \tag{4}$$

A standard duality argument shows that (4) holds iff $\sup_n |\theta(z_n)| < 1$ and the family $\{k_n\}$ of reproducing kernels, $k_n(\zeta) \stackrel{\text{def}}{=} (1-\overline{\theta(z_n)}\theta(\zeta))(1-\overline{z}_n\zeta)^{-1}$, forms an unconditional basis in its closed linear hull. The latter statement is known [HNP] to be equivalent to

$$\operatorname{dist}(\theta, BH^\infty) < 1, \qquad B = B_{\{z_n\}}. \tag{5}$$

Thus (4) and (5) are equivalent. However, (5) is still rather implicit, and one would think of more explicit conditions implying (4). Theorems 3 and 4 below contain simple sufficient conditions. The proofs will be based on the following results due to Øyma [Ø].

**Theorem B.** *Let $\{z_k\}$ be an i.s. in $\mathbb{D}$ and assume $w_n \to 0$. Then there exists a unique $H^\infty$ function $f$ of minimal norm such that $f(z_n) = w_n$ $(n = 1, 2, \ldots)$. This function is a constant times an inner function and has analytic continuation across $\mathbb{T}\setminus \operatorname{Clos}\{z_n\}$.*

**Theorem C.** *Assume that $\{z_n\}$ is an i.s. and $z_n \to 1$ nontangentially. Then the unique function $f$ of Theorem* B *is a constant times a Blaschke product.*

For an arbitrary inner function $\theta$ let $S_\theta$ denote its *singular support*, i.e., the smallest of all closed subsets $E$ of $\mathbb{T}$ such that $\theta$ is analytic across $\mathbb{T}\setminus E$.

**Theorem 3.** *Suppose $\theta$ is inner, $\{z_n\}$ is an i.s., $\lim_{n\to\infty}\theta(z_n)=0$, and $\operatorname{Clos}\{z_n\} \cap \mathbb{T} \subsetneqq S_\theta$. Then we have (4).*

*Proof*. By Theorem B there exists a unique function $f$ such that $f \in \theta + BH^\infty$, $\|f\|_\infty = \operatorname{dist}(\theta, BH^\infty)$. Moreover, $f$ is inner (up to a constant factor) and analytic across $\mathbb{T}\setminus \operatorname{Clos}\{z_n\}$. Since $\operatorname{Clos}\{z_n\} \cap \mathbb{T}$ is strictly contained in $S_\theta$, one infers that $\theta$ is not analytic across $\mathbb{T}\setminus \mathrm{Clos}\{z_n\}$. Hence $f \neq \theta$. In other words, $\theta$ is not the unique function of minimal norm that interpolates the values $\theta(z_n)$ in $z_n$. Thus $\|f\|_\infty < \|\theta\|_\infty = 1$, and one arrives at (5), which is the same as (4). □

**Theorem 4.** *Let* $\theta(z) = \exp(-\int(\zeta + z)(\zeta - z)^{-1}\, d\mu(\zeta))$, *where* $\mu$ *is a positive Borel singular measure on* $\mathbb{T}$. *Then* $\mu$*-almost all points* $\zeta$ *of* $\mathbb{T}$ *enjoy the following property: if* $\{z_n\}$ *is an i.s. and* $z_n \to \zeta$ *nontangentially, then* (4) *holds.*

*Proof.* It is well known [Gar, Chapter ii] that the nontangential limit of $\theta$ at $\zeta$ equals 0 for $\mu$-almost all $\zeta \in \mathbb{T}$. Let $\zeta_0$ be such a point, and let $\{z_n\}$ be an i.s. tending to $\zeta_0$ nontangentially. From Theorem C one sees that the unique extremal function in $\theta + B_{\{z_n\}}H^\infty$ is a constant times a Blaschke product, whereas $\theta$ is singular. Therefore $\theta$ is not extremal. Hence (5) holds, and so does (4). □

The next proposition is, in fact, a mere combination of Theorems 1 and 2.

**Theorem 5.** *There exist absolute constants* $\delta_0$ $(0 < \delta_0 < 1)$ *and* $N$ $(N \in \mathbb{N})$ *with the following property: For each inner function* $\theta$ *one can find an i.s.* $\{z_n\}$ *such that* $\delta(B_{\{z_n\}}) \geq \delta_0$, $T_{\{z_n\}}K^2_\theta \neq l^2$, *but* $T_{\{z_n\}}K^2_{\theta^N} = l^2$.

