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UB Riskcenter Working Paper Series

University of Barcelona

Research Group on Risk in Insurance and Finance www.ub.edu/riskcenter

Working paper 2015/03 \\ Number of pages 30

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Helena Chuliá, Montserrat Guillén and Jorge M. Uribe

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MORTALITY AND LONGEVITY RISKS IN THE UNITED KINGDOM: DYNAMIC FACTOR MODELS AND COPULA-FUNCTIONS

BY

HELENA CHULIÁ, MONTSERRAT GUILLÉN AND JORGE M. URIBE

ABSTRACT

We present a methodology to forecast mortality rates and estimate longevity and mortality risks. The methodology uses Generalized Dynamic Factor Models fitted over the differences of the log-mortality rates. We compare prediction performance with models previously proposed in the literature, such as the traditional Static Factor Model fitted over the level of log-mortality rates. We also construct risk measures by the means of vine-copulae simulations, taking into account the dependence between the idiosyncratic components of the mortality rates. The methodology is implemented to project the mortality rates of the United Kingdom, for which we consider a portfolio and study longevity and mortality risks.

KEYWORDS

Longevity, mortality forecasting, factor models, vine-copulae, Value at Risk.

1. INTRODUCTION AND BACKGROUND

Longevity risk (LR), understood as a downward deviation of the population mortality rates from a forecasted mortality trend, has become a topic of great academic interest in recent times, specifically for the actuarial literature. Such interest is completely justified, given the fact that longevity risk impacts different products and contracts, in which agents from the public and private sectors are involved. For instance, longevity risk appears subjacent to immediate or deferred annuities; enhanced and impaired annuities; guaranteed annuity options; lifetime mortgages and, more importantly, defined-benefit pension schemes (Richards and Jones, 2004).

Within the last category it becomes relevant not only for insurance companies or private employers, who promise a pension on retirement, based on the employee's final salary, but also, for institutions within the public sector, which typically offer generous final-salary benefits, albeit largely unfunded. MacMinn, Brockett and Blake (2006) and Dushi, Friedberg and Webb (2010) provide further insights about the topic of LR in the context of defined-benefit pension schemes. Particularly important it is for European countries that have experienced important reductions in mortality rates in the last century and much of them affect age groups above 65 years.

The study of longevity risk has been approached from different perspectives. On the one hand many authors have documented the importance of longevity risk in terms of its expected impacts on the solvency of insurance companies vulnerable to it (Hanewald, Post and Gründl, 2011; Hári, De Waegenaere, Melenberg and Nijman, 2008; Olivieri, 2011). On the other hand, many others have developed analytical frameworks aiming to provide

hedging strategies and instruments to be used by practitioners at insurance companies and pension funds facing LR. These strategies involve using financial instruments that range from longevity bonds to survivor swaps (Blake, Boardman and Cairns, 2014; Cairns, 2011; Dahl, Melchior and Møller, 2008; Lorson and Wagner, 2014; Ngai and Sherris, 2011; Wong, Chiu and Wong, 2014; among others).

The central point in understanding longevity risk is related to the stochastic nature of mortality rates. As such, their forecast is subject to uncertainty and statistical confidence judgments. The stochastic modeling of mortality rates has been well documented in the literature (Cairns et al., 2011; Continuous Mortality Investigation, 2004, 2005, 2013) and recommended by the regulators in recent times (Hollmann, Mulder and Kallan, 2000).

Among the available alternatives, factor models are an attractive approach, due to the low frequency (i.e. annual) of the mortality data and the relative high number of specific mortality rates to be forecasted. Specific mortality rates, which discriminate between ages and sex, are thought to be more appropriate to deal with mortality projections than aggregate rates, due to the heterogeneity of the population regarding mortality aspects. Factor models allow the researcher to reduce the dimensionality problem and to construct more accurate forecasts. The strategy consists in making the mortality rates dependent on few unobserved stochastic factors, for example, stochastic trends. By doing this, the number of estimated parameters in the model is significantly reduced and optimal first-efficient forecasts are possible.

