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OBSTRUCTIONS TO THE INTEGRABILITY OF HAMILTONIAN SYSTEMS FROM DIFFERENTIAL GALOIS THEORY

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Abstract

Differential Galois Theory opens the door to a fairly recent field of study. Revisiting the ideas behind Galois theory of algebraic equations in polynomials, we will learn on the development of an analogous approach applied to differential equations. We will provide links with Dynamical Systems in terms of integrability, in particular, of Hamiltonian Systems. Finally, we will apply our results to a particular example and design our own original strategy to apply the theory.

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1 Introduction

The project

Dynamical systems captured my attention since the very beginning because they link apparently unrelated subjects. As a student of Physics and Mathematics, I appreciate the beauty of connecting symmetries of Physical Dynamical Systems with abstract group and algebraic theory of Galois. In this work, we use The Residue Theorem for integral computations, we endow polynomials a Topology, we make use of Algebraic Geometry and Commutative algebra, we reformulate Theory of Differential Equations, we endow paths around singularities a matrix group using Homotopy classes, we create numerical and symbolical computational routines and we give criteria for Hamiltonian systems to be integrable. The concept of integrability by quadratures, whether speaking of roots in polynomials or integrals in differential equations, hides the essence of symmetries at its core in the transformations that preserve the roots of the equations describing the system.

Memory structure

This project aims to give criteria for the integrability of a dynamical system. To achieve this, the project is structured in several blocks.

The first block is strongly influenced by the spirit of [6] about *Differential Galois Theory*, with an approach from (Differential) Ring Theory and then extending to quotient fields. We complement it with [5], which covers many relevant aspects. The first sub-block reminds the most elementary concepts of *Algebraic Structures*, with the aim of giving a consistent definition of the Zariski *Topology* on polynomial ideals, a key concept in the approach to Differential Galois Theory. Then it builds axiomatically a theory of differential rings in section 5, differential operators and Picard-Vessiot theory in section 6. With notions of algebraic groups from section 4, we define the differential Galois Group in section 7.

The Hamiltonian formalism is reviewed in detail in chapter 8 towards defining classical integrability. Then it is compared with new concepts of integrability from Morales-Ramis-Simó Theory in section 10 using the notions of the monodromy group and variational equations.

Finally, in section 11, an example of Hamiltonian dynamical system is proposed containing an integrable case and a chaotic case. It is analysed with several tools like invariant manifolds and Melnikov integrals to ensure the integrability or not of each case. Then, we design symbolic and numerical tools to apply the theory in search for obstructions to integrability reaching satisfactory results.

2 Preliminary notions

2.1 Ideals

We shall start from the very beginning introducing some elementary algebraic structures the reader may skip to finally define the Zariski Topology and its use in algebraic geometry.

Definition 2.1. We define a *group* as any set G equipped with any binary operator: $(G, *)$ satisfying the following properties:

1. closure: $a * b \in G \ \forall a, b \in G$,
2. associativity: $a * (b * c) = (a * b) * c \ \forall a, b, c \in G$,
3. identity element existence: $\exists e \in G$ so that $a * e = e * a = a \forall a \in G$,
4. inverse: $\forall a \in G \exists a^{-1} \in G$ so that $a * a^{-1} = a^{-1} * a = e$

Definition 2.2. A commutative group is one that satisfies the property $a * b = b * a \forall a, b \in G$.

Definition 2.3. A ring is a set R equipped with two operators, $(R, +, *)$, we'll call addition and product satisfying the following properties:

1. $(R, +)$ is a group.
2. distributiveness: $\forall a, b, c \in R$:

   $$\begin{aligned} \text{left} \quad & a * (b + c) = a * b + a * c, \\ \text{right} \quad & (a + b) * c = a * c + b * c, \end{aligned}$$

3. $(R, *)$ satisfies all the group properties except the inverse element existence.

If the multiplication is commutative, R is called Abelian or commutative.

Some references reserve the property of multiplicative identity element existence for 'unitary rings'. Additionally, if non-zero elements have a multiplicative inverse, the ring R is instead called a field.

Definition 2.4. An ideal I of a ring R is an additive group closed under product.

An ideal I is called prime if $a * b \in I \Rightarrow a \in I$ or $b \in I$, in other words, if no element of I is the product of two elements outside it.

We say, however, an element $a \in R$ to be prime when the condition a dividing bc, a product of two elements of R implies that a divides one of them at least.

Not to be confused with the above, we say an non-zero, non-unit element a is irreducible if it cannot be obtained as the product of two non-unit elements.

Definition 2.5. A proper ideal I of a ring R is any ideal other than the trivial ones R and $\emptyset$.

Definition 2.6. A commutative ring D is called an integral domain if no element in D divides zero.

Definition 2.7. A finitely generated ideal I of R is generated by a (not generally unique) list a_i: $I = \{\sum_i c_i a_i, c_i \in R\}$.

Definition 2.8. Principal ideals are those generated solely by one element.

Definition 2.9. A principal domain is an integral domain where every proper ideal is a principal ideal. This hypothesis is sufficient for the following definition.

Definition 2.10. A unique factorization domain is one where every non-zero, non-unit element has a unique decomposition in prime elements or irreducible elements.

Definition 2.11. A proper ideal I of R is called maximal if there is no ideal in R containing I.

Observation 2.1. Maximal ideals are prime ideals. The converse fails.

Observation 2.2. In an integral domain, prime elements are irreducible.

Definition 2.12. We denote $R[X]$ the ring of polynomials in the indeterminate X and coefficients in R.

The next sections referring to theoretical aspects of the formalism of *Differential Galois Theory* are extracted predominantly from [6]. This subject might pose some difficulties for undergraduate students. We have strongly based the structure of this part on the reference, developing the subject from a ring theory perspective and extending to fields. Some "heavy" proofs have been skipped, and, in the other hand, some more "tangible" aspects have been highlighted, expanding examples in detail, complementing some checks for clarification and aiming for meaningful conclusions.

2.2 Zariski topology

We denote by C an algebraically closed field. Also denote $\mathbb{A}^n = C^n = C \times \cdots C$ the affine n-space.

Definition 2.13. We define an affine variety as the common set of zeros in $\mathbb{A}^n$ of a finite collection of polynomials. That is, an algebraic variety inside an affine space.

Definition 2.14. If $\mathcal{F} \subseteq C[X_1, \ldots X_n]$, let $\mathcal{V}(F) \subseteq \mathbb{A}^n$ be the common zeros of elements in $\mathcal{F}$:

$$\mathcal{V}(F) = \{a \in \mathbb{A} | f(a) = 0, \forall f \in \mathcal{F}\}$$

By the *Hilbert Basis Theorem*, see [6], each ideal of $C[X_1, \ldots, X_n]$ has a finite set of generators, and thus, $\mathcal{V}(F)$ is an affine variety.

Definition 2.15. Analogously, to each set $S \subset \mathbb{A}$ we associate the ideal of polynomials vanishing on S:

$$\mathcal{I}(S) = \{f \in C[X_1, \ldots X_n] | f(a) = 0, \forall a \in S\}$$

It is indeed an ideal since the sum and product of polynomials that vanish in S also does. The identity elements are those of $\mathbb{C}$.

Observation 2.3. We have the inclusions $I \subset \mathcal{I}(\mathcal{V}(I))$ and $S \subset \mathcal{V}(\mathcal{I}(S))$.

Proof. That is: the set of polynomials vanishing on the common zeros of an ideal of polynomials I contains I, and the common zeros of all the vanishing polynomials on a set S contains S. Indeed,

- Every polynomial from I vanishes on the common zeros of the polynomials of I.
- Every element in S is a zero of the polynomials that vanish on all S. □

Definition 2.16. We define the radical $\sqrt{I}$ of an ideal I as

$$\sqrt{I} := \{f \in \mathbb{C}[X_1, \ldots X_n] | \exists r \geq 1 | f(X)^r \in I\}$$

It is an ideal containing I (take $r = 1$). The famous *Hilbert Nullstellensatz Theorem* states that it coincides with the above set

$$\sqrt{I} = \mathcal{I}(\mathcal{V}(I))$$

The inclusion $\sqrt{I} \subset \mathcal{I}(\mathcal{V}(I))$ is relatively simple to check:

In fact, since set the of zeros of polynomials in $\sqrt{I}$ is the same as that of I, then $\mathcal{V}(\sqrt{I})) = \mathcal{V}(I))$.

Since $\sqrt{I}$ is an ideal, by observation 2.3, $\sqrt{I} \subset \mathcal{I}(\mathcal{V}(\sqrt{I})) = \mathcal{I}(\mathcal{V}(I))$.

The other inclusion of the Theorem is not proved here, but can be found in any book of elementary algebraic geometry. As an example, the radical ideal of $m\mathbb{Z}$ is $r\mathbb{Z}$, where r is the product of all distinct prime factors of m, as the radical ideal consists of taking all the positive roots.

Corollary 2.2.1. *$\mathcal{V}$ and $\mathcal{I}$ set a bijective correspondence between the collection of all radical ideals of $C[X_1, \ldots, X_n]$ and the collection of all affine varieties of $\mathbb{A}^n$.*

Proposition 2.1. *The correspondence $\mathcal{V}$ satisfies the following equalities:*

1. $\mathbb{A}^n = \mathcal{V}(0)$ *The zero polynomial ideal vanishes in all the affine space.*

   $\emptyset = \mathcal{V}(C[X_1, \ldots, X_n])$ *There is no common root to every possible polynomial in $C[X_1, \ldots, X_n]$.*

2. *If I and J are two ideals of $C[X_1, \ldots, X_n]$, then $\mathcal{V}(I) \cup \mathcal{V}(J) = \mathcal{V}(I \cap J)$.*

3. If I_α is an arbitrary collection of ideals of $C[X_1, \dots, X_n]$, then $\bigcap_\alpha \mathcal{V}(I_\alpha) = \mathcal{V}(\sum_\alpha I_\alpha)$

satisfying the axioms of closed sets in a topology.

Definition 2.17. A ring R is called *Noetherian* if it satisfies the ascending chain condition on ideals. That is

$$\forall \{I_k\}_k | I_1 \subseteq \dots \subseteq I_{k-1} \subseteq I_k \subseteq I_{k+1} \subseteq \cdots, \exists n | I_n = I_m$$

This definition is equivalent to R having a maximal ideal, and also equivalent to the condition that every right ideal I in R is of the form $I = a_1 R + \cdots + a_n R$.

Definition 2.18. A topological space is called *Noetherian* if open subsets satisfy the ascending condition.

Hilbert's basis theorem implies the descending chain condition on closed sets and, therefore, the ascending chain condition on open sets. Hence, $\mathbb{A}^n$ is a Noetherian topological space, implying too that it is compact: every open covering admits a finite subcovering. Separability condition does not hold, however.

Definition 2.19. A topological space is said to be irreducible if it cannot be written as the union of two proper, non empty, closed subsets.

A Noetherian topological space can be written in its finitely many irreducible (maximal) components.

Proposition 2.2. *A closed set V in $\mathbb{A}^n$ is irreducible if and only if its ideal $\mathcal{I}(V)$ is prime. In particular, $\mathbb{A}^n$ is irreducible.*

Proof. Take $I = \mathcal{I}(V)$, with V irreducible and $f_1, f_2 \in I \subset C[X_1, \dots, X_n]$. Then $\forall x \in V$, $f_1(x) f_2(x) = f_1 f_2(x) = 0$, so x must be a zero of f_1 or f_2, that is, $x \in \mathcal{V}((f_1) \cup (f_2))$, equivalently, $V \subset \mathcal{V}((f_1) \cup (f_2))$. But V is irreducible, so it must be contained in one of both ideals generated by f_i (Zarisky-closed), that is, $f_1 \in I$ or $f_2 \in I$. I contains no elements consisting of the product of two elements outside I: it is prime.

Now we'll prove V reducible implies I not prime. Suppose V is the union of two sets closed in V, say V_1, V_2. If none covers V, $V \not\subset V_i, i = 1, 2$ then $\mathcal{I}(V_i) \not\subset \mathcal{I}(V) = I$, since the correspondence inverts inclusion order. Thus, we can find two $f_i \in \mathcal{I}(V_i) \setminus I$ (no f_i vanishing in V). But their product does vanish in V: if $x \in V$ then $x \in V_i$, for $i = 1$ or $i = 2$, so $f_i(x) = 0$ and $f_1(x) f_2(x) = 0$, that is, $f_1 f_2 \in I$, and I is not prime. □

Definition 2.20. A *principal open set* of $\mathbb{A}^n$ is the set where a certain polynomial doesn't vanish.

$$O_p := \{a \in \mathbb{A}^n : p(a) \neq 0\} = \mathbb{A}^n \setminus V((p))$$

By arbitrary union and finite intersection combination of them we can obtain any other Zariski-open set, forming thus a basis of the Zariski topology.

A subspace of a topological space is irreducible if and only if its closure is.

Since the only closed set containing O_p is $O_0 = \mathbb{A}^n$, the closure of a principal open set is the whole affine set A^n, which is irreducible, so principal open sets are irreducible.

If V is closed in $\mathbb{A}^n$, each polynomial defines a C-valued function on V allowing different polynomials to define the same function. There is a one-to-one correspondence between polynomial functions on V and the residue class ring

$$C[X_1, \ldots, X_n]/\mathcal{I}(V).$$

Recall that it means the set of equivalence classes respect to the relation $[p] = [q] \Leftrightarrow p - q \in \mathcal{I}(V) \Leftrightarrow (p - q)(a) = 0, \forall a \in V$.

Definition 2.21. We denote this ring with $C[V]$ and call it the coordinate ring of V.

It is a finitely generated algebra over C (in the polynomial sense). The following definitions will be used in section 6.

Definition 2.22. Let R be a fixed commutative ring. An (associative unital) R-algebra is an additive abelian group A with the structure of both a ring and an R-module in such a way that the scalar multiplication satisfies

$r \cdot (xy) = (r \cdot x)y = x(r \cdot y), \forall r \in R, x, y \in A.$

Definition 2.23. Suppose that R is a ring and 1_R is its multiplicative identity. A left R-module M consists of an abelian group $(M, +)$ and an operation $\cdot : R \times M \to M$ such that for all $r, s \in R$ and $x, y \in M$ we have:

1. $r \cdot (x + y) = r \cdot x + r \cdot y$,
2. $(r + s) \cdot x = r \cdot x + s \cdot x$,
3. $(rs) \cdot x = r \cdot (s \cdot x)$,
4. $1_R \cdot x = x$

Observation 2.4. The coordinate ring $C[V]$ is reduced (without non-zero nilpotent elements) because $\mathcal{I}(V)$ is a radical ideal (equal to its radical).

Proof. If an ideal J in a ring R is radical then R/J is reduced.

Let $x + J \in J/R$ such that $(x + J)^n = 0_{R/J}$, then as well $x^n + J = 0_{R/J}$ and so $x^n \in J$ and since J is radical, it is generated by its distinct prime roots, $x \in J$, and so $x + J = 0_{R/J}$ is not non-zero. □

If V is an affine variety and $f \in C[V]$, we define $O_f = \{P \in V : f(P) \neq 0\}$, an open set of V.

Now, if V is irreducible, equivalently $\mathcal{I}(V)$, $C[V]$ is also an integral domain (no zero divisors) since it comes from taking quotient by the prime ideal $\mathcal{I}(V)$.

We may then consider its field of fractions $C(V)$ called the *function field* of V. Elements f of $C(V)$ are called *rational functions* of V, and can be written as the quotient of two elements of $C[V]$, not uniquely in general. We have to take care the the denominator doesn't vanish at a point P to give f a well defined value at P, in which case we say f is *regular* at P. We trivially define the domain of f as the set of points where it is regular.

If $f \in C[X_1, \ldots, X_n]$, the points of the principal open sets $O_f = \{x \in \mathbb{A}^n : f(x) \neq 0\}$ are in 1-1 correspondence with the closed sets of $\mathbb{A}^{n+1}$ defined by $\{(x_1, \ldots, x_n, x_{n+1}) : f(x_1, \ldots, x_n)x_{n+1} - 1 = 0\}$. That is, the principal open sets have an affine variety structure and its coordinate ring is $C[O_f] = C[X_1, \ldots, X_n, 1/f]$.

3 Tangent Space of an affine variety

Definition 3.1. If $f(X_1, \ldots, X_n) \in C[X_1, \ldots, X_n]$ and $x = (x_1, \ldots, x_n) \in \mathbb{A}^n$, we define the differential of f at x

$$d_x f = \sum_{i=1}^{n} (\partial f / \partial X_i)(x)(X_i - x_i)$$

Clearly from the definition, for $f, g \in C[X_1, \ldots, X_n]$, we have

1. $d_x(f + g) = d_x f + d_x g$, linearity,
2. $d_x f(f \cdot g) = d_x f \cdot g + f \cdot d_x g$, the Leibnitz rule.