*Proof.* Let $\{z_n\}$ be constructed as in the proof of Theorem 1, and let $N$ be chosen as in the proof of Theorem 2. □

We conclude with a "geometric" result pertaining, in a way, to the subject. Let $\varphi \in L^\infty$, and let $T_\varphi$ denote the *Toeplitz operator* with symbol $\varphi$:

$$(T_\varphi f)(z) \stackrel{\text{def}}{=} \int \frac{\varphi(\zeta)f(\zeta)}{1 - z\bar{\zeta}}\, dm(\zeta) \qquad (z \in \mathbb{D}).$$

We treat $T_\varphi$ as a bounded operator going from $H^1$ to $H^p$, $0 < p < 1$, and consider its kernel $K(\varphi) \stackrel{\text{def}}{=} \operatorname{Ker} T_\varphi$, which is assumed to be nontrivial from now on. Theorem 6 below provides an explicit characterization of the extreme points in the unit ball of $K(\varphi)$. Recall that $K(\varphi)$ is endowed with $L^1$-norm, and so *ball*$(K(\varphi)) = \{f \in H^1 : \|f\|_1 \leq 1,\ T_\varphi f = 0\}$. Once the symbol $\varphi$ is fixed, we set $\tilde{f} \stackrel{\text{def}}{=} \overline{z\varphi f}$ and note that $K(\varphi) = \{f \in H^1 : \tilde{f} \in H^1\}$. Finally, if $f$ is an $H^1$ function, $I_f$ stands for the inner factor in the canonical factorization of $f$.

**Theorem 6.** *Let* $f \in K(\varphi)$, $\|f\|_1 = 1$. *The following statements are equivalent.*

(i) $f$ *is an extreme point of ball*$(K(\varphi))$.

(ii) *The inner functions* $I_f$ *and* $I_{\tilde{f}}$ *are relatively prime* (*i.e., they have no common inner divisors*).

*Proof.* We need the following lemma, which can be found in [Gam, Chapter V, §9].

**Lemma.** *Suppose* $X$ *is a closed subspace of* $H^1$, *and let* $f \in X$, $\|f\|_1 = 1$. *The function* $f$ *is an extreme point in the unit ball of* $X$ *iff any function* $h$ *in* $L^\infty_\mathbb{R}$ *for which* $fh \in X$ *is constant almost everywhere.* (*We write* $h \in L^\infty_\mathbb{R}$ *to mean that* $h$ *is a real-valued function in* $L^\infty$.)

In fact, Gamelin considers the case $X = H^1$, but the same proof works in case $X$ is an arbitrary subspace of $H^1$. In particular, one can take $X = K(\varphi)$.

We prove now that (i) implies (ii). Let $f$ be an extreme point, and let the inner function $u$ be the greatest common divisor of $I_f$ and $I_{\tilde f}$. We have then $\widetilde{fu} = \overline{z\varphi}\overline{fu} = \tilde f\overline{u}$, and the latter function belongs to $H^1$ because $u$ divides $I_{\tilde f}$. Hence $fu \in K(\varphi)$. Similarly one sees that $f\overline{u} \in K(\varphi)$. It follows that the product $f \cdot \operatorname{Re} u = \frac{1}{2} f(u + \overline{u})$ is also in $K(\varphi)$. Since $f$ is an extreme point, the above lemma yields $\operatorname{Re} u = \text{const}$, whence $u = \text{const}$ and so the functions $I_f$ and $I_{\tilde f}$ are relatively prime.

Conversely, let (ii) hold. Suppose we have $fh \in K(\varphi)$ for a function $h$ in $L^\infty_{\mathbb{R}}$. Setting $g \overset{\text{def}}{=} fh$ we get $\tilde g = \overline{z\varphi}\overline{f}h = \tilde f h \in H^1$. Thus $h = g/f = \tilde g/\tilde f$, whence $\tilde f g = f \tilde g$ and a similar equality holds for the inner factors: $I_{\tilde f} I_g = I_f I_{\tilde g}$. Recalling that $I_f$ and $I_{\tilde f}$ are relatively prime, we conclude that $I_f$ divides $I_g$; that is, $I_g/I_f \in H^\infty$. Therefore the bounded function $h$, $h = g/f$, can be also written in the form $h = g_1/f_1$, where both $f_1$ and $g_1$ are in $H^1$, and besides $f_1$ is outer. This means $h \in H^\infty$. Since $h$ belongs also to $L^\infty_{\mathbb{R}}$, it cannot help being constant. Thus, assuming (ii) holds, we have shown that the inclusion $fh \in K(\varphi)$ implies $h = \text{const}$. The statement (i) now follows by Lemma. □