In very recent times, the literature has also started to recognize the importance of the dependence relationships in different spectral frequencies among the mortality rates, in order to make estimations 'in' and 'out'-of-sample more accurate. For instance, some works highlight the importance of cointegration among the variables and the need for pre-testing for unit roots and multivariate cointegration before using traditional factor models or Vector Autogregessions (VAR) models (Njenga and Sherris, 2011; Torri, 2011). These relations can be thought as arising in the low frequency domain of the multivariate spectral density of the specific mortality rates.

However, the dependence patterns emerging in the high and medium parts of the spectra have not been much explored, even if some studies have documented the importance of dependence relationships between contiguous categories of mortality rates or their improvements (D'Amato, Haberman, Piscopo and Russolillo, 2012; Denton, Feaver and Spencer, 2005; Wills and Sherris, 2008; Lin, Wang and Tsai, 2015).

In this article we propose a methodology that accommodates advances in two different fronts, point estimation and forecast based on Generalized Dynamic Factor Models, and the construction of alternative confidence scenarios for those forecasts. The task in the second part involves the estimation of the multivariate probability density function of the forecasted mortality rates and not only their first moment. Having done that, we will be able of calculating risk measures, which will take into account linear and non-linear dependences among mortality rates variations, by the means of vine-copulae. By doing so, we will provide a robust alternative to measure longevity (and mortality) risk using distorted risk measures, such as Value at Risk, or Tail-Value at Risk.

The methodology is applied to forecast mortality rates and to estimate risk measures for the United Kingdom using data from 1950 to 2011 provided by the Human Mortality Database. We expand the current literature in several directions. First we compare the performance of four different factor models in forecasting mortality rates. One is a traditional (Static) Factor Model that assumes the presence of one common stochastic trend

in the data; better known in the actuarial literature as the Lee-Carter Model. A second model works with non-common trends, but stationary and common dynamic factors fitted on the differences of the log-mortality rates; a third one is a mix between the two above and it is a Dynamic Factor Model (DFM) over the log-mortality rates. Lastly we propose a Generalized Dynamic Factor Model (GDFM) over the differences of log-mortality rates. GDFM are widely employed in the econometrics literature, but they are still not present in the actuarial literature, in spite of their good properties, widely documented in the estimation of unobservable factors (Forni, Hallin, Lippi and Reichlin, 2000, 2004) and more recently in forecasting (Forni, Hallin, Lippi and Reichlin, 2005).

Second, we incorporate explicit modeling of the dependence structure in the construction of scenarios for mortality rates projections. We also propose VaRs and Tail-VaRs as a way to measure longevity risk, and we highlight how they should be estimated in this context. This could be useful to assess suitable capital requirements for the operation of firms exposed to such a risk.

Finally we connect the econometrics literature about DFM, unit roots and cointegration, with the actuarial literature regarding mortality forecasting. Thus, we reference some crucial findings in this branch of econometrics that could certainly enrich the currently ongoing analysis about longevity risk, within the actuarial science.

Our work is also related to Plat (2011). In this study the author estimates the VaR of a portfolio consisting of 45,000 male and 36,000 female policyholders of age 65 and older. His approach relies on a two steps algorithm: first he estimates the common trends on a dynamic basis (using blocks of 30 years length), and then he estimates the stochastic variation around the trend projection. In this way he is capable of simulating different scenarios of operation for the insurance company and calculating the Net Asset Value (NAV) of the company in each case. This allows him to estimate the VaR of the company at different levels of confidence. It should be notice that different from Plat, we do not assume a cointegration relationship operating on the 220 mortality rates of our empirical exercise. Therefore our forecasting exercise only includes the modeling of the stochastic volatility around the trend, alternative to the second step in Plat's algorithm.

This document is organized as follows. First we present the main theoretical points regarding our methodological approach and we discuss the models and the estimation strategy. In section 3 we explore the relationship between the models explained in section 2 and some o the most popular factor models for the forecasting of mortality rates in the actuarial literature. In section 4 we describe our data in terms of their stochastic time-series properties, such as the presence of unit roots and variability explained by the firsts principal components. In the section 5 we present our main results and discuss our principal findings. Lastly, we conclude.