Definition 3.2. If V is an affine variety in $\mathbb{A}^n_C$ and $x \in V$, we define the *tangent space* to V at the point x as the linear variety

$$Tan(V)_x = \{x \in \mathbb{A}^n_C | d_x f = 0, \forall f \in \mathcal{I}(V)\} = \mathcal{V}(d_x \mathcal{I}(V))$$

Example 3.1. A linear variety, that is, a variety given one-degree polynomials $\mathcal{I}(V)$, is equal to its tangent space at any point, since $d_x(\mathcal{I}(V) = \mathcal{I}(V)$ and in this case $V = \mathcal{V}(\mathcal{I}(V))$.

For a finite set of generators $\{f_i\}$ of $\mathcal{I}(V)$, $d_x f$ generate $\mathcal{I}(Tan(V)_x)$, the ideal of polynomials vanishing on the tangent space.

Next we give an intrinsic definition of the tangent space. For a variety $V \subset \mathbb{A}^n$ and $x \in V$, take $M_x = \mathcal{I}(x)$, the maximal ideal of $C[V]$ vanishing at x.

For arbitrary $f \in C[X_1, \ldots, X_n]$, $d_x f$ can be seen as a linear function on $\mathbb{A}^n$ with origin x, hence on $Tan(V)_x \subset \mathbb{A}^n$. Since $d_{Tan(V)_x}(\mathcal{I}(V)) = \{0\}$, then $d_x f$ (f

arbitrary), is determined on $Tan(V)_x$ by the image of taking f modulo $\mathcal{I}(V)$; that is, the image of f in $C[V] = C[X_1, \ldots, X_n]/\mathcal{I}(V)$.

d_x can be seen as a C-linear map from $C[V]$ to the dual space of $Tan(V)_x$. Since $C[V] = C \oplus M_x$ as C-vector spaces and $d_x(C) = 0$, we may restrict d_x from M_x to $(Tan(V)_x)^*$ the dual of the tangent space, or simply called the cotangent space. Taking quotient by its kernel, M_x^2, we obtain an isomorphism with the image:

Proposition 3.1. *The map d_x defines an isomorphism from M_x/M_x^2 to the cotangent space at x.*

Proof. A linear function g on the tangent space at x (an element of the dual of the tangent space) is just the restriction of a linear function on A^n with origin at x given by a linear polynomial $f(X_1, \ldots, X_n)$, so $d_x f = f = g$. Thus the image is "total" and the map is surjective.

Now, let us check that the Kernel of d_x is M_x^2. Indeed, if $f \in Ker(d_x)$, for $f \in M_x$, we consider a non constant polynomial representative $\tilde{f}$ of f before taking the quotient $\pi : C[X_1, \ldots, X_n] \to M_x$ that maps $\tilde{f} \mapsto f$.

Since $d_{Tan(V)_x} f = 0$, $d_x f$ equals some representative in terms of the generators of $\mathcal{I}(Tan(V)_x)$: $d_x f = \sum a_i d_x f_i$ for some $a_i \in C, f_i \in \mathcal{I}(V)$. If we set $g = \tilde{f} - \sum a_i f_i$, another representative of f, we see $d_x g = d_x f - \sum a_i d_x f_i = 0$ in the whole $\mathbb{A}^n$, hence identically zero, but g being non constant. Its linear (and constant) part in the Taylor expansion (the differential) is zero, so g is a product of polynomials vanishing at x, telling us that it belongs to the square ideal of $C[X_1, \ldots, X_n]$, recalling $\pi(C[X_1, \ldots, X_n] = M_x$, so $\pi(g) = f \in M_x^2$. □

4 Algebraic Groups

Definition 4.1. Let C denote an algebraically closed field of zero characteristic.

An algebraic group over C is an algebraic variety G defined over C, endowed with a group structure such that the maps of translation of two elements and element inversion are (continuous) morphisms of varieties.

The general linear group $GL(n, C)$ is the group of all invertible $n \times n$ matrices with entries in C with matrix multiplication.

It corresponds to the principal open subset (of the whole affine space $\mathbb{A}^{n^2}$)of non-vanishing determinant matrices.

Viewed as an affine variety, its coordinate ring is generated by the restricted n^2 coordinate functions X_{ij} , and attaching $1/det(X_{ij})$.

Closed subgroups of $GL(n, C)$ are algebraic groups. This will later apply to the Galois Group of a differential extension, which we shall define later.

The direct product of two or more algebraic groups consists of the direct product of groups endowed with the Zariski topology and it is an algebraic group. We use this concept for the next observation.

4.1 The identity component

Observation 4.1. Let G be an algebraic group, the only irreducible (in which any two non-empty open sets intersect) component of G contains the unit element e.

Proof. Let $X_1, \ldots, X_m$ be the distinct irreducible components containing e. The product $X_1 \cdots X_m$ is again an irreducible subset which must contain e and lie in some X_i, while clearly every X_i lies in $X_1 \cdots X_n$, so $m = 1$. □

Definition 4.2. We call this unique irreducible component containing e the identity component of G, denoted G^0.

We will prove one of the following three lemmas for G an algebraic group.

Lemma 4.1.0.1. *G^0 is a normal subgroup of G of finite index.*

Proof. To prove that it is a subgroup we only need to check that it is closed under its binary operation. For that we see that, for $x \in G^0$, $x^{-1}G^0$, which is isomorphic to G^0, is then irreducible containing e: $x^{-1}G^0 = G^0$, ie, $G^0G^0 = G^0$.

To prove that it is normal, we check invariance under conjugation: $xG^0x^{-1} = G^0$ since it is also irreducible containing e. Since G is noetherian, it has finitely many irreducible components, hence G^0 is of finite index in G. □

Lemma 4.1.0.2. *Every closed subgroup of finite index in G contains G^0 .*

Lemma 4.1.0.3. *Every finite conjugacy class of G has at most as many elements as $[G : G^0]$.*

5 Differential rings

Definition 5.1. A *derivation* of a ring A is a map $d : A \to A$ such that

1. $d(a + b) = d(a) + d(b)$
2. $d(ab) = d(a)b + ad(b)$

The second property induces the Leibniz rule.

Notation. We may denote $a' = d(a), d'' = d^2(a), \ldots, a^{(n)} = d^n(a)$.

Proposition 5.1. *If A is an integral domain, a derivation d of A extends to the quotient field $Qt(a)$ in a unique way.*

Proof. Recall that A being an integral domain means it is commutative and having no non-zero zero divisors allows us to divide by non-zero elements. So for $\frac{a}{b} \in \mathbb{Q}(t$ we define $(\frac{a}{b})' = \frac{a'b - ab'}{b^2}$ and check that is independent of the class representative:

$$\left(\frac{ac}{bc}\right)' = \frac{(ac)'bc - ac(bc)'}{b^2c^2} = \frac{(a'c + ac')bc - ac(b'c + bc')}{b^2c^2}$$
$$= \frac{a'cbc + ac'bc - acb'c - acbc'}{b^2c^2} = \frac{a'cbc - acb'c}{b^2c^2} = \frac{a'b - ab'}{b^2}$$

Now we can check the proposed definition satisfies the definition of derivation:

1. $$\left(\frac{a}{b} + \frac{c}{d}\right)' = \left(\frac{ad + bc}{bd}\right)' = \frac{(ad + bc)'bd - (ad + bc)(bd)'}{b^2d^2}$$
$$= \frac{(ad)'bd + (bc)'bd - ad(bd)' - bc(bd)'}{b^2d^2} = \frac{(ad)'bd - ad(bd)'}{b^2d^2} + \frac{(bc)'bd - bc(bd)'}{b^2d^2}$$
$$= \left(\frac{ad}{bd}\right)' + \left(\frac{bc}{bd}\right)' = \left(\frac{a}{b}\right)' + \left(\frac{c}{d}\right)'$$

2. $$\left(\frac{a}{b} \cdot \frac{c}{d}\right)' = \left(\frac{ac}{bd}\right)' = \frac{(ac)'bd - ac(bd)'}{b^2d^2} = \frac{(a'c + ac')bd - ac(b'd + bd')}{b^2d^2}$$
$$= \frac{a'b - ab'}{b^2d^2}cd + \frac{c'b - cd'}{b^2d^2}ab = \left(\frac{a}{b}\right)' \cdot \frac{c}{d} + \frac{a}{b} \cdot \left(\frac{c}{d}\right)'$$ □

Definition 5.2. A differential ring is a commutative ring with identity endowed with a derivation. Naturally, differential rings which are fields are called differential fields.

It is our particular interest to understand the ring of analytic functions in the complex plane with the usual derivative, which is a differential ring. Since it is an integral domain, the derivation extends to its quotient field, called the field of *meromorphic functions*.

If A is a ring, we can extend the derivation of A to $A[X]$ by assigning to X' an arbitrary value in $A[X]$, or if A is a field, we can extend its derivation to the field $A(X)$ of rational functions.

We now have the differential ring $A\{X\}$ of polynomials in X and its derivatives, called the differential polynomials. Like before, if A is a differential field, $A\{X\}$ is an integral domain and we can extend its derivation uniquely to its quotient field denoted $A\langle X\rangle$ whose elements are *differential rational functions of X*.

Definition 5.3. The elements in a differential ring A with zero derivative form the subring C_A called the ring of *constants*.

Observation 5.1. For a field K, C_K is also a field:

Proof. $a \in C_K \Rightarrow d(a) = 0 \Rightarrow 0 = d(1) = d(aa^{-1}) = d(a)a^{-1} + ad(a^{-1}) = ad(a^{-1})$
$\Rightarrow d(a^{-1}) = 0 \Rightarrow a^{-1} \in C_K$ □

Definition 5.4. Let I be an ideal of a differential ring A. We say that I is a differential ideal if it is closed under derivation: $d(I) \subset I$.

Observation 5.2. If I is a differential ideal of the differential ring A, we can define a derivation in the quotient ring A/I by $d(\bar{a}) = \overline{d(a)}$ independently on the choice of the representative in the coset and indeed defines a derivation in A/I:

$d(\bar{a}) = (d(a+I) = d(a) + d(I) = d(a) = \overline{d(a)}$

Definition 5.5. A differential morphism between differential rings is a morphism that commutes with the derivation.

If I is a differential ideal, the morphism $A \to A/I$ is a differential morphism.

Proposition 5.2. *If $f : A \to B$ is a differential morphism, then Ker f is a differential ideal and the isomorphism $\overline{f} : A/Ker\ f :\to Im\ f$ is a differential one.*

Proof. $a \in \text{Ker} f \Rightarrow f(a') = f(a)' = 0' = 0 \Rightarrow a' \in \text{Ker } f$ so Ker$f$ is a differential ideal.

The isomorphism property of f comes directly from linear algebra. We now check that $\bar{f}$ commutes with derivation. For every $a \in A$

$$(\bar{f}(\bar{a}))' = (f(a))'$$

since f restricted to the quotient evaluates elements by f. Now, since f is a differential morphism, it commutes with derivation

$$f(a)' = f(a') = \bar{f}(\bar{a'}) = \bar{f}(\bar{a}')$$

□

Definition 5.6. An inclusion $A \subset B$ of differential rings is an extension of differential rings if the derivation of B restricts to the derivation of A.

Definition 5.7. For $S \subset B$, we denote by $A\{S\}$ the differential A-subalgebra of B generated by S over A, that is, the smallest subring of B containing A, S and $d(S)$.

For differential field extensions $K \subset L$, and for a subset S of L, we define analogously $K\langle S\rangle$ the differential subfield of L generated by S over K.

From now on we will reserve notation K and L for differential fields forming an extension.

Definition 5.8. An algebraic field extension $L|K$ is said to be separable if it every element $\theta \in L$ is separable, that is θ is the root of a polynomial in $K[X]$ without multiple roots, equivalently, if the irreducible polynomial of θ over K has no multiple roots.

Observation 5.3. For characteristic zero constant fields, a root θ of $P(X) \in \mathbb{C}[X]$ is multiple if, and only if, the derived polynomial vanishes in θ.

Proof. If $P(\theta) = 0$, let $P(X) = h(X)(X-\theta)^m, h(X) \in C[X], h(\theta) \neq 0$, then
$P'(X) = h'(X)(X-\theta)^m + h(X)m(X-\theta)^{m-1} = (h'(X)(X-\theta)+h(X)m)(X-\theta)^{m-1}$
Now if $m > 1$, a multiple root, then $P'(\theta) = 0$, and if $m = 1$, $P'(\theta) = h(X) \neq 0$. □

Proposition 5.3. *If $K \subset L$ is a separable algebraic field extension, the derivation of K extends uniquely to L. Additionally, K-automorphism of L are differential K-automorphisms.*

Proof. We will prove the first part for finite extensions and then extend the proof.

If $K \subset L$ is a finite separable extension, by the *Primitive Element Theorem*, see [7] we have $L = K(\alpha)$, for some $\alpha \in L$.

Now let $P(X)$ be the irreducible polynomial of α over K, $P(\alpha) = 0$, and deriving

$$0 = d(P(\alpha)) = P^{(d)}(\alpha) + P'(\alpha)\alpha' = 0,$$

where $P^{(d)}$ results from deriving only the coefficients of P. Isolating,

$$\alpha' = -P^{(d)}(\alpha)/P'(\alpha)$$

Remember that separability implies that the denominator doesn't vanish, so the derivation of α is unique. Elements of L are rational polynomial expressions of α.

L is isomorphic to the quotient of $K[X]$ by the irreducible polynomial ideal generated by P: $L \cong K[X]/(P)$.

We will check now that defining a derivation in $K[X]$ of X in a similar way as we did for α makes (P) a differential (irreducible) ideal and so the quotient $K[X]/(P)$ a differential ring. Indeed, we define

$$X' := -P^{(d)}(X)h(X), \text{ with } h \in K[X]: \ \ h(X)P'(X) \equiv 1 \text{ mod P}$$

But if, say $h(X)P'(X) = 1 + k(X)P(X)$, then $1 - P'(X)h(X) = k(X)P(X)$, and

$$d(P(X) = P^{(d)}(X) + P'(X)X',$$

which by definition is $P^{(d)}(X) + P'(X)(-P^{(d)}h(X))$, and taking common factor,

$$P^{(d)}(X)(1 - P'(X)h(X)) = -P^{(d)}(X)k(X)P(X) \in (P)$$

By applying *Zorn's Lemma*, see [7], the general algebraic case can be obtained.

Now, for the second part, let σ a K-automorphism of L, we will check that $\sigma^{-1}d\sigma$ is also a derivation of L extending that of K, and by uniqueness, $\sigma^{-1}d\sigma = \sigma$, so $d\sigma = \sigma d$ and σ is a differential automorphism.

Indeed, if d is a derivation in L, for $a, b \in L$, the derivation axioms hold for $\sigma^{-1}d\sigma$:

1. $d\sigma(a + b) = d(\sigma(a) + d\sigma(b)) = d(\sigma(a)) + d(\sigma(b))$ and applying σ^{-1} left,

$$\sigma^{-1}d\sigma(a + b) = \sigma^{-1}d\sigma(a) + \sigma^{-1}d\sigma(b)$$

2. $d\sigma(a \cdot b) = d(\sigma(a) \cdot \sigma(b)) = d(\sigma(a)) \cdot \sigma(b) + \sigma(a) \cdot d(\sigma(b))$ and like before,

$$\begin{aligned}\sigma^{-1}d\sigma(a \cdot b) &= \sigma^{-1}d\sigma(a) \cdot (\sigma^{-1}\sigma)(b) + (\sigma^{-1}\sigma)(a) \cdot \sigma^{-1}d\sigma(b)\\ &= \sigma^{-1}d\sigma(a) \cdot (b) + (a) \cdot \sigma^{-1}d\sigma(b)\end{aligned}$$

Also, since σ fixes K, $\sigma^{-1}d\sigma_{|K} = d$ so it extends d. ☐

A remark is needed concerning the characteristic of K. If it is $p > 0$, like $\mathbb{F}_p(T)$ endowed with $T' = 1$, and $P(X) = X^p - a \in K[X]$, with $a \notin K^p$ and $P(\alpha) = 0$, we have $P'(X) = pX^{p-1} - a' = -a'$ and $0 = d(P(\alpha)) = P^{(d)} + P'(\alpha)\alpha' = -a' - a'\alpha' = -a'(1 + \alpha)$

If $a' = 0$, a is a constant, this always holds for every value of α'. Otherwise, it is not possible ot extend the derivation of K to L.

Definition 5.9. We say an element $\alpha \in L$ is

1. a *primitve (or integral) element* over K if $\alpha' \in K$.

   Its name finds explanation in that, say $\alpha' = a \in K$, then α is a solution of the differential equation over K: $d(\alpha) - a = 0$

2. an *exponential element* over K if $\alpha'/\alpha \in K$

   Its name finds explanation in that, say $\alpha'/\alpha = d(log(\alpha)) = z' \in K$, with $z = log(\alpha)$ primitive, then $\alpha = e^z$.