**Corollary 1.** *If $f$ is any function in $K(\varphi)$ with $\|f\|_1 = 1$, it can be written in the form $f = \frac{1}{2}(f_1 + f_2)$, where $f_1$ and $f_2$ are extreme points of ball$(K(\varphi))$.*

*Proof.* The desired representation is obtained by setting $f_{1,2} = f\{1 \pm \operatorname{Re}(\lambda u)\}$, where $u$ is again the greatest common divisor of $I_f$ and $I_{\tilde f}$ and $\lambda$ is a complex number with $|\lambda| = 1$ for which $\int |f| \operatorname{Re}(\lambda u)\, dm = 0$, The details are very much the same as in [Gar, Chapter IV, Theorem 5.1] where the needed assertion is proved for the special case $\varphi \equiv 0$, in which case $K(\varphi) = H^1$. □

**Corollary 2.** *The extreme points of ball$(K(\varphi))$ form a dense subset (with respect to $L^1$ norm) of the unit sphere of $K(\varphi)$, provided $\varphi \not\equiv 0$.*

*Proof.* For a fixed function $f$ in $K(\varphi)$ with $\|f\|_1 = 1$, set $f_\alpha \overset{\text{def}}{=} f - \alpha\mathscr{O}_f$, where $\alpha \in \mathbb{D}$ and $\mathscr{O}_f$ stands for the outer factor of $f$. One easily verifies that $f_\alpha \in K(\varphi)$, $I_{f_\alpha} = (I_f - \alpha)/(1 - \overline{\alpha} I_f)$, and $I_{\tilde f_\alpha} = I_{\tilde f}$. By Frostman's theorem (see [Gar, Chapter ii]) $I_{f_\alpha}$ is a Blaschke product for $\alpha \in \mathbb{D}\backslash A$, where $A$ is a certain set of zero logarithmic capacity. Now let $\{z_j\}$ be the zero sequence of $I_{\tilde f}$, and let $\{\alpha_n\}$ be a sequence of points in $\mathbb{D}\backslash(A \cup \{I_f(z_j)\})$ tending to 0. As readily seen, the inner factors of $f_{\alpha_n}$ and $\tilde f_{\alpha_n}$ are relatively prime for $n = 1, 2, \ldots$. Therefore the unit norm functions $f_{\alpha_n}/\|f_{\alpha_n}\|_1$ are extreme points of *ball*$(K(\varphi))$. They also converge to $f$ in $L^1$ norm, and the proof is complete. □

*Remarks.* 1. As mentioned above, $K(0) = H^1$ and so, formally speaking, Theorem 6 extends the well-known theorem of de Leeuw and Rudin that says the extreme points of *ball*$(H^1)$ are precisely the outer functions of unit norm. To derive this from Theorem 6 one should notice that $\varphi \equiv 0$ implies $\tilde f \equiv 0$ for all $f$ in $H^1$ and define $I_{\tilde f}$ to be also 0. Note that the statement of Corollary 2 fails if $\varphi \equiv 0$; the $L^1$-closure of the extreme points of *ball*$(H^1)$ is known to contain precisely those unit norm functions in $H^1$ that have no zeros in $\mathbb{D}$. (See [H, Chapter 9]).

2. In the case that $\varphi = \overline{\theta}$, $\theta$ being an inner function, $K(\varphi)$ becomes the star-invariant subspace $K^1_\theta \stackrel{\text{def}}{=} H^1 \cap \overline{z}\theta\overline{H^1}$ that can be also defined as $\mathrm{Clos}_{L^1} K^2_\theta$. For this special case Theorem 6 and its corollaries have been previously obtained by the author [Dya1]. See also [Dya2] for some information on the exposed points in $K^1_\theta$. In particular, it is proved there that Corollary 2 remains valid after $K(\varphi)$ is replaced by $K^1_\theta$ and the word "extreme" is replaced by "exposed".

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