2. METHODOLOGY

In this section we present the forecasting methodologies employed in the empirical section. We also explain how to estimate longevity risk, in order to incorporate the information about dependence in mortality series in the estimation. We follow in most of 2.1 and 2.2 subsections the notation and presentation by Bai and Ng (2008).

2.1 Factor Models

Let N be the number of cross-sectional units and T be the number of time series observations. In our case, we have 220 cross-sectional units (mortality rates for ages from 0 to 109+ years, for males or females). If we consider males and females separately then $N = 110$. For $i = 1, \dots, N$ and $t = 1, \dots, T$. The Static Factor Model (SFM) is defined as:

$$\begin{aligned} x_{it} &= \lambda_i F_t + e_{it} \\ x_{it} &= C_{it} + e_{it}, \end{aligned} \tag{2.1}$$

where e_{it} is referred to as the idiosyncratic error and λ_i is referred to as the factor loadings. This is a vector of weights that unit i puts on the corresponding r static common factors F_t. $C_{it} = \lambda_i F_t$ refers to the common component of the model. If we define $X_t = (x_{1t}, x_{2t}, \dots, x_{Nt})'$ and $\Lambda = (\lambda_1, \dots, \lambda_N)'$, in vector form, for each period, we have:

$$\underset{(N \times 1)}{X_t} = \underset{(N \times r)}{\Lambda} \underset{(r \times 1)}{F_t} + \underset{(N \times 1)}{e_t}, \tag{2.2}$$

where $e_t = (e_{1t}, e_{2t}, \dots, e_{Nt})'$. Notice that although the model specifies a static relationship between x_{it} and F_t, F_t itself can be a dynamic vector process. In the case that F_t and X_t are jointly stationary (i.e. either each series is stationary or *all* of them are cointegrated), F_t can be tough to evolve according to a vector autoregressive (VAR) process:

$$A(L)F_t = u_t, \tag{2.3}$$

where $A(L)$ is a polynomial of the lag operator. The static factor model is implemented in several studies aiming to forecast mortality rates, as the one by Alonso (2008). Note that if we set $L = 1$ and $A(1) = I_r$, where I_r is the identity of order r, the model in (2.3) becomes a Multivariate Radom Walk (MRW). In that case, we further require that $r = 1$, we are in presence of the popular model proposed by Lee and Carter (1992).

In the general case F_t could contain stationary and non-stationary factors as in Peña and Poncela (2006) or Bai and Ng (2004). Nevertheless, for empirical applications it is convenient to restrict the attention to the cases where *all* the factors in F_t are stationary or the case where *all* of them evolve following the same stochastic trends (by assumption). This is especially true for models with very large N, in which the implementation of traditional cointegration tests based on VAR representations of the original variables are not suitable.

The static model can be compared with the Dynamic Factor Model (DFM), defined as:

$$x_{it} = \lambda_i(L) f_t + e_{it}, \tag{2.4}$$

where $\lambda_i(L) = (1 - \lambda_{i1}L -, \dots, -\lambda_{is}L^s)$ is a vector of dynamic factor loadings of order s. In the case when s is finite, we refer to it as a Dynamic Factor Model, whereas a Generalized Dynamic Factor Model allows s to be infinite. Stock and Watson (2010)

provide examples of the former and Forni et al. (2000) introduce the latter. In either case, the (dynamic) factors f_t evolve according to:

$$f_t = C(L)\varepsilon_t, \tag{2.5}$$

where ε_t are iid errors. The dimension of f_t, denoted q, is the same as the dimension of ε_t.

One additional classification of the models stated in equations (2.2)-(2.3) and (2.3)-(2.4) regards to whether the idiosyncratic disturbances in (2.2) or (2.4) are allowed to be weekly correlated or not. When they are not, it becomes an exact factor model, on the contrary, when they are allowed; the model is an approximate factor model.