Definition 5.10. A linear *differential operator* $\mathcal{L}$ with coefficients in K is a polynomial in the derivation operator of K d:

$$\mathcal{L} = \sum_{i=0}^{n} a_i d^i, \text{ with } a_i \in K$$

and d^0 the identity operator. If $a_n \neq 0$, we say $\mathcal{L}$ has degree n, and if it is 1 we say it is monic.

Linear differential operators over K form a (non-commutative) ring $K[d]$, where d satisfies $da = a' + ad$ for $a \in K$. As in usual ring polynomials, $deg(\mathcal{L}_1\mathcal{L}_2) = deg(\mathcal{L}_1) + deg(\mathcal{L}_2)$ implying that there are no non-trivial invertibles in $K[d]$:

if $\mathcal{L}_1(X)\mathcal{L}_2(X) = 1$, taking degrees $deg\mathcal{L}_1 + deg\mathcal{L}_2 = 0$ and since they are both non-negative, the degrees are both bound to be zero.

To every operator $\mathcal{L}$ we associate the differential equation $\mathcal{L} = 0$.

The Euclides algorithm provides us a division left or right, as in algebraic polynomials.

6 Picard-Vessiot extensions

Consider a monic differential operator $\mathcal{L}$ over the differential field $K \subset L$. The set of solutions of $\mathcal{L}(Y) = 0$ in L is a C_L-vector space of dimension no bigger than $deg(\mathcal{L})$, as we shall see.

Definition 6.1. Let $y_1, \ldots, y_n \in K$ its wrońskian is $W(y_1, \ldots, y_n) = |d^{(i-1)}y_j|_{1 \leq i,j \leq n}$

Proposition 6.1. $W \neq 0 \Leftrightarrow$ *$y_1, \ldots, y_n \in K$ are linearly independent over C_K.*

Proof. $\Rightarrow$) Assume $y_1, \ldots, y_n$ to be linearly dependent over C_K, say

$$\sum_{j=1}^{n} c_j y_j = 0, \;\; c_j \in C_K \text{ not all zero.}$$

Applying $d^{(i-1)}$, we obtain for each $i = 1, \ldots, n$ that

$$\sum_{j=1}^{n} c_j y_j^{(i-1)} = 0$$

saying that the columns inside W are linearly dependent and thus $W = 0$.

$\Leftarrow$) If $W = 0$, we obtain $\sum_{j=1}^{n} c_j y_j^{(i-1)} = 0$ with $c_j \in K$ not all zero for each $i = 1, \ldots, n$. We will assume $c_1 = 1$ and $W(y_2, \ldots, y_n) \neq 0$. Applying $d(\cdot)$ to each equation (i) we obtain

$$\sum_{j=1}^{n} c_j y_j^{(i)} + \sum_{j=2}^{n} c_j' y_j^{(i-1)} = \sum_{j=2}^{n} c_j' y_j^{(i-1)} = 0$$

but from equation $(i-1)$ the first term vanishes, giving a homogoneous linear equations system in $c_2', \ldots, c_n'$ with determinant $W(y_2, \ldots, y_n) \neq 0$, with unique solution $c_2' = \ldots, c_n' = 0$, that is, all C_i constants, that is, $y_1, \ldots, y_n$ linearly dependent over C_K.

□

We are now allowed to talk about linear independence over the constant field without ambiguity (independently of the it).

Corollary 6.0.1. *An equation will never admit a number greater than its order of linearly independent solutions in L over its field of constants.*

Definition 6.2. Such a set of solutions is called a *fundamental set of solutions.*

This set generates the space of solutions of the mentioned equation.

Definition 6.3. Keeping the above notation for a homogeneous linear differential equation, we say $K \subset L$ is a Picard-Vessiot extension if the following two conditions are satisfied:

1. $L = K\langle y_1, \ldots, y_n \rangle$. The fundamental set of solutions of $\mathcal{L}(Y) = 0$ generates L over K.

2. $C_K = C_L$. We incorporate no constants.

This definition is analogous to the notion of a splitting field of a polynomial: the minimal field generated by its solutions. Condition 2 in the above definition guarantees minimality, as we shall check in the following examples.

Example 6.1. Let k be a differential field and consider z a solution of the differential equation $\mathcal{L}(Y) = Y' - Y = 0$. If, instead of considering the natural extension $K = k\langle z \rangle$, we adjoin $L = K\langle y \rangle$ a second indeterminate, also a solution of $\mathcal{L}$, (not respecting minimality), condition 1 is respected. However, $y/z \in L$ is a new constant incorporated to L:

$$\left(\frac{y}{z}\right)' = \frac{y'z - yz'}{z^2} = \frac{yz - yz}{z^2} = 0 \Rightarrow \left(\frac{y}{z}\right) \in C_L \setminus C_K$$

Below we will only give the idea behind the proof that for a differential field K with algebraically closed field of constants C, there exists a Picard-Vessiot extension L of K for a given homogeneous linear differential equation L over K. The extension is defined up to differential K-isomorphism.

The idea for existence is to construct a differential K-algebra ,R, containing a full set of solutions of L, and then quotient R by a maximal (thus prime) ideal, making it an integral domain with no new constants.

Now we adress its uniqueness. First, we prove that the image by differential morphism σ from a Picard-Vessiot extension L of K onto another differential extension with the same field of constants as K is unique with respect to σ and L.

Proposition 6.2. *Let L_1, L_2 be Picard-Vessiot extensions of K for the aforementioned equation $\mathcal{L}(Y) = 0$, of order n. Let $K \subset L$ be an extension with $C_K = C_L$. Let $\sigma_i : L_i \to L$ differential K-morphisms, for $i = 1, 2$. Then its images coincide $\sigma_1(L_1) = \sigma_2(L_2)$.*

Proof. Let us define $V_i = \{y \in L_i : \mathcal{L}(Y) = 0\}$ and $V = \{y \in L : \mathcal{L}(Y) = 0\}$. V_i and V are C_K-vector spaces ($C_K = C_L$) of dimensions n and at most n, respectively. Since σ_i is a differential morphism (commutes with derivation), we will check that $\sigma_i(V_i) \subset V$, and so we have equality $\sigma_1(V_1) = \sigma_2(V_2) = V$. But since each $L_i = K\langle V_i \rangle$ and images by differential morphism are determined by differential generators, we get $\sigma_1(L_1) = \sigma_2(L_2)$.

Finally, we had to check that $\sigma_i(V_i) \subset V$. Indeed,

$$y \in V_i \Rightarrow \mathcal{L}(y) = 0 \text{ and } y \in L_i \Rightarrow \mathcal{L}(\sigma_i(y)) = \sigma_i(\mathcal{L}(y)) = \sigma_i(0) = 0 \Rightarrow \sigma_i(y) \in V$$

□

We can use the proposition in the following way.

Corollary 6.0.2. *Let $K \subset L \subset M$ be differential fields with $K \subset L$ Picard-Vessiot and $C_M = C_K$, then any differential K-automorphism σ sends L onto itself: $\sigma(L) \subset L$.*

From algebraic field extension theory we know this is an equivalent definition of normal extension, see [7], which is a remarkable fact.

Corollary 6.0.3. *Algebraic Picard-Vessiot extensions are normal algebraic extensions.*

7 Differential Galois Group

With the tow last corollaries in mind , we are ready to define the differential Galois group, check analogous properties its polynomial counterpart, and see it as a linear algebraic group.

Definition 7.1. For $K \subset L$ a differential field extension, the group $G(L|K)$ consisting of differential morphisms from L onto itself fixing K is called differential Galois group of the extension $K \subset L$. We can also denote it $Gal_K(\mathcal{L})$ for Picard-Vessiot extensions for equation $\mathcal{L}$.

Now will merely state two important facts for our construction of the Galois group.

Proposition 7.1. 1. *Given a non trivial Picard-Vessiot extension $K \subset L$ and an element $x \in L \setminus K$, there exists a differential K-automorphism of L not fixing x. In other words, elements fixed by the all $Gal(L|K)$ are no more than those of K.*

2. *For any two consecutive Picard-Vessiot extensions $K \subset L \subset M$, every $\sigma \in Gal(L|K)$ can be extended to a differential automorphism of M.*

Corollary 7.0.1. *Analogously to algebraic Galois theory, for a Picard-Vessiot extension $K \subset L$, $L^{Gal(L|K)} = K$. That is, the subfield of L fixed by the action of the Galois group of $L|K$ is exactly K.*

Proof. It is clear $K \subset L^{\text{Gal}(L|K)}$, since every element in K is fixed by the K-automorphisms of L, by definition. From part 1., we see the other inclusion.

□

We shall see the Galois group now as a linear algebraic group.

Observation 7.1. With the above notation, $G_K(\mathcal{L})$ is isomorphic to a subgroup of $GL(n, \mathbb{C})$ over C_K.

Indeed, for a a fundamental set of solutions of $\mathcal{L}$ $y_1, \ldots, y_n$ and a morphism $\sigma \in Gal(\mathcal{L})$, the image of y_j is again a solution of $\mathcal{L}$: since σ commutes with derivation, $\mathcal{L}(\sigma(y_j)) = \sigma(\mathcal{L}(y)) = \sigma(0)$, and thus a linear combination of the fundamental solutions $\sigma(y_j) = \sum_{i=0}^{n} c_{ij} y_i$. This defines a map from the Galois group of $\mathcal{L}$ to the space of matrices c_{ij}. Since the Galois group sends a fundamental set of solutions to another one, their wronskian is non-zero and equal to the product of the previous wronskian (non-zero) and the transformation c_{ij} determinant, so this matrices are non singular, thus the second space being the General Linear Group over K. This map is a morphism: The image of the sum $\sigma_1 + \sigma_2$ is $c_{ij,1} + c_{ij,2}$ defined through $(\sigma_1 + \sigma_2)(y_j) = \sigma_1(y_j) + \sigma_2(y_j) = \sum_{i=1}^{n} (c_{ij,1} + c_{ij,2}) y_i$. Same for product and identity, and it is injective since it is linear and sends non-zero elements to non-zero elements.

Later we shall also see that the Gal($\mathcal{L}$) is Zariski-closed in $GL(n, \mathbb{C})$.

Below we show the most important examples of Picard-Vessiot extensions: adjunction of an integral (or primitive element of K) and an exponential of an integral(or simply an exponential of K).

Example 7.1. Let $L = K\langle\alpha\rangle$ with $\alpha' = a \in K$, a not a derivative in K (not a trivial extension). We proof that α is transcendent over K: if α was algebraic over K then let $P(X) = X^n + \sum_{i=1}^{n} b_i X^{n-i}$ the irreducible polynomial of L over K, deriving $P(\alpha) = 0$, every coefficient in α powers must vanish (otherwise giving a polynomial vanishing on α over K of order less than the irreducible one). In particular, the $(n-1)$ order term $n\alpha^{n-1}\alpha' + b_1'\alpha^{n-1} = \alpha^{n-1}(na + b_1')$ coefficient, giving $a = \alpha' = -\frac{b_1'}{n} = \left(-\frac{b_1}{n}\right)'$, a derivative in K!

If we had not chosen P monic, then we would have reached to $a = \alpha' = -\frac{b_1'}{nb_0} = \left(-\frac{b_1}{nb_0}\right)'$, so we have proven, in fact, that no polynomial of α in K can be constant. Neither can a polynomial fraction, say $\frac{f}{g}$. Assume g monic, of minimal degree ≥ 1; differentiating, $\frac{f'(\alpha)g(\alpha) - f(\alpha)g'(\alpha)}{g(\alpha)^2} = 0 \Rightarrow \frac{f(\alpha)}{g(\alpha)} = \frac{f(\alpha)'}{g(\alpha)'}$, obtaining $g(\alpha)'$ nonzero (recall that no polynomial in α can be constant), and of degree less than g (a contradiction), since it is monic and so the highest coefficient is constant $d(1) = d(1\cdot 1) = 1d(1) + d(1)1 = 2d(1) \Rightarrow d(1) = 0$ making the highest order term vanish, as we have done before.

Since the filed $L = K\langle\alpha\rangle$ consists of rational functions on α, we conclude that $C_K = C_L$.

We take the $\{1, \alpha\}$ set of fundamental solutions of $\mathcal{L}(Y) = Y'' - \frac{a'}{a}Y' = 0$, with wronskian $1 \cdot \alpha' - \alpha \cdot 0 = a \neq 0$ (not a derivative). $K \subset L$ is a Picard-Vessiot extension for the equation $\mathcal{L}$.

Let σ a differential K-automorphism of L. Since $\alpha' = a \in K$, we put $\sigma(\alpha) - \alpha = c \in L$ and

$$a = \sigma(a) = \sigma(\alpha') = (\sigma(\alpha))' = (\alpha + c)' = a + c' \Rightarrow c' = 0 \Rightarrow c \in C_L = C_K.$$

Each mapping $\sigma : \alpha \mapsto \alpha + c, 1 \mapsto 1$, induces a differential K-automorphism of L, so the Galois group is $G(L|K) \cong C_K \cong \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \right\} \subset \mathrm{GL}(2, \mathbb{C})$.

Example 7.2. Let $L = K\langle\alpha\rangle$, with $\alpha'/\alpha = a \in K \setminus \{0\}$.

We see that $K\langle\alpha\rangle = K(\alpha)$: the smallest differential field containing K, α, α' coincides with the smallest differential field containing K, α, in that order. The inclusion $K(\alpha) \subset K\langle\alpha\rangle$ is clear and also $\alpha \in K\langle\alpha\rangle \Rightarrow \alpha' = \alpha \cdot a \in K(\alpha) \Rightarrow \alpha \in K(\alpha)$ gives the other inclusion.

As well, α is a fundamental set of solutions of $Y' - aY = 0$. We assume $C_K = C_L$.

If α is algebraic over K and let $P(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0$ its irreducible polynomial, differentiating,

$$0 = (P(\alpha))' = P^{(d)}(\alpha) + P'(\alpha)\alpha' = P^{(d)}(\alpha) + P'(\alpha)\alpha a = an\alpha^n + \sum_{k=0}^{n-1}(aka_k)\alpha^k,$$

a polynomial over K of degree $n = deg(P)$ vanishing on α, thus a multiple of the irreducible P by a factor $\lambda = \dfrac{an}{1} = \dfrac{a'_k + aka_k}{a_k}$, thus $a'_k = a(n-1)a_k$ for $k = 0, \ldots, n-1$. Now,

$$\left(\frac{\alpha^{n-1}}{a_k}\right)' = \frac{(n-k)\alpha^{n-k-1}\alpha' a_k - \alpha^{n-k}a'_k}{a_k^2} = \frac{(n-k)\alpha^{n-k-1}\alpha a a_k - \alpha^{n-k}a(n-k)a_k}{a_k^2} = 0.$$

In particular, for $k = 0$, $\left(\dfrac{\alpha^n}{a_0}\right) = c$ with $c' = 0$, $c \in C_L = C_K \subset K$. $\alpha^n = a_0 c = b \in K$. Then $P(X)$ divides $X^n - b$ (monic of $deg = n$) and so $P(X) = X^n - b$.

For $\sigma \in$Gal$L|K)$, we have

$$\sigma(\alpha)' = \sigma(\alpha') = \sigma(\alpha a) = \sigma(\alpha)\sigma(a) = a\sigma(\alpha)$$

$$\left(\frac{\sigma(\alpha)}{\alpha}\right)' = \frac{\sigma(\alpha)'\alpha - \sigma(\alpha)\alpha'}{\alpha^2} = \frac{a\sigma(\alpha)\alpha - \sigma(\alpha)a\alpha}{\alpha^2} = 0$$

Thus, $\sigma(\alpha) = c\alpha$ for some $c \in C_K = C_L$. If $\alpha^n = b \in K$ (α is algebraic), then

$$b = \sigma(b) = \sigma(\alpha^n) = (\sigma(\alpha))^n = (c\alpha)^n = c^n\alpha^n = c^n b \Rightarrow c^n = 1$$

c must be a unity root and Gal$(L|K)$ a finite cyclic group.

If α is transcendent over K, the map $\sigma(\alpha) = c\alpha$ is a differential K-automorphism of L for each $c \in C_K$ and hence Gal$(L|K)$ is isomorphic to the multiplicative group of C_K. □

Next we will see that $G(L|K)$ is a Zariski-closed subgroup of GL$(n, \mathbb{C})$, that is, there exists an ideal polynomials all vanishing at it, more precisely, on the entries of the matrices that give the linear relations between the images by Galois elements of the generators and the generators.

The second part of the next proposition tells us that any non singular matrix with entries roots of such polynomials corresponds to a Galois element generator image matrix. That is, these polynomials vanish on no more than matrices defined that way. More formally:

Proposition 7.2. *Let $L = K\langle y_1, \ldots, y_n\rangle$ be a Picard-Vessiot extension of K. There exists a set S of polynomials $F(X_{ij})$ over C_K ,with $1 \le i, j \le n$ such that*

1. *If σ is a differential K-automorphism of L and $\sigma(y_j) = \sum_{i=1}^{n} c_{ij}y_i$, then $F(c_{ij}) = 0, \forall F \in S$.*

2. *Given a matrix* $(c_{ij}) \in GL(n, \mathbb{C})$ *with* $F(c_{ij}) = 0 \forall F \in S$, *there exists a differential* K*-automorphism* σ *of* L *such that* $\sigma(y_j) = \sum_{i=1}^{n} c_{ij} y_i$.

Proof. Let $K\{Z_1, \ldots, Z_n\}$ be the ring of differential polynomials in n indeterminates over K. We define

$$\begin{aligned} \phi\colon K\{Z_1, \ldots, Z_n\} &\to L = K\langle y_1, \ldots, y_n\rangle \\ Z_j &\mapsto y_j \\ P(Z_1, \ldots, Z_n) &\mapsto P(y_1, \ldots, y_n) \end{aligned}$$

the evaluating (differential) morphism, in fact a K-automorphism since it fixes zero-degree polynomials.

$\Gamma = Ker(\phi)$ is a differential polynomial ideal of $K\{Z_1, \ldots, Z_n\}$. Since $K\{Z_1, \ldots, Z_n\}/\Gamma \cong \phi(K\{Z_1, \ldots, Z_n\}$ is an integral domain, Γ is prime. Let $L[X_{ij}]$, with $1 \leq i, j \leq n$ be the ring of polynomials in the indeterminates X_{ij} and endowing the trivial derivation $X'_{ij} = 0$. We consider the following commutative diagram.

$$\Gamma \xrightarrow{\phi} 0$$

$$Z_j \xrightarrow{\phi} y_j$$

$$\begin{array}{ccccc} \Gamma & & Z_j & & K\{Z_1, \ldots, Z_n\} \xrightarrow{\phi} L \\ \downarrow f & & \downarrow f & & \downarrow f \qquad\qquad \downarrow \sigma \\ \Delta & & \sum_{ij} y_i & & L[X_{ij}] \xrightarrow{\phi^*} L \end{array}$$

$$X_{ij} \xrightarrow{\phi^*} c_{ij}$$

Let Δ be the image of Γ by the left vertical arrow f and let $\{w_k\}$ be a basis of L as a C_K vector space. $\Delta = \{\sum a_k w_k, a_k \in S\}$ for $S \subset C[X_{ij}]$ the set of polynomial-type coefficients of $\{w_k\}$.

1. Let σ be a differential K-automorphism of L, for each $p \in \Delta$, $\phi^*(p) = \sum w_k a_k(c_{ij})$ coincides by the preimage of p in Γ sent to 0 by ϕ and again to 0 by σ, so every coefficient $a_k \in S$ vanishes at c_{ij}.

2. Consider a matrix $(c_{ij}) \in$GL$(n, \mathbb{C})$ such that $F(c_{ij}) = 0, \forall F \in S$.

   We see $\Gamma \subset Ker(\phi^* \circ f)$:

   $$P(Z_j) \in \Gamma \Rightarrow \phi^*(f(P(Z_j))) = \sigma(\phi(P(Z_j))) = \sigma(0) = 0 \Rightarrow P(Z_j) \in Ker(\phi^* \circ f)$$

, so we have a K-morphism

$$\sigma\colon K\{y_1,\ldots,y_n\}\to K\{y_1,\ldots,y_n\}$$
$$y_j\mapsto\sum_i c_{ij}y_i$$

It is injective: we will consider a non-zero element u in its kernel I and reach contradiction, whether it is algebraic or transcendent.

If $\sigma(u(y_1,\ldots,y_n))=0$, u algebraic with $0=Irr(u,K)=b_0+\sum_{i=1}^{m}b_ku^k$, $b_k\in K$, then $\sigma(Irr(u,K))=b_0+\sum_{i=1}^{m}b_k\sigma(u)^k=\sigma(b_0)=0\Rightarrow b_0\in I\Rightarrow K\subset I\Rightarrow I=K\{y_1,\ldots,y_n\}$.

On the other hand, if u is transcendent, defining the transcendence degree trdeg$[L:K]$ as the largest cardinal of a subset of L algebraically independent over K, for example trdeg$[\mathbb{Q}(\sqrt{2},\pi):\mathbb{Q}]=1$, we have:

trdeg$[K\{y_1,\ldots,y_n\}:K]>$trdeg$[K\{\sigma y_1,\ldots,\sigma y_n\}:K]$,

trdeg$[K\{y_j,\sigma y_j\}:K]=$trdeg$[K\{y_j,c_{ij}\}:K]$=trdeg$[K\{y_j\}:K]$ and

trdeg$[K\{y_j,\sigma y_j\}:K]=$trdeg$[K\{\sigma y_j\}:K]$, a contradiction.

σ is also surjective since the image contains $y_1,\ldots,y_n$ and the matrix (c_{ij}) is invertible (every element has a preimage).

Hence, σ is bijective and can be extended to a K-automorphism of the field $L=K\langle y_1,\ldots,y_n\rangle$

□

We give now a remarkable result for our comparison between polynomial and differential Galois theory concerning dimensions: the dimensions of the algebraic variety G is equal to the Krull dimension of its coordinate ring $C[G]$.

The dimension of a topological space X is the supremum number of distinct irreducible closed subsets of X in ascending inclusion order. Clearly, the dimension of an affine variety is the maximum of dimensions of its irreducible components. For a ring R, we define the Krull dimension of R as we did for X but now for distinct prime ideals. But if $V\subset\mathbb{A}^n$ is an affine variety, irreducible closed susbsets of V correspond to prime ideals of the ring $C[X_1,\ldots,X_n]$ containing $\mathcal{I}(V)$, and taking quotient, to prime ideals of the coordinate ring $C[V]$. And, by *Noehter's Normalization Lemma*, if V is irreducible, the Krull dimension of $C[V]$ is equal to the transcendence degree trdeg$[C(V):C]$.

We formulate then the next proposition.

Proposition 7.3. *Let $K\subset L$ be a Picard-Vessiot extension, then*

$$dim\ G(L|K)=trdeg[L:K]$$

Definition 7.2. A differential field extension $K \subset L$ is called normal if

$$\forall x \in F \setminus K \; \exists \sigma \in G(L|K) \mid \sigma(x) \neq x$$

Next we will state the fundamental theory of Picard-Vessiot theory, which we will not prove.

Theorem 7.0.1. *Let $K \subset L$ be a Picard-Vessiot extension with $G(L|K)$ its differential Galois group.*

1. *The correspondences*

$$H \mapsto L^H, F \mapsto G(L|K)$$

   between Zariski-closed subgroups H of $G(L|K)$ and differential intermediate fields $K \subset F \subset L$ are mutually inverse, bijective, inclusion inverting maps.

2. *$F|K$ is a Picard-Vessiot extension of K if and only if the subgroup $H := G(L|F)$ is normal in $G(L|K)$. In this case, the restricition morphism*

$$G(L|K) \to G(F|K)$$
$$\sigma \mapsto \sigma_{|F}$$

   induces an isomorphism $G(L|K)/G(L|F) \cong G(F|K)$.

8 Monodromy Group

Definition 8.1. For a linear differential equation

$$Y^{(n)} + a_1(z)Y^{(n-1)} + \cdots + a_{n-1}(z)Y' + a_n(z)Y = 0 \tag{8.1}$$

with $a_i(z) \in \mathbb{C}(z)$, a point is called *regular* if the functions a_i have no pole in P; otherwise P is called *singular*. A point at infinity ∞ is regular only if 0 is regular for the equation for $x = z^{-1}$.

Definition 8.2. If $P \in \mathbb{C}$ (resp. $P = \infty$) is a singular point for (8.1), we consider the limit $\lim_{z\to P}(z-P)^i a_i(z)$ (resp. $\lim_{z\to P} z^i a_i(z)$). If this limit exists and is fintite for $i = 1, \ldots, n$, the point P is a *regular singular* point for (8.1). The equation (8.1) is called Fuchsian if all points in $\mathbb{P}^1(\mathbb{C})$ are regular or regular singular points.

Any analytic solution of (8.1) in the neighborhood of a regular point can be analytically continued along any path in $\mathbb{C}$ not passing through any singular point. Let S be the set of singular points of (8.1) and $z_0 \in \mathbb{P}^1 \setminus S$, let $f_1, \ldots, f_n$ linearly independent analytic solutions in the neighborhood of z_0, and let $\gamma \in \pi_1(\mathbb{P}^1 \setminus S, z_0)$, the fundamental homotopy group.

By analytic continuation along γ, we obtain $\widetilde{f_1}, \ldots, \widetilde{f_n}$, which are solutions of (8.1) too. We then have a matrix $M(\gamma) \in \mathbf{GL}(n, \mathbb{C})$ such that

$$\begin{pmatrix} \widetilde{f_1} \\ \vdots \\ \widetilde{f_n} \end{pmatrix} = M(\gamma) \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}.$$

The mapping

$$\begin{aligned} \rho \colon \pi_1(\mathbb{P}^1 \setminus S) &\to \mathbf{GL}(n, \mathbb{C}) \\ \gamma &\mapsto M(\gamma) \end{aligned}$$

is a group homomorphism. Its image M is called the *monodromy group* of (8.1) and it is determined up to conjugation. Since an element of the differential Galois group of the differential equation (8.1) is determined by the images of a fundamental set of solutions, and analytic continuation preserves analytic relations, we can see M as a subgroup of the differential Galois group of the differential equation. It can be shown that the matrix $M(\gamma)$ depends only on the homotopy class of the path γ.

For a Fuchsian differential equation, it is proved in [6] that the differential Galois group is the Zariski closure of the monodromy group.

8.1 Matrix Differential Equations

Definition 8.3. Let K be a differential field, a matrix $A \in gl(n, K)$ and R a differential ring containing K. A matrix $Z \in G(n, R)$ such that $Z' = AZ$ is called a fundamental solution matrix of $Y' = AY$.

The same study of the evolution of the fundamental solution matrix A along analytic continuation into A_γ may be done to obtain the monodromy group of matrices $M(\gamma)$ such that

$$A_\gamma = M(\gamma)A$$

Remark 8.1.1. *From now on, we shall work with this object rather than the vector-form monodromy. One can switch from vector to matrix differential equations without loss of generality, as it is detailed in [5]. Matrix differential equations will arise naturally from first order variational equations, after this parentheses of Hamiltonian Formalism introduction.*

9 The Hamiltonian Formalism

9.1 Newton's Equations

9.1.1 1 Degree of freedom

A system with one degree of freedom is a system described by the differential equation in the real line

$$\ddot{x} = f(x) \tag{9.1}$$

The kinetic energy is the quadratic form

$$T = \frac{1}{2}\dot{x}^2 \tag{9.2}$$

The potential energy is the function

$$U(x) = -\int_{x_0}^{x} f(\epsilon)\, d\epsilon \tag{9.3}$$

The total energy is the sum

$$E = T + U \tag{9.4}$$

wisely chosen so that is a first integral, or a preserved magnitude of the system:

$$\frac{dE}{dt} = \dot{x}\ddot{x} + \frac{dU}{dx}\frac{dx}{dt} = \dot{x}f(x) - f(x)\dot{x} = 0 \tag{9.5}$$

9.1.2 Phase Flow

Equation (9.1) is equivalent to the system of two equations

$$\dot{x} = y \qquad \dot{y} = f(x) \tag{9.6}$$

We can consider the plane with coordinates x, y and call it the *phase plane*, consisting of *phase points*, which we can denote as $z = (x, y)^\top$ satisfying the phase space vector field field equation

$$\dot{z} = \begin{pmatrix} y \\ f(x) \end{pmatrix} \tag{9.7}$$

A solution to which is the motion $\varphi : \mathbb{R} \to \mathbb{R}^2$, called the *phase curve*, such that

$$x = \varphi(t) \qquad y = \dot{\varphi}(t) \tag{9.8}$$

The phase curve lies entirely on a constant energy level $E(x, y) = h$, or equivalently, $(x, y) \in E^{-1}(h)$.

This notion will motivate the definition of integrability for arbitrary dimensional systems, where one can obtain the phase curve as the preimage of a set of well behaving first integrals.

Back to our 1-dimensional problem, we isolate

$$\frac{dx}{dt} = \dot{x} = \sqrt{2(E - U(x))} \Rightarrow \int_{t_0}^{t} dt = t - t_0 = \int_{x_0}^{x} \frac{dx}{\sqrt{2(E - U(x))}} \tag{9.9}$$

9.1.3 Conservative Force Fields

Definition 9.1. A (force) vector field $\vec{F}$ is said to be conservative if it can be written as the gradient of some (potential) scalar field U:

$$\vec{F} = -\vec{\nabla}_x U \tag{9.10}$$

Definition 9.2. The work of a field $\vec{F}$ along a path γ between two points M_0, M is defined as the integral

$$W(\gamma, M, M_0) = \int_{M_0}^{M} \vec{F} \cdot d\vec{l} \tag{9.11}$$

Theorem 9.1.1. *A vector field $\vec{F}$ is conservative if and only if its work along a path between two points depends only on the two endpoints.*

Proof. If the work is well defined for the endpoint, then changing its sign we obtain the potential energy of F. Conversely, taking U the potential energy of the conservative vector field $\vec{F}$, its work results $U(M_0) - U(M)$, irrespective of the path. □

9.1.4 Central fields

Definition 9.3. A vector field in the Euclidian plane is called *central* with center at 0 if it is invariant with respect to a group of motions of the plane which fix 0.

Theorem 9.1.2. *Every central field is conservative, and its potential energy depends only on the distance to the center of the field: $U = U(r)$.*

Proof. Set $F(r) = \phi(r)\vec{r}/r$, then $W = \int_{r(M_0)}^{r(M)} \phi(r)\, dr$, irrespective of the path. □

The first Newton law sets the acceleration $\ddot{\vec{x}} = f(\vec{x}) = \vec{F}/m$ to be the force per mass unit. For a unit mass particle in a central field, its motion is defined by the radial equation

$$\ddot{\vec{r}} = \vec{\phi}(r)\vec{e}_r \tag{9.12}$$

Definition 9.4. The *angular momentum* of a unit mass point relative to the origin 0 is the vector product

$$\vec{M} = \vec{r} \times \dot{\vec{r}}$$

Theorem 9.1.3. *Under motions in a central field, the angular momentum $\vec{M}$ is conserved*

Proof. $\dot{\vec{M}} = \dot{\vec{r}} \times \dot{\vec{r}} + \vec{r} \times \ddot{\vec{r}}$. But central field equation imply $\vec{r}, \ddot{\vec{r}}$ are collinear. □

Corollary 9.1.1. *Every orbit in a central field motion is planar.*

Proof. The equation of this plane is $\vec{M} \cdot \vec{r} = (\vec{r} \times \dot{\vec{r}}) \cdot \vec{r} = 0$, $\vec{M}$ constant. □

Theorem 9.1.4. *For a conservative field with axial symmetry around the z axis, the moment of velocity relative to this axis is conserved.*

Proof. $M_z = \vec{e_z} \cdot (\vec{r} \times \dot{\vec{r}})$, then $\dot{M}_z = \vec{e_z} \cdot (\dot{\vec{r}} \times \dot{\vec{r}}) + \vec{e_z} \cdot (\vec{r} \times \ddot{\vec{r}})$. Since $\ddot{r} = \vec{F}$, $r, \ddot{r}$ lie in a plane containing the z axis, so the second term is also zero. □

9.2 Variational Principles

Remark 9.2.1. *We will work with functionals restricting to those having image in $\mathbb{R}$ and domain the infinite-dimensional space of curves.*

Definition 9.5. A functional is called *differentiable* if $\phi(\gamma + h) - \phi(\gamma) = F + R$, with F linear on h, and $R(h, \gamma) = O(h^2)$, in the sense that $|h| < \epsilon, |dh/dt| < \epsilon \Rightarrow |R| < C\epsilon^2$. The linear part F has already been introduced before as the differential, while h is referred to as the variation of the curve.

Let $\gamma = \{(t, x) : x = x(t), t_0 \leq t \leq t_1\}$ be a curve in the (t, x) plane; $\dot{x} = dx/dt$; $L = L(a, b, c)$ a differentiable function of three variables.

For example, if we chose $L = \sqrt{1 + b^2}$, ϕ is the curve length.

Theorem 9.2.1. *The functional* $\phi = \int_{t_0}^{t_1} L(x, \dot{x}, t)\, dt$ *is differentiable and its derivative is*

$$F(h) = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \right) h\, dt + \left(\frac{\partial L}{\partial \dot{x}} h \right)_{t_0}^{t_1} \tag{9.13}$$

Proof.

$$\begin{aligned}\phi(\gamma + h) - \phi(\gamma) &= \int_{t_0}^{t_1} (L(x + h, \dot{x} + \dot{h}, t) - L(x, \dot{x}, t))\, dt \\ &= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x} h + \frac{\partial L}{\partial \dot{x}} \dot{h} \right) dt + O(h^2) = F(h) + R\end{aligned}$$

Integrating by parts the $\dot{h}$ term, with $u = \dfrac{\partial L}{\partial \dot{x}}$, $v = h$,

$$\int_{t_0}^{t_1} \frac{\partial L}{\partial \dot{x}} \dot{h}\, dt = -\int_{t_0}^{t_1} h \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) dt + \left(h \frac{\partial L}{\partial \dot{x}} \right)_{t_0}^{t_1}$$

And we can now group into the three terms of the proposed $F(h)$. □

Definition 9.6. An extremal of a differentiable function $\phi(\gamma)$ is a curve γ such that $F(h) = 0$, $\forall h$.