2.2. Identification

We can rewrite the model in (2.4) in static form, simply by redefining the vector of factors to contain the dynamic factors and their lags, and the matrix of loads according. In this case both, SFM and DFM, can be presented in matrix form as:

$$\underset{(N \times T)}{X} = \underset{(N \times r)}{\Lambda}\underset{(r \times T)}{F} + \underset{(N \times T)}{e}, \tag{2.6}$$

where $X = (X_1, \dots, X_N)$ and $F = (F_1, \dots., F_T)$. Clearly F and Λ are not separately identifiable. For any arbitrary $(r \times r)$ invertible matrix H, $F\Lambda' = FHH^{-1}\Lambda' = F^*\Lambda'^*$, where $F^* = FA$ and $\Lambda^* = \Lambda \mathrm{H}^{-1}$, the factor model is observationally equivalent to $X = F^*\Lambda'^* + e$. Therefore r^2 restrictions are required to uniquely fix F and Λ (Bai and Wang, 2012). Many alternatives are available in the literature to achieve this goal. For example, Harvey (1990), Zuur, Fryer, Jolliffe, Dekker and Beukema (2003) and Holmes, Ward and Scheuerell (2014) propose the following procedure: In the first $N - 1$ rows of Λ**,** the λ-value in the j-th column and i-th row is set to zero if $j > i$. The intercept is constrained so that each of the time series in F_t has a mean equal to zero across the time (from $t = 0$ to $t = T$). The matrix of second moments $[e'e]$, is set equal to the identity matrix of order N, I_N.

Notice that the estimation of the factors using Principal Components (PC) or Singular Value Decomposition (SVD), by construction, impose the normalization that $\frac{\Lambda'\Lambda}{N} = I_r$ and $F'F$ being diagonal, which are enough to guarantee identification (up to a column sign rotation).

2.3. Generalized Dynamic Factor Model

GDFM was originally proposed by Forni and Reichlin (1998) and Forni et al. (2000). But it was till Forni et al. (2005) that it could be used to forecasting proposes. It is a generalization of the DFM because it allows for a richer dynamic structure in the factors and it does not assume mutual orthogonality of the idiosyncratic components e_{it}.

In forecasting exercises the GDFM differs from traditional static and dynamic models because it uses a twofold strategy. Following Forni et al. (2005) the first step enables us to place smaller weights on variables having larger idiosyncratic components. In this way the idiosyncratic error contained in the linear combination is minimized. As we will see, in

general, this is exactly why better forecasting in the great majority of mortality rates can be obtained using this model, but it makes the model weak facing 'outliers' or 'extreme variation' in mortality rates. We will discuss further this point in section 3.

2.4. Factors estimation

There are different alternatives to estimate models in equations (2.1)-(2.3) and (2.2)-(2.4). One of them consists in using PC or equivalently SVD, to estimate the factors and their loadings.

There is another way of estimating (2.2)-(2.3), under the assumption of Gaussian errors and possibly a MRW structure. It arises by noticing that (2.2) and (2.3) can be tough of as a State Space representation, where the transition equation is a first order Markov process. In this case equation (2.3) is the hidden state vector, unobservable by definition, and (2.1) is the output, or measurement equation. Therefore, the model can be estimated by Maximum Likelihood, either, by using and Expectations Maximization (EM) algorithm, or other numeric-optimization algorithms (Hamilton, 1994; Holmes et al., 2014). As it is well known, in this context the Kalman Filter is an optimal estimator of the parameters in the model, and the Kalman Smoother can be used to estimate the unobservable factors. We prefer here the factors estimation through PC, in the interest of parsimony and due to the documented advantages of this method in terms of model specification (Stock and Watson, 2002; Bai and Ng, 2008; Bates, Plagborg-Møller, Stock and Watson, 2013).