Theorem 9.2.2. *The curve* $\gamma : x = x(t)$ *is an extremal of the functional* $\phi(\gamma)$ *if and only if, along* γ*, the Euler-Lagrange equation holds:*

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0 \tag{9.14}$$

Proof. Firstly, the term of F consisting of $\left(h \dfrac{\partial L}{\partial \dot{x}} \right)_{t_0}^{t_1}$ vanishes since $h(t_0) = h(t_1)$. Thus if $F(h) = 0$ for any continuous h with $h(t_0) = h(t_1)$, and setting $f(t) := \dfrac{d}{dt} \left(\dfrac{\partial L}{\partial \dot{x}} - \dfrac{\partial L}{\partial x} \right)$, a lemma shows that if $\int_{t_0}^{t_1} f(t)\, h(t)\, dt = 0$, then $f = 0$.

To prove the lemma, assume $\exists t^* \in (t_0, t_1) | f(t^*) > 0$, then, since f is continuous, take a ball $B(t^*, d)$ where $f(t) > c$, and construct h such that $h = 0$ outside $B(t^*, d)$, $h > 0$ inside $B(t^*, d)$ and $h = 1$ inside $B(t^*, d/2)$, giving the integral a lower bound of the rectangle area $dc > 0$, a contradiction.

Conversely, if $f = 0$, clearly $F = 0$ too. □

One may compare the Newton's equations

$$\frac{d}{dt}(m_i \dot{\vec{r}_i}) + \frac{\partial U}{\partial r_i} = 0 \tag{9.15}$$

with the Euler-Lagrange equation above, to see that

Theorem 9.2.3. *Motions of the Newton's equation system coincide with extremals of the functional*

$$\phi(\gamma) = \int_{t_0}^{t_1} L\,dt, \text{ taking } L = T - U,$$

the difference between kinetic and potential energy.

Proof. Applying the last theorem,simply taking $L = T - U$ and write its Euler-Lagrange partial derivatives equations: the kinetic term

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_i}\right) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}_i}\right) = \frac{d}{dt}(m_i r_i)$$

goes to the left-hand side of Newton's equation and the potential energy term

$$\frac{\partial L}{\partial r_i} = -\frac{\partial U}{\partial r_i}$$

to the right-hand side. □

Let $\vec{q} = (q_1, \ldots, q_{3n})$ be any coordinates in the configuration space of a system of n mass points. Then, the evolution of $\vec{q}$ with time is subject to the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{q}}}\right) - \frac{\partial L}{\partial \vec{q}} = 0$$

We refer to q_i as the generalized coordinates, $\dot{q}_i$ the generalized velocities, $\frac{\partial L}{\partial \dot{\vec{q}}}$ as the generalized momenta p_i, $\frac{\partial L}{\partial \vec{q}}$ as the generalized forces Q_i, and to $\int_{t_0}^{t_1} L(\vec{q}, \dot{\vec{q}}, t)\,dt$ as the action.

Definition 9.7. A cyclic coordinate q_i is one for which $\frac{\partial L}{\partial q_i} = 0$

Theorem 9.2.4. *The generalized moment p_i to a cyclic coordinate q_i is conserved.*

Proof. By Lagrange equation, $\dot{p}_i = \frac{dp_i}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} = 0$ □

9.3 Hamilton's equations

We consider the system of Lagrange equations $\dot{\vec{p}} = \frac{\partial L}{\partial \vec{q}}$, where $\vec{p} = \frac{\partial L}{\partial \dot{\vec{q}}}$ with a given Lagrangian function $L : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ which we will assume to be convex with respect to the second argument $\dot{\vec{q}}$.

For a convex function $f(x)$, we define its Legendre transform

$$F(p) = pf'^{-1}(p) - f(f'^{-1}(p)) \tag{9.16}$$

Theorem 9.3.1. *The system of Lagrange equations is equivalent to the system of 2n first order equations, called the Hamilton equations*

$$\dot{\vec{p}} = -\frac{\partial H}{\partial \vec{q}}$$
$$\dot{\vec{q}} = \frac{\partial H}{\partial \vec{p}},$$

where $H(\vec{p}, \vec{q}, t) = \vec{p} \cdot \dot{\vec{q}} - L(\vec{q}, \dot{\vec{q}}, t)$ *is the Legendre transform of the Lagrangian function viewed as a function of* $\dot{\vec{q}}$.

Proof. First, we check that H is the Legendre transform of $f(\vec{x}) := L(\dot{\vec{q}})$.

$$f'(x_i) = \frac{\partial L}{\dot{q}_i} = \frac{\partial}{\partial q_i}\left(\frac{1}{2} m_i \dot{q_i}^2\right) = m_i \dot{q}_i = m_i x \Rightarrow f'^{-1}(p_i) = p_i/m_i = \dot{q}_i = x_i$$

$$f(f'^{-1}(p_i)) = f(x_i) = L(\dot{q}_i)$$

Reconstructing for all components and applying (9.16) results the proposed H.

Now we check the equivalence: differentiating both separately:

$$dL = \sum_i \left(\frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i\right) + \frac{\partial L}{\partial t} dt$$

reexpressing the term

$$\frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i = p_i d\dot{q}_i = d(p_i \dot{q}_i) - \dot{q}_i dp_i$$

so that

$$\begin{aligned} dH = d\left(-L + \sum_i p_i \dot{q}_i\right) &= \sum_i \left(-\frac{\partial L}{\partial q_i} dq_i + \dot{q}_i dp_i\right) - \frac{\partial L}{\partial t} dt \\ &= \sum_i \left(\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i\right) + \frac{\partial H}{\partial t} dt \end{aligned}$$

and simply identifying differential base coefficients, applying Lagrange equations

$$\begin{cases} \dfrac{\partial H}{\partial q_i} = & -\dfrac{\partial L}{\partial q_i} = -\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_i}\right) = -\dot{p}_i \\ \\ \dfrac{\partial H}{\partial p_i} = & \dot{q}_i \\ \\ \dfrac{\partial H}{\partial t} = & -\dfrac{\partial L}{\partial t} \end{cases}$$

□

Corollary 9.3.1. *Expanding the differential of H and applying Hamilton's equations*

$$\frac{dH}{dt} = \frac{\partial H}{\partial \vec{p}}\dot{\vec{p}} + \frac{\partial H}{\partial \vec{q}}\dot{\vec{q}} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial \vec{p}}\left(-\frac{\partial H}{\partial \vec{q}}\right) + \frac{\partial H}{\partial \vec{q}}\left(\frac{\partial H}{\partial \vec{p}}\right) + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$$

and so for an autonomous system H is constant.

We can introduce again for hamiltonian systems the symmetries point of view, a particular case of the Noether's Theorem: for every cyclic coordinate q_i of H, there is a correspondent constant coordinate p_i:

Corollary 9.3.2. $\frac{\partial H}{\partial q_i} = 0$ *cyclic* $\Rightarrow \dot{p}_i = \frac{\partial H}{\partial q_i} = 0$ *constant.*

9.4 Liouville Theorem

Definition 9.8. The 2n-dimensional space with coordinates $p_1, \ldots, p_n; q_1, \ldots, q_n$ is called the phase space.

Definition 9.9. The phase flow is the one-parameter group of transformations of phase space

$$g^t : (\vec{p}(0), \vec{q}(0)) \to (\vec{p}(t), \vec{q}(t))$$

for $(\vec{p}(t), \vec{q}(t))$ the solution to Hamilton equation's system.

The phase flow is clearly a group for the composition operation, corresponding to the additive group of time $(\mathbb{R}, +)$: $e_g = g^0$, $(g^t)^{-1} = g^{-t}$ and $g^{t_1} \circ g^{t_2} = g^{t_1+t_2}$.

Theorem 9.4.1. *The phase flow is volume preserving:*

$$vol(g^t D) = vol(D).$$

More generally, given a system of ordinary differential equations

$$\dot{\vec{x}} = \vec{f}(\vec{x})$$

whose solution may be extended to the whole time axis. Let $\{g^t\}$ be its group of transformations:

$$g^t(\vec{x}) = \vec{x} + \vec{f}(\vec{x})t + O(t^2), \text{ as } t \to 0$$

Denote $D(t) = g^t D(0)$ the region, initially $D(0)$, transformed by the phase flow g^t, and $v(t)$ the volume of $D(t)$.

We also introduce the divergence of a vector field $\vec{f}$ defined as

$$div(\vec{f}) = \vec{\nabla} \cdot \vec{f} = \sum_i \frac{\partial f_i}{\partial x_i} = Tr\left(\frac{\partial f_i}{\partial x_j}\right)_{ij} = Tr\left(\frac{\partial \vec{f}}{\partial \vec{x}}\right)$$

Theorem 9.4.2. *If $div(\vec{f}) = 0$, then g^t preserves volume: $v(t) = v(0)$.*

Proof.

Lemma 9.4.0.1. *The rate of change of volume at time 0 is the integral of the divergence along the initial region $D(0)$.*

$$(dv/dt)|_{t=0} = \int_{D(0)} div\vec{f}\, d\vec{x}$$

Proof. By the Jacobian formula

$$v(t) = \int_{D(0)} \left| \frac{\partial g^t(\vec{x})}{\partial \vec{x}} \right| d\vec{x}$$

and

$$\frac{\partial g^t(\vec{x})}{\partial \vec{x}} = I + \frac{\partial \vec{f}}{\partial \vec{x}} t + O(t^2), \text{ as } t \to 0$$

so

$$(dv/dt)|_{t=0} = \int_{D(0)} d\vec{x}\, \frac{d}{dt} \left| I + \left(\frac{\partial \vec{f}}{\partial \vec{x}} t \right) \right| =_{(*)} \int_{D(0)} d\vec{x} \left(\sum_i \frac{\partial f_i}{x_i} \right) = \int_{D(0)} div(\vec{f})\, d\vec{x}$$

, where for $(*)$ we have used the following lemma. □

Lemma 9.4.0.2. *For a square matrix A we have $det(I + tA) = 1 + tTr(A)t + O(t^2)$*

Proof. This is is left as an exercise in [8], here we use the characteristic polynomial expansion

$det(B - tI) = (-1)^n t^n + (-1)^{n-1} Tr(B) t^{n-1} + \ldots + det(B)$, for $B = -t^2 A$

$det(B - tI) = t^n det(B/t - I) = t^n det(-At - I) = (-t)^n det(I + tA)$

$det(I + tA) = (-1)^n t^{-n}((-t)^n + (-t)^{n-1} Tr(-t^2 A) + \ldots + det(-t^2 A) = 1 + tTr(A) + \ldots + O(t^2)$ □

Now that lemmas are proven, if the divergence vanishes, so does its integral. □

This is precisely the sufficient condition we shall check for hamiltonian systems:

We construct the vector $\vec{z}$ of $2n$ generalized coordinates concatenating $\vec{q}$ and $\vec{p}$ satisfying the equation

$$\dot{\vec{z}} = J \vec{\nabla}_{\vec{z}} H =: f(\vec{z}), \text{ for } J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

Now we can express the divergence from Hamilton's equations

Proof.

$$div(\vec{f}) = \frac{\partial}{\partial \vec{p}} \left(\frac{-\partial H}{\partial \vec{q}} \right) + \frac{\partial}{\partial \vec{q}} \left(\frac{-\partial H}{\partial \vec{p}} \right) = 0$$

□

9.5 Integrability

Definition 9.10. A function F is said to be a first integral of a system if its value is preserved along the solutions of the system.

Definition 9.11. The Poisson bracket $\{F, H\}$ of functions F and H given on a symplectic manifold (M^{2n}, ω^2) [8] is the derivative of the function F in the direction of the phase flow with Hamiltonian function H:

$$\{F, H\} = \frac{d}{dt}\bigg|_{t=0} F(g_H^t(x))$$

We use now the isomorphism between 1-forms and vector fields on a symplectic manifold defined by

$$\omega^2(\vec{u}, J\omega^1) = \omega^1(\vec{u})$$

which gives us that the velocity of the phase flow g_H^t is JdH. Consequently, the Poisson bracket of the functions F, H is equal to the following

$$\{F, H\} = dF(JdH) = w^2(JdH, JdF)$$

Now, in the canonical space $\mathbb{R}^{2n}$ of coordinates $(\vec{p}, \vec{q})$ endowed with the form $w^2(\vec{u}, \vec{v}) = (J\vec{u}, \vec{v})$ the Poisson bracket takes the form

$$\{F, H\} = \sum_{i=1}^{n} \frac{\partial H}{\partial q_i}\frac{\partial F}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial F}{\partial q_i} = \vec{F}_z J \vec{H}_z$$

Proposition 9.1. *A function F is a first integral of a system with hamiltonian function H if and only if it commutes via Poisson Bracket with H. That is, $\{F, H\} = 0$ is identically equal to zero.*

Proof. Expanding the differential and applying Hamilton equations

$$dF = \sum_{i=1}^{n} \frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial p_i}\dot{p}_i = \sum_{i=1}^{n} \frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} + \frac{\partial F}{\partial p_i}\left(-\frac{\partial H}{\partial q_i}\right) = \{F, H\}$$

or in a more compact notation $dF = \vec{F}_z \cdot \dot{\vec{z}} = \vec{F}_z J \vec{H}_z = \{F, H\}$ □

This motivates the following definition.

Definition 9.12. Two functions f_1, f_2 on a symplectic manifold are in involution if their Poisson bracket is equal to zero. A set of functions $\{f_1, \ldots, f_n\}$ is in involution if every pair in this set is in involution.

Theorem 9.5.1 (Liouville-Arnold Theorem)**.** *Consider a system of $2n$-dimensional phase space M, consider a set of functions $F_1 = H, \ldots, F_n$ in involution. Consider a level set of these functions*

$$M_f = \{x : F_i(x) = f_i, i = 1, \ldots, n\}$$

Assume that the functions F_i are functionally independent on M_f, ie, the 1-forms df_i, $i = 1, 2, \ldots, n$, are linearly independent at each point of M_f. Then

- *M_f is a smooth manifold which is invariant under the phase flow with Hamiltonian $H = F_1$.*

- *If the manifold M_f is compact and connected, then it is diffeomorphic to the n-dimensional torus $T^n = S^1 \times \ldots \times S^1 = \{\vec{\varphi} = (\varphi_1, \ldots, \varphi_n) \ mod\ 2\pi\}$*

- *The phase flow with Hamiltonian H is*

$$\frac{d\vec{\varphi}}{dt} = \vec{\omega}$$

- *Most importantly, the canonical equations with Hamiltonian H can be integrated by quadratures.*

10 Morales-Ramis-Simó Theorem

Next we will give the notion of integrability from Morales-Ramis-Simó theory. We will assume the first integrals to be rational functions and the space M to be a Zariski-open set of a complex projective space.

Definition 10.1. A system of linear ordinary differential equations

$$\dot{\xi} = A\xi \tag{10.1}$$

with $A \in Mat(m, K)$ a matrix with coefficients on a differential field K, is said to be integrable if its general solution can be obtained combining quadratures (integrals), exponential of quadratures and algebraic functions.

Equivalently, if the Picard-Vessiot extension of K, $L := K\langle u_{ij}\rangle$, differentially generated by the coefficients u_{ij} of the fundamental solution matrix of (10.1), is what is known as a Liouville extension.

Theorem 10.0.1. *A linear differential equation is integrable if, and only if, the identity component G^0 of its Galois group G is a solvable group. In particular, if the identity component is commutative, the equation is integrable.*

Theorem 10.0.2. *If the Hamiltonian system*

$$\dot{x} = X_H(x) \tag{10.2}$$

is completely integrable with meromorphic first integrals in a neighbourhood of Γ, the Riemann surface immersed in M defined by a particular solution that is not an equilibrium point, then the identity component G^0 of the Galois group of the first order variational equation of (10.2) is commutative. In particular, its monodromy group is virtually commutative.

10.1 Variational Equations

For the sake of easing comparison, we stated the *Morales-Ramis-Simó Theorem* next to the *Liouville-Arnold Theorem.* We will now define some necessary notions concerning the variational equations and the idea behind the *Morales-Ramis-Simó Theorem.*

Variational equations arise naturally from ordinary equations when studying the behaviour of a flow on the tangent space around a point. Consider a system of dimension n in its Cauchy standard form:

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases} \tag{10.3}$$

After a time lapse t the initial point x_0 will be transported in a motion tangent to the flow f at each point. Now we ask about the evolution of a point close to x only differing by a small vector ξ:

$$\dot{x} + \dot{\xi} = (x+\xi)^{\cdot} = f(x+\xi) =_{(*)} f(x) + Df(x)\xi + O(\|\xi\|^2), \tag{10.4}$$

where at $(*)$ we used Taylor expansion. Cancelling terms from (10.3), we get the *first order variational vector equations*:

$$\begin{cases} \dot{\xi} = Df(x)\xi \\ \xi(0) = \xi_0 \end{cases} \tag{10.5}$$

It is well known that the flow tangent vector is always a solution of equation (10.5):

$$\dot{f(x)} = \frac{f(x)}{dt} = Df(x)\frac{dx}{dt} = Df(x)f(x)$$

For this reason, Poincaré sections are taken transversal to the flow and sometimes only the normal part of the variational equations is considered.

To effectively analyse the motion of the vector ξ we can write it as a linear transformation from its initial value

$$\xi(t) = A(t)\xi_0 \ .$$

From (10.5)

$$\begin{cases} \dot{A}\xi_0 = \dot{\xi} = Df(x)\xi = Df(x)A\xi_0 \\ A(0)\xi_0 = \xi(0) = \xi_0 = I_n\xi_0 \end{cases}$$

since the definition of ξ_0 is general, we can cancel it to obtain the *first order variational matrix equations*:

$$\begin{cases} \dot{A} = Df(x)A \\ A(0) = I_n \end{cases} \tag{10.6}$$

$A(t)$ gives information about the evolution of any vector ξ_0 into $\xi(t)$ by flow f in time t, that is the whole tangent space on x_0. In algebraic terms, the columns of A are the n linearly independent solutions of the first order variational vector equations; starting at distinct canonical coordinates : the columns of the identity.

The same procedure of varying the initial conditions may be done at any order in the following form: for a solution x of (10.3), we take

$$\begin{cases} x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \epsilon^3 x_3(t) + \dots \\ x_0 \quad = x_0^{(0)} \quad + \epsilon x_1^{(0)} \quad + \epsilon^2 x_2^{(0)} \quad + \epsilon^3 x_3^{(0)} \quad + \dots \end{cases}$$

deriving,

$$\begin{aligned} \sum_{j\geq 0} \dot{x_j}(t)\epsilon^j = \dot{x}(t) = f(x) = f\left(\sum_{j\geq 0} x_j(t)\epsilon^j\right) = f\left(x_0(t) + \epsilon \sum_{j\geq 0} x_j(t)\epsilon^j\right) =_{(*)} \\ = f(x_0) \\ + Df(x_0)(\epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \dots) \\ + \frac{1}{2!}D^2 f(x_0)(\epsilon^2(x_1, x_1) + 2\epsilon^3(x_1, x_2) + \dots) \\ + \frac{1}{3!}D^3 f(x_0)(\epsilon^3(x_1, x_1, x_1) + \dots) \\ + \dots \end{aligned}$$

and identifying powers of ϵ leads to equations

$$\epsilon^0 : \ \dot{x_0} = f(x_0) \tag{10.7}$$
$$\epsilon^1 : \ \dot{x_1} = Df(x_0)x_1 \tag{10.8}$$
$$\epsilon^2 : \ \dot{x_2} = Df(x_0)x_2 + \frac{1}{2}D^2 f(x_0)(x_1, x_1) \tag{10.9}$$
$$\epsilon^3 : \ \dot{x_3} = Df(x_0)x_3 + D^2 f(x_0)(x_1, x_2) + \frac{1}{3}D^3 f(x_0)(x_1, x_1, x_1) \tag{10.10}$$

Note that the successive objects $D^j f(x_0) : (\mathbb{R}^n)^j \to R^n$ are differential j-forms.

10.1.1 Idea behind Morales-Ramis-Simó Theorem

Consider two consecutive motions of the form

$$\begin{aligned} (1)\ \dot{x} = Ax,\ x \in \mathbb{R}^n,\ A \in \mathcal{M}_{n\times n}(\mathbb{R}) \\ (2)\ \dot{y} = By,\ y \in \mathbb{R}^n,\ A \in \mathcal{M}_{n\times n}(\mathbb{R}). \end{aligned}$$

We integrate for (1) with time t: $x(t) = e^{At}x_0$, and then (2) too:

$$y(t) = e^{Bt}x(t) = e^{Bt}e^{At}x_0 =^? e^{(A+B)t}x_0,$$

which is correct only when $[A, B] = AB - BA = 0$.

First integrals F, G induce flows of the motion of a point in a system with Hamiltonians F, G in time t, τ: $\varphi^F(t), \varphi^G(\tau)$, which commute if their Poisson Brackets do $0 = \{F, G\} := (\nabla F)^T J_n (\nabla G)^T$, see [8], page 211 about involution.

If we now take paths γ_i in complex time $t \in \gamma_i \subseteq \mathbb{C}$ between four points with identical image by a solution $z(t)$ of our system around some regular singularity of $z(t)$, the monodromy matrices M_{γ_i} obtained by integrating the variational equations (at first order $\dot{A} = DFA, A(0) = I$, or the group obtained at any order) form a group describing the behaviour of flows along the image solutions $z(\gamma_i)$ that return to the same point $z(\gamma_i(0))$ modifying its neighbourhood structure. It is natural to think that this subgroup of the Galois group of the equation, containing the identity, shall be commutative: $M_{\gamma_1} M_{\gamma_2} M_{\gamma_1}^{-1} M_{\gamma_2}^{-1} = I$.

11 Example of Hamiltonian System

11.1 Integrable System

Consider a Hamiltonian in the canonical couples $(x_i, y_i), i = 1, 2$

$H_0(x_1, x_2, x_3, x_4) = F_1 + \nu F_2$, $F_1 = x_1y_2 - x_2y_1$, $F_2 = (R_1^2 - R_2^2)/2 + R_2^4/4$, with $\nu > 0$ a parameter and $R_1^2 = x_1^2 + x_2^2$, $R_2^2 = y_1^2 + y_2^2$.

a) The Poisson bracket is $\{F_1, F_2\} = \sum_{i=1}^{n} \frac{\partial F_1}{\partial x_i}\frac{\partial F_2}{\partial y_i} - \frac{\partial F_1}{\partial y_i}\frac{\partial F_2}{\partial x_i}$, where $n = 2$. We check

$$\begin{cases} \frac{\partial F_1}{\partial x_1} = y_2, \quad \frac{\partial F_1}{\partial x_2} = -y_1, \quad \frac{\partial F_1}{\partial y_1} = -x_2, \quad \frac{\partial F_1}{\partial y_2} = x_1 \\ \frac{\partial F_2}{\partial x_1} = x_1, \quad \frac{\partial F_2}{\partial x_2} = x_2, \quad \frac{\partial F_2}{\partial y_1} = -y_1 + y_1^3 + y_1y_2^2, \quad \frac{\partial F_2}{\partial y_2} = -y_2 + y_2^3 + y_2y_1^2 \end{cases}$$

$\Rightarrow \{F_1, F_2\} = y_2y_1(R_2^2 - 1) + x_2x_1 + (-y_1)y_2(R_2^2 - 1) - x_1x_2 = 0$

We check integrability: the existance of a maximal set of first integrals that commute with each other via the Poisson bracket and are functionally independent almost everywhere.

The gradients of F_1 and F_2 are linearly independent vectors except for the set $\{R_2^2 - 1 = (\frac{x_2}{y_1})^2 = (\frac{x_1}{y_2})^2\}$ of zero measure.

F_1, F_2 are indeed first integrals since $\{H, F_2\} = \{F_1 + \nu F_2, F_2\} = \{F_1, F_2\} + \nu\{F_2, F_2\} = 0 + 0$, and same for $\{H, F_1\} = 0$.

b) The origin is a fixed point of complex saddle-type. We write the Hamilton-Jacobi equations $\dot{q}_i = \frac{\partial H}{\partial p_i}$, $\dot{p}_i = -\frac{\partial H}{\partial q_i}$

$$\begin{cases} \dot{x_1} = -x_2 + \nu y_1(R_2^2 - 1) \\ \dot{x_2} = x_1 + \nu y_2(R_2^2 - 1) \\ \dot{y_1} = -y_2 - \nu x_1 \\ \dot{y_2} = y_1 - \nu x_2 \end{cases}$$

Obviously, if all coordinates are zero, the flow is zero. The origin is a fixed point. To examine its behaviour around the origin we'll find the linear part and study its matrix M. Let $\vec{r} = (q_1, q_2, p_1, p_2)$, we have a flow, independent of the time, like $\dot{\vec{r}} = \vec{f}(\vec{r}) \approx \vec{f}(\vec{0}) + D\vec{f}(\vec{r})|_{\vec{0}}(\vec{r} - \vec{0}) = M\vec{r}$, where $M_{i,j} = \dfrac{\partial \dot{r}_i}{\partial r_j}|_{\vec{0}} = J_4D^2H$, with $J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Then $M = \begin{pmatrix} 0 & -1 & -\nu & 0 \\ 1 & 0 & 0 & -\nu \\ -\nu & 0 & 0 & -1 \\ 0 & -\nu & 1 & 0 \end{pmatrix} = -\begin{pmatrix} J_2 & \nu I_2 \\ \nu I_2 & J_2 \end{pmatrix}$

To calculate its eigenvalues, by blocks, $det(M - \lambda I_4) = det((-J_2 - \lambda I_2)^2 - \nu^2 I_2) = det(\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix}^2 - \nu^2 I_2) = det(\begin{pmatrix} \lambda^2 - (\nu^2 + 1) & 2\lambda \\ -2\lambda & \lambda^2 - (\nu^2 + 1) \end{pmatrix})$

$= (\lambda^2 - (\nu+1))^2 + 4\lambda^2 = 0 \Leftrightarrow \lambda^2 = \pm 2\lambda i + \nu^2 + 1 \Leftrightarrow \lambda = \frac{\pm 2i \pm \sqrt{-4+4(\nu 2+1)}}{2} =$
$\pm\nu \pm i$

, a complex saddle. The corresponding eigenvectors form a base:

$$\begin{cases} \lambda_1 = +i+\nu \to v_1 = (-i,-1,+i,+1) \\ \lambda_2 = +i-\nu \to v_2 = (+i,+1,+i,+1) \\ \lambda_3 = -i-\nu \to v_3 = (-i,+1,-i,+1) \\ \lambda_4 = -i+\nu \to v_4 = (+i,-1,-i,+1) \end{cases}$$

Those with eigenvalue positive real part correspond to the unstable manifold W^u: v_1, v_4, and those with negative real part, to the stable one W^s: v_2, v_3. However, in this case, the point doesn't have an open neighbourhood homeomorphic to an open in $\mathbb{R}^4$ (a manifold).

c) We see first that $F_i(v_j) = 0$ for $i = 1, 2$ and$j = 1, 2, 3, 4$.

$$\begin{cases} F_1(v_1) = (-i)1 - (-1)i = 0 \\ F_1(v_2) = i(-1) - 1i = 0 \\ F_1(v_3) = -i1 - 1(-i) = 0 \\ F_1(v_4) = i1 - (-1)(-i) = 0 \end{cases},$$

$$F_2 = \frac{x_1^2 + x_2^2 - (y_1^2 + y_2^2)}{2} + \frac{(y_1)^4 + (y_2)^4 + 2y_1^2 y_2^2}{4}$$

$$F_2(v_1) = \frac{(-i)^2 + (-1)^2 - i^2 - 1^2}{2} + \frac{(-i)^4 + (-1)^4 + 2i^2 1^2}{4} = 0$$

Since $v_i = (\pm i, \pm 1, \pm i, \pm 1)$ and F_2 depends on the square of its components, it is zero for the rest of eigenvectors.

Since F_1, F_2 are first integrals, they will be constantly zero along the orbits containing the origin, which form the unstable manifold and stable manifold (backwards in time). Thus, both manifolds will be contained in the preimages $F_1^{-1}(0) \cap F_2^{-1}(0)$. Next, we will see that they coincide.

Making a suitable change of coordinates,

$$\begin{cases} x_1 = R_1 \cos(\theta_1) \\ x_2 = R_1 \sin(\theta_1) \\ y_1 = R_2 \cos(\theta_2) \\ y_2 = R_2 \sin(\theta_2) \end{cases}$$, imposing $F_1 = 0$, $F_2 = 0$, for points not in the origin,

$F_1 = x_1 y_2 - x_2 y_1 = R_1 R_2(\cos(\theta_1)\sin(\theta_2) - \cos(\theta_2)\sin(\theta_1)) = R_1 R_2 \sin(\theta_2 - \theta_1) = 0 \Leftrightarrow \theta_2 = \theta_1$ or $\theta_2 = \theta_1 + \pi$, a line in the angle plane.

$F_2 = \dfrac{R_1^2 - R_2^2}{2} + \dfrac{R_2^4}{4} = 0 \Leftrightarrow R_1 = \pm\sqrt{R_2^2 - R_2^4/2}$. This curve is the separatrix of F_2 in the radius plane of $F_1 = 0$. We observe the typical structure of Liouville-integrable problems on levels of $F_2 = c$. For $c < 0$ we obtain two center circles around $(R_2, R_1) = (\pm 1, 0)$, for $c = 0$, the separatrix, and for $c > 0$ a larger ellipse containing it.

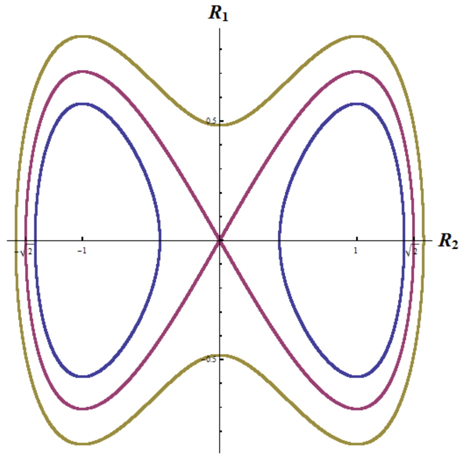

Figure 1: Curves in the radius plane of unperturbed motion

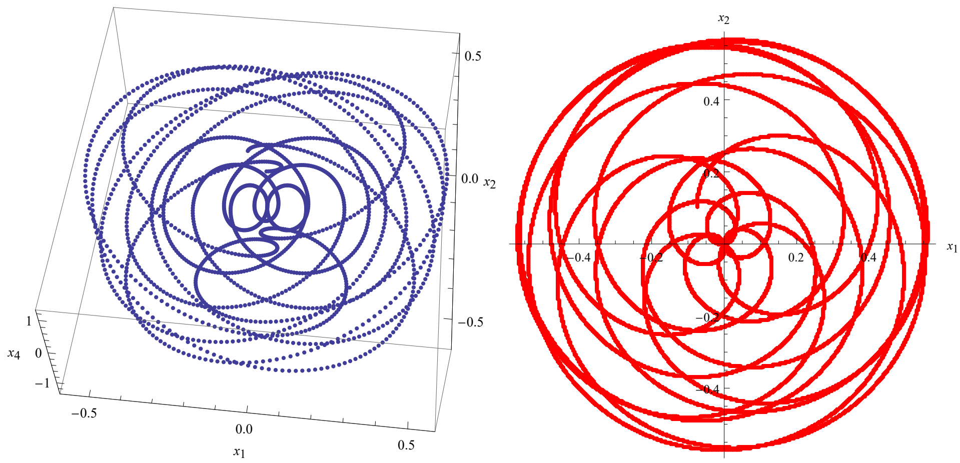

Figure 2: Orbit for $F_2 < 0$

The orbits in (x_1, x_2, y_1, y_2) of the curves of the above figure lie on an Arnold torus in $\mathbb{R}^4$. For example, for the curve of $F_2 < 0$, we plot the points (x_1, x_2, y_2) and observe they form the intersection of a torus of $\mathbb{R}^4$ into $\mathbb{R}^3$. We also project it into the plane (x_1, x_2) to observe its quadruple symmetry.

It is a common fact about the orbits of $F_1 = 0, F_2 \neq 0$ plotted in the position space that they traverse the region $R_1 = 0$, but, unlike the separatrix, never the origin $R_1 = R_2 = 0$, that is, they return to the positions origin with non-zero velocity. This fact as well as the bounded range of R_1, R_2 gives rise to these symmetries observed.

An orbit of $F_2 < 0$ exhibits a triple symmetry (see figure below).

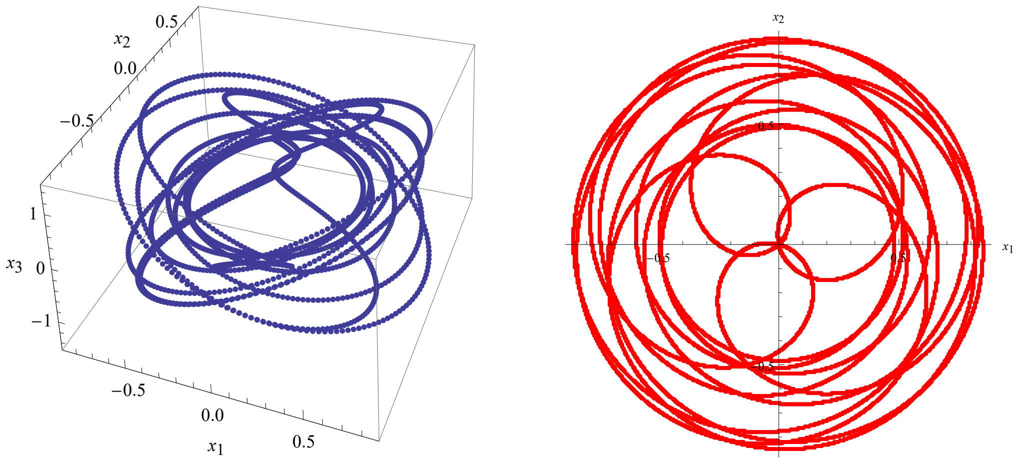

Figure 3: Orbit for $F_2 > 0$

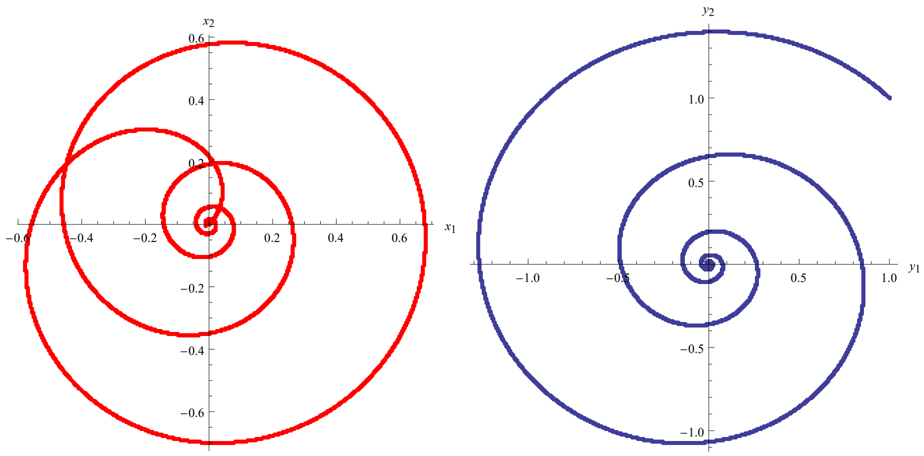

Figure 4: Orbit for $F_2 = 0$

Finally, we plot an orbit $F_1 = F_2 = 0$ lying on the separatrix: it starts at origin position with $R_2 = \sqrt{2}$ modulus velocity, goes all the way up to R_1^{max} through the stable manifold and back to $R_1 = 0$, this time with modulus velocity $R_2 = 0$, a fixed point.

We proceed to show that there is no fixed point inside the separatrix apart from the origin, so the orbits do not get "stuck" along it, but they follow it back to the origin, concatenating the unstable with the unstable manifolds.

We will check this fact seeing that radial velocities are never zero except for the origin.

$$2R_2 dR_2 = 2y_1 dy_1 + 2y_2 dy_2 \Rightarrow \frac{\partial R_2}{\partial y_i} = y_i / R_2 \Rightarrow \vec{\nabla}_{\vec{y}} R_2 = \vec{y}/R_2$$

$$\dot{R}_2 = \frac{\partial R_2}{\partial y_1}\dot{y}_1 + \frac{\partial R_2}{\partial y_2}\dot{y}_2 = \vec{\nabla}_{\vec{y}} R_2 \cdot \dot{\vec{y}} = \frac{1}{R_2}(y_1(-y_2 - \nu x_1) + y_2(y_1 - \nu x_2))$$
$$= \frac{-\nu}{R_2}(x_1 y_1 + x_2 y_2) = -\nu R_1 \cos(\theta_2 - \theta_1) = \mp \nu R_1 = 0 \Leftrightarrow R_1 = 0$$

By the same procedure, $\dot{R}_1 = \nu(R_2^2 - 1)R_2 \cos(\theta_2 - \theta_1) = \pm\nu(R_2^2 - 1)R_2 = 0 \Leftrightarrow R_2 = \pm 1, 0$ but $(R_1, R_2) = (0, \pm 1)$ doesn't belong to the curve. The only solution for $\dot{R}_1 = \dot{R}_2 = 0$ is the origin.

Next, we give an explicit solution for the separatrix solution. We already know the curves in the angle plane and radius plane, but now we will give it in terms of time. Using

$$\begin{cases} \dot{y_1} = -\dfrac{\partial H}{\partial x_1} = -(y_2 + \nu x_1) \\ \\ \dot{y_2} = -\dfrac{\partial H}{\partial x_2} = -(-y_1 + \nu x_2) \\ \\ \theta_2 = \arctan(\dfrac{y_2}{y_1}) \end{cases}$$

, we have

$$\dot{\theta}_2 = \frac{\dot{y_2}y_1 - y_2\dot{y_1}}{y_1^2 + y_2^2} = \frac{(y_1 - \nu x_2)y_1 - (y_2 + \nu x_1)y_2}{y_1^2 + y_2^2} = \frac{\nu F_1 + R_2^2}{R_2^2} = 1$$

So simply $\theta_2(t) = t + \beta$, with β an initial phase, and $\theta_1(t) = \pm\theta_2(t)$.

For the radius, we recover

$$\begin{cases} \dot{R}_2 = \mp R_1 \\ R_1 = R_2\sqrt{1 - \dfrac{R_2^2}{2}} \end{cases} \Rightarrow \quad \frac{dR_2}{R_2\sqrt{1 - \dfrac{R_2^2}{2}}} = dt$$

and so

$$R_2(t) = \frac{\sqrt{2}}{\cosh(\nu t)}, \quad R_1(t) = \frac{\sqrt{2}}{\cosh(\nu t)}\sqrt{1 - \frac{1}{\cosh^2(\nu t)}} = \frac{\sqrt{2}\sinh(\nu t)}{\cosh^2(\nu t)}$$

choosing necessarily positive radius sign and so $\theta_1(t) = -\theta_2(t)$

If we use a Poincaré section defined by the maximum of R_2, which is $\sqrt{2}$, then the intersection with the manifolds is at the single point of the curve $(R_1, R_2) = (0, \sqrt{2})$, that is, $x_1 = x_2 = 0, R_2^2 = y_1^2 + y_2^2 = 2$.

11.2 Small Perturbation

Now we introduce a perturbation to $H = H_0 + \epsilon y_1^5$, which will only affect the field of $\dot{x_1} = \dfrac{\partial H}{\partial y_1} = \dfrac{\partial H_0}{\partial y_1} + 5\epsilon y_1^4$.

We compute the image of the circle around the origin $(y_1, y_2) = \rho(\cos(\theta), \sinh(\theta))$ by the Poincaré map in the unstable manifold , with $x_i = -y_i, i = 1, 2$, and stable manifold with $x_i = y_i, i = 1, 2$.

For the unperturbed case, the circle shape would be preserved. However, the perturbation modifies the image from a circle into a twisted circle in the positions x_1, x_2. Velocities y_1, y_2 stay in a circle of radius $\sqrt{2}$.

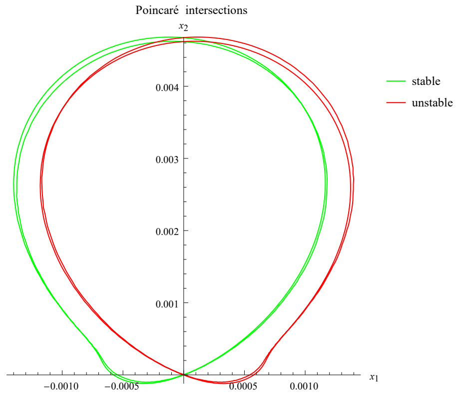

Figure 5: Deformed circle by perturbed flow into Poincaré section

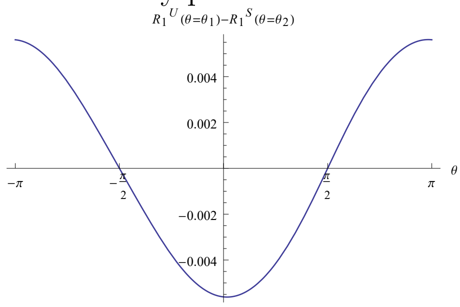

Figure 6: Radius difference along angle

There is an evident symmetry between them, which we can visualize with the curve of the radius $R_2 = \sqrt{y_1^2 + y_2^2}$ difference between stable and unstable along the angle $\theta_2 = \arctan(y_2/y_1)$.

11.3 Energy Splitting via Melnikov Integral

Let us now give a measure of the splitting from the perturbation $H = H_0 + \epsilon y_1^5$. We know that for the unperturbed case the unstable and stable manifolds coincide $W^U = W^S$ and that they intersect the Poincaré section $(R_1, R_2) = (0, \sqrt{2})$ in a circle each. With the perturbation, these circles break. However, the energy $H_0 = F_1 + \nu F_2$ is still constant so the variations of F_1 and F_2 are related via

$$\Delta H = \Delta F_1 + \nu \Delta F_2 + \Delta(\epsilon y_1^5) = 0 \Rightarrow \Delta F_2 = -\frac{\Delta F_1}{\nu} - \frac{\epsilon}{\nu}\Delta(y_1^5)$$

Now, since the perturbation only affects the variable $\dot{x_1}$ by a new term $5\epsilon y_1^4$

$$\dot{x_1} = \frac{\partial H}{\partial y_1} = \frac{\partial H_0}{\partial y_1} + 5\epsilon y_1^4$$

then it will cause a variation to the quantity $F_1 = x_1 y_2 - x_2 y_1$ by a term $5\epsilon y_1^4 y_2$:

$$\dot{F_1} = \dot{x_1} y_2 + x_1 \dot{y_2} - \dot{x_2} y_1 - x_2 \dot{y_1} = 0 + 5\epsilon y_1^4 y_2$$

Let $z = (x_1, x_2, x_3, x_4)^T$ and let z_0 the solution of H_0, then the solution to $H_0 + \epsilon H_1$ will be of the form $z_0(t) + \epsilon z_1(t) + \epsilon^2 z_2(t) + \ldots$

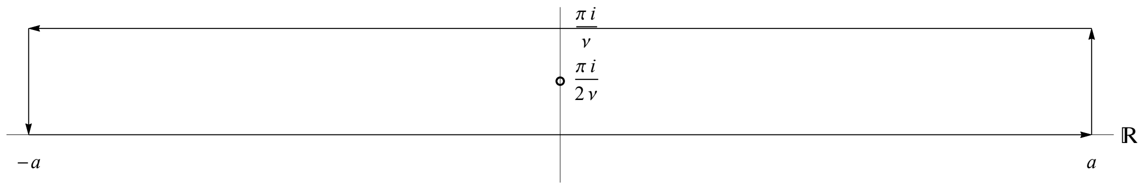

Figure 7: Path γ

We now integrate the variation of F_1 along the stable manifold, that is, going back in time from $(R_1, R_2, \theta_1, \theta_2) \approx (0, \sqrt{2}, -\beta, \beta)$:

$$\Delta F_1^(W^S) = \int_{t=0}^{-\infty} 5\epsilon y_1^4 y_2|_{z_0+\epsilon z_1}\, dt = 5\epsilon \int_{t=0}^{-\infty} y_1^4 y_2|_{z_0}\, dt + O(\epsilon^2)$$

For the unstable,

$$\Delta F_1^(W^U) = 5\epsilon \int_{t=0}^{\infty} y_1^4 y_2|_{z_0}\, dt \ ,$$

so the total variation, or splitting, is

$$\boxed{\Delta F_1^(W^U) - \Delta F_1^(W^S) = 5\epsilon \int_{-\infty}^{\infty} \frac{\sqrt{2}\cos^4(t+\beta)\sin(t+\beta)}{\cosh^5(\nu t)}\, dt = \Gamma(\epsilon, \nu, \beta)},$$

known as a *Melnikov* integral. It has a singularity at $\cosh(\nu t) = 0 \Leftrightarrow \cos(-i\nu t) = 0$ $\Leftrightarrow -i\nu t = \frac{\pi}{2} + k\pi$, giving $t_k = \frac{i\pi}{2\nu} + \frac{ik\pi}{\nu}$ the first of which is $\boxed{t_0 = \frac{i\pi}{2\nu}}$.

To compute Γ, we proceed by breaking the power, ignoring ν,

$$\begin{aligned} \cos^4(t) &= \left(\frac{e^{it}+e^{-it}}{2}\right)^4 = (e^{4it} + 4e^{2it} + 6 + 4e^{-2it} + e^{-4it})\frac{1}{2^4} \\ &= \frac{2\cos(4t) + 8\cos(2t) + 6}{16} = \frac{\cos(4t) + 4\cos(2t) + 3}{8} \end{aligned}$$

We also observe that $\lim_{t\to\infty} \cosh(\nu t) = \infty$ making the integral vanish on its boundaries.

Γ is an integral along the real line, but it can be seen as a term of the integral along the closed curve γ around the singularity t_0 reaching a sufficiently large (take limits after) segment around 0 of $\mathbb{R}$, say $[-a, a]$, vertical segments $[\pm a, \pm a + 2t_0]$ and another horizontal segment $[-a + 2t_0, a + 2t_0]$.

We can apply the residue theorem for a closed rectifiable curve γ avoiding the singularities $a_1, \ldots, a_n$ with index $\mathrm{Ind}(\gamma, a_k)$ if a_k lies inside γ:

$$\oint_\gamma f(z)\, dz = 2\pi i \sum_{k=1}^{n} \mathrm{Ind}(\gamma, a_k)\, \mathrm{Res}(f, a_k)$$

for our case, the only singularity being t_0 of trivial index one. Once we know the residue,

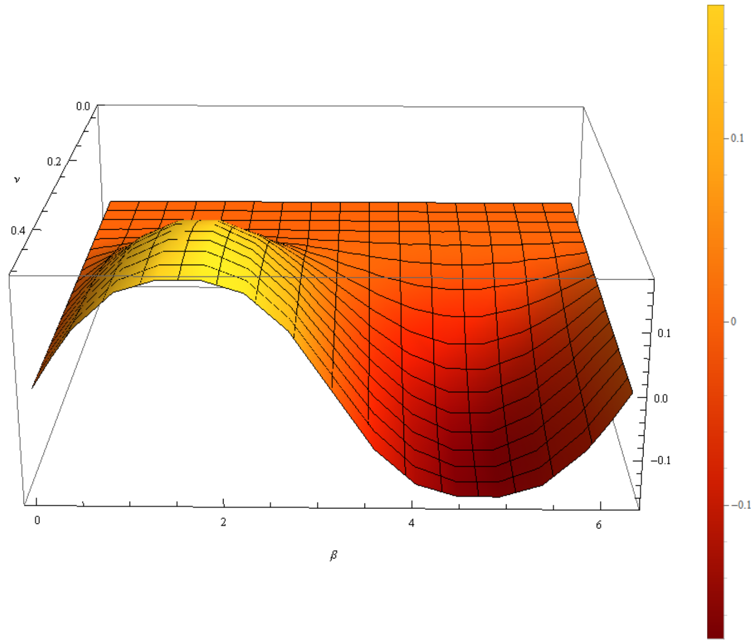

Figure 8: Melnikov integral Γ

$$-\frac{i\sin\left(\beta+\frac{i\pi}{2\nu}\right)\left(\left(\nu^2+25\right)\left(9\nu^2+25\right)\cos\left(4\beta+\frac{2i\pi}{\nu}\right)+4\left(9\nu^4+130\nu^2+217\right)\cos\left(2\beta+\frac{i\pi}{\nu}\right)+3\left(9\nu^4+90\nu^2+145\right)\right)}{192\nu^5}$$

one finds the relation between the integrals along both horizontal segments by noting that

$$\cos\left(t+\frac{i\pi}{\nu}\right)=\cos(t)\cosh\left(\frac{\pi}{\nu}\right)-\sin(t)\sinh\left(\frac{\pi}{\nu}\right)$$

where the first term compensates by symmetry when integrating, so there is some proportionality relation by factor $\sinh\left(\frac{\pi}{\nu}\right) \to e^{-\frac{c}{\nu}} \to 0$ rapidly as $\nu \to 0$. This behaviour is known as practical stability: the system doesn't notice perturbations for relatively small values of the parameter ν. See [10] for details on the subject.

Back to our calculations, the F_1 variation gives a Melnikov integral of exact value

$$\Gamma=\frac{\pi\left(2\left(\nu^2+1\right)\left(9\nu^2+1\right)\sin(\beta)\operatorname{sech}\left(\frac{\pi}{2\nu}\right)+27\left(\nu^2+1\right)\left(\nu^2+9\right)\sin(3\beta)\operatorname{sech}\left(\frac{3\pi}{2\nu}\right)+\left(\nu^2+25\right)\left(9\nu^2+25\right)\sin(5\beta)\operatorname{sech}\left(\frac{5\pi}{2\nu}\right)\right)}{384\nu^5}$$

In the following graphic we can see its behaviour with $\beta \in [0, 2\pi]$ and $\nu \in [0.01, 0.5]$. It is clear now that it vanishes for $\nu = 0$ or $\beta = k\pi$, it is antisymmetric around $\beta = \pi$ and symmetric between $\beta \in [0, \pi]$ around $\beta = \pi/2$. It is also extremely small for $\nu < 0.2$, because it behaves negative-denominator-exponentially: $\Gamma \propto e^{-\frac{\pi}{\nu}}$ due to the previously mentioned factor $\sinh(\pi/\nu)$.

11.4 Variational Equations

11.4.1 Setting of the problem

We consider the solution $z(t)$ of $\dot{x} = F(x)$ in the separatrix $F_1 = F_2 = 0$ for the unperturbed case, and its first variational equation $\dot{A} = Df|_{z(t)}A$, $A(0) = I_4$ where

$$Df = \begin{pmatrix} 0 & -1 & \nu(3y_1^2 + y_2^2 - 1) + 20\epsilon y_1^3 & \nu(2y_1y_2) \\ 1 & 0 & \nu(2y_1y_2) & \nu(y_1^2 + 3y_2^2 - 1) \\ -\nu & 0 & 0 & -1 \\ 0 & -\nu & 1 & 0 \end{pmatrix}$$

The unperturbed case, $\epsilon = 0$, allows us to greatly simplify Df:

$$Df = -\begin{pmatrix} J_2 & \nu(I_2 - M) \\ \nu I_2 & J_2 \end{pmatrix}, \text{ with } M = \begin{pmatrix} y_1^2 + y_2^2 & 2y_1y_2 \\ 2y_1y_2 & y_1^2 + y_2^2 \end{pmatrix} + 2\begin{pmatrix} y_1^2 & 0 \\ 0 & y_2^2 \end{pmatrix}$$

Polar coordinates $(R, \theta) = (R_2, \theta_2) = (\sqrt{y_1^2 + y_2^2}, \arctan(y_2/y_1)$ substitution leads to

$$M = R^2\begin{pmatrix} 1 & \sin(2\theta) \\ \sin(2\theta) & 1 \end{pmatrix} + 2R^2\begin{pmatrix} \cos^2(\theta) & 0 \\ 0 & \sin^2(\theta) \end{pmatrix}$$

Using $\begin{cases} 1 + 2\cos^2(\theta) = 2 + cos(2\theta) \\ 1 + 2\sin^2(\theta) = 2 - cos(2\theta) \end{cases}$ we obtain $M = R^2\begin{pmatrix} 2 + \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & 2 - cos(2\theta) \end{pmatrix}$

Starting at $A = I_4$ and integrating $\dot{A}$ according to the variational equation along closed paths $\gamma_i : [0, 2\pi] \to \mathbb{C}$ on complex time avoiding (around) the singularities of the solution $z(t)$ located at the imaginary time points $\pm t_0 = \pm\frac{\pi i}{2\nu}$ should give us the monodromy group of matrices $M_{\gamma_i} = \oint_{\gamma_i} dA = \oint_{\gamma_i} \dot{A}dt = \oint_{\gamma_i} Df(t)A(t)dt$ defined up to homotopy class of γ_i.

If this evolution of A from the identity matrix at $\gamma_i(0)$ to M_{γ_i} according to the matrix (vector) field defined by Df is not commutative with respect to paths γ_i, γ_j of different homotopy class, we will conclude that we found an obstruction to integrability.

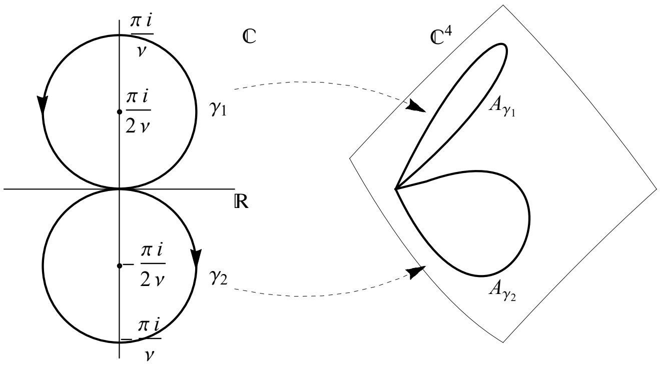

Figure 9: Monodromy skecth

11.4.2 Monodromy group commutativity

For our case, we choose a circular path γ_1 with center t_0, starting at the origin in counter clockwise direction. The parametrization is

$$t(\tau) = \gamma_1(\tau) = t_0 + \|t_0\| e^{i\tau - \pi/2} = \frac{\pi i}{2\nu} - i\frac{\pi}{2\nu}e^{i\tau} = \frac{\pi i}{2\nu}(1 - e^{i\tau}) = t_0(1 - e^{i\tau}),\ \ \tau \in [0, 2\pi]$$

so the variational equation transforms into

$$\frac{dA}{d\tau} = \frac{dA}{dt}\frac{dt}{d\tau} = t_0 i e^{i\tau}\dot{A} = \frac{\pi}{2\nu}e^{i\tau}Df(z(t(\tau)))A(t(\tau)).$$

We compute the integral numerically with a *Runge-Kutta method* with independent variably τ using step size 10^{-6}. The submatrix M of the matrix $Df(z(t(\tau)))$ is proportional $R_2^2 \sim \operatorname{sech}^2(\nu t)$, which takes large values quickly in the vicinities of $\frac{\pi i}{2\nu}$, making our numerical calculations impossible. For this reason, we have chosen the path circles to be the largest possible: of radius $\|t_o\| = \frac{\pi}{2\nu}$. We choose $\nu = 1/2$ to make F_1 visible enough, so now $\|t_0\| = \pi$ and the differential equation to integrate is just

$$\boxed{\frac{dA}{d\tau} = \pi e^{i\tau} Df\left(z(t(\tau))\right) A(t(\tau))}.$$

Note that we should also consider the terms of M $\cos(2\theta_2(t(\tau)))$ and $\sin(2\theta_2(t(\tau)))$ at $\tau = \pi \rightarrow t = 2\pi i$ reaching $\cosh(4\pi) \sim \sinh(4\pi) \sim 10^5$.

For $\epsilon = 0$, we obtain a norm $\left\|A_{\gamma_1(2\pi)} - I_4\right\| = 5.1 \cdot 10^{-8}$
and a determinant $\det(A_{\gamma_1(2\pi)}) = 1 + 4.1 \times 10^{-9} + 3.7 \cdot 10^{-9} i$,
sufficiently close to the identity, thus commuting with anything, since the inverse path is $\gamma_1^{-1}(t(\tau)) = \gamma_1(t(-\tau))$ and will have the inverse monodromy matrix $A_{\gamma_1}^{-1} = I_4^{-1} = I_4$. Surely, at $\tau = 2\pi$, $A_{\gamma_1}A_{\gamma_2}A_{\gamma_1}^{-1}A_{\gamma_2}^{-1} = I_4 A_{\gamma_2} I_4 A_{\gamma_2}^{-1} = I_4$, irrespective of A_{γ_2}. No obstructions to integrability were found, as we expected.

We plot the evolution of the real and imaginary part of the entries of A:

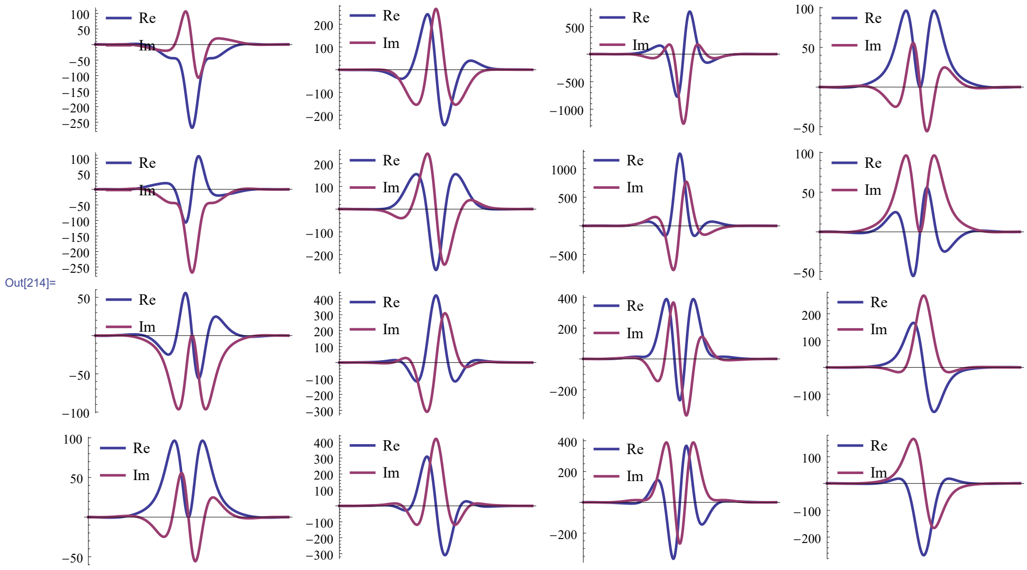

Figure 10: Evolution along τ of $A(\gamma_1(\tau))$ with $\epsilon = 0$

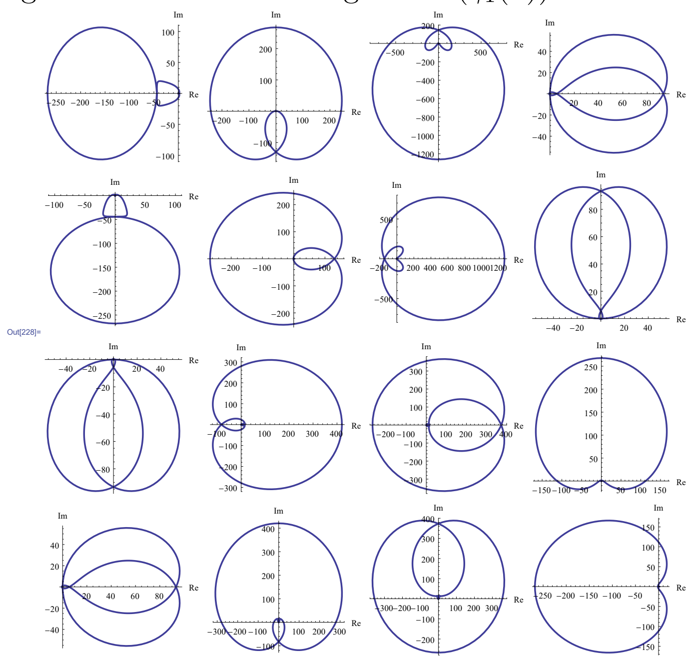

Figure 11: $A(\gamma_1(\tau))$ as complex curves with $\epsilon = 0$

Now, we shall proceed identically for the perturbed case with parameter $\epsilon = 10^{-10}$.

Reducing the notation $M_i = A_{\gamma_i(2\pi)}$, we obtain a norm $\|M_1 - I_4\| = 144$ and a determinant $\det(M_1) = 1 + 2.6 \times 10^{-6} + 2.1 \cdot 10^{-5} i$, indicating that M_1 went far from the identity but the transformation was volume-preserving as expected, discarding suspicions on big numerical errors.

We plot the evolution of A_{γ_1}, only curves in $\mathbb{C}$ with complicated patterns.

Out[125]=

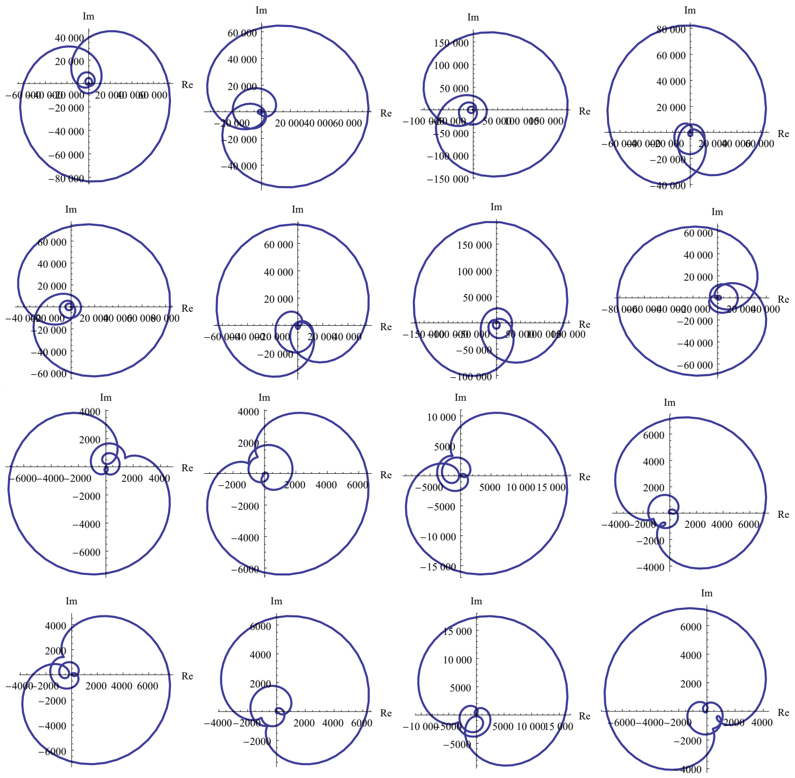

Figure 12: $A(\gamma_1(\tau))$ as complex curves with $\epsilon = 10^{-10}$

$$M_1 = \begin{pmatrix} 0.998826 - 46.8898i & 24.6789 + 0.622651i & 110.411 - 0.00285502i & -0.786424 + 30.6693i \\ 13.0177 - 0.332734i & 0.998622 + 5.18694i & 0.784054 + 30.6692i & -6.54117 - 0.000924993i \\ 19.9134 - 0.00042518i & -0.264263 + 10.4751i & 1.00104 + 46.8898i & -13.0177 - 0.333772i \\ 0.264092 + 10.4751i & -4.11271 - 0.000259718i & -24.6789 + 0.622305i & 0.999689 - 5.18695i \end{pmatrix}$$

For the second path, $\gamma_2(t(\tau)) = \gamma_2(-t(\tau))$, we obtain a matrix M_2 with the same determinant and difference with identity norm as M_1. In fact,

$$M_2 = \begin{pmatrix} 0.998826 - 46.8898i & -24.6789 - 0.622651i & -110.411 + 0.00285502i & -0.786424 + 30.6693i \\ -13.0177 + 0.332734i & 0.998622 + 5.18694i & 0.784054 + 30.6692i & 6.54117 + 0.000924993i \\ -19.9134 + 0.00042518i & -0.264263 + 10.4751i & 1.00104 + 46.8898i & 13.0177 + 0.333772i \\ 0.264092 + 10.4751i & 4.11271 + 0.000259718i & 24.6789 - 0.622305i & 0.999689 - 5.18695i \end{pmatrix}$$

We compute $M_2^{-1}M_1^{-1}M_2M_1 \neq I_4$ with norm $\sim 10^8$, very far from commuting. However, many observations can be made at the sight of these matrices, which satisfy a number of symmetries, reflecting the monodromy group structure. If we define $C = \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$ and denote $\circ$ the entry-wise matrix product, then the following rules hold:

1. $M_2 = M_1 \circ C$. That is, M_2 is M_1 switching sign out of the two diagonals.

2. Furthermore, each M_i satisfies itself $\begin{cases} Re(M_i^{-1}) = Re(M_i) \circ C \\ Im(M_i^{-1}) = -Im(M_i) \circ C \end{cases}$, $i = 1, 2$.

3. From the two above rules, $\begin{cases} M_1^{-1} = \bar{M}_2 \\ M_2^{-1} = \bar{M}_1 \end{cases}$. Since matrix inversion and complex conjugation commute, $M_2^{-1}M_1^{-1}M_2M_1 = \bar{M}_1\bar{M}_2M_2M_1$.

4. The pair of matrices $B_1 = M_1M_2$ and $B_2 = M_2M_1$ satisfy the same rules as the pair (M_1, M_2)

5. The two matrices $\frac{M_1 \pm M_2}{2}$ are disjoint projections of M_1, orthogonal by matrix product.

This set of rules provides information about the monodromy group structure, which is itself interesting, although our main goal is to observe that it is not Abelian and poses an obstruction to integrability.

12 Conclusions

Along with this project, we have successfully connected concepts like Galois Theory and physical symmetries, all through the scope of Dynamical Systems. There is much more we can learn at a theoretical level of both mathematical and physical parts of this recent field of study. However, this tools have proven useful to shed light on the historically unanswered problem of integrability by quadratures. We have investigated a dynamical system Hamiltonian problem and designed our own original strategy to apply the theory with satisfactory results.

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