The GDFM uses a two-step estimation strategy discussed in Forni et al. (2005) in the context of forecasting. First, the variance-covariance matrices of the common and the idiosyncratic components in equation (2.1) are estimated, by using the first q dynamic principal components operating on the spectral density of x_{it}. Then the information coming from the first step is used to extract linear combinations of the x's that are more efficient than standard principal components. Particularly:

$$\hat{C}_t = \left[\Gamma_0^C \hat{Z}'\left(\hat{Z}\hat{\Gamma}_0\hat{Z}'\right)^{-1}\right]\left(\hat{Z}X_t\right), \tag{2.7}$$

where $\hat{C}_t$ is the estimation of the common component, Γ_0^C and $\hat{\Gamma}_0$ are contemporaneous-covariance matrices of the common components and the x's, respectively. The first matrix is estimated based on spectral density methods. $\hat{Z}$ are generalized eigenvectors and therefore $\hat{Z}X_t$ are the generalized principal components (GPC).

2.5. Point forecasts

With the factors at hand, the forecasting of the mortality rates by linear regression techniques is straightforward. The factors estimated in a first regression stage by SVD, PC or GPC can be employed in a second regression. Consider forecasting x_{it+1} using all the data in X_t and treat F_t as observed. If e_{it} follows an autoregression and the errors are Gaussian, then:

$$\begin{aligned} E[x_{it+1}|X_t, f_t, X_{t-1}, f_{t-1}, \dots] &= E[\lambda_i(L)f_{t+1} + e_{it+1}|X_t, f_t, X_{t-1}, f_{t-1}, \dots] = \\ = E[\lambda_i(L)f_{t+1}|X_t, f_t, X_{t-1}, f_{t-1}, \dots] &+ [e_{it+1}|X_t, f_t, X_{t-1}, f_{t-1}, \dots] = \\ &= \alpha(L)f_{t+1} + \delta(L)x_{it}\ . \end{aligned} \tag{2.8}$$

In the SFM case L disappears. In the forecasting equation (2.8) we also could include some other covariates. It also would be possible to index in time the matrix Λ, such that Λ_t contains the time-varying loads of the system. However, as noted by Bates, et al. (2013), the original DFM or SFM seem to behave very well in the presence of parameter instability and therefore, time-varying extensions do not provide deeper insights into the model's structure, but instead consume degrees of freedom and demand additional restrictions to be imposed on it.

Following Stock and Watson (2006) the h-step ahead forecast can be performed directly by projecting x_{it+h} onto the estimated factors lagged h periods, that is, by estimating β in the equation:

$$X_{t+h} = \beta F_t + e_{t+h}. \tag{2.9}$$

Unknown factors can be replaced by their (consistent) estimations $\hat{\mathrm{F}}_\mathrm{t}$ following Stock and Watson (2006). Direct forecast can be potentially less efficient than iterated forecast (solving the full DFM or SFM forward using the KF), but it is also more robust facing model misspecification. The importance of model misspecification in the particular context of the estimation and forecasting of mortality rates have been documented by Stallard (2006).

Notice that β equals Λ when $h = 0$. Alternatively in the context of GDFM:

$$X_{t+h} = \left[\Gamma_h^C \hat{Z}'\left(\hat{Z}\hat{\Gamma}_0\hat{Z}'\right)^{-1}\right]\left(\hat{Z}X_t\right) + e_{t+h}. \tag{2.10}$$

In the empirical application we ignore the forecasting of the idiosyncratic component in equation (2.8), given that it seems pretty much as a white noise process, so we concentrate in the estimation of equations (2.9) and (2.10), based entirely on the information provided by the common factors, leaving the dependence relationship to affect only the 'simulated scenarios' of the risk measures as explained in section 2.6.

2.6. Dependence between the idiosyncratic components

In spite of the preferred forecast method, up to this subsection, we only have considered the dependence in the common factors. Nothing has been said about the possible dependence in the idiosyncratic components. The dependence arising in the 'noisy-high' frequency of the spectra is key to the estimation of 'unexpected' movements in the time series. Therefore it is crucial for the estimation of risk. In this document we approach it by the means of copula functions, and thus we are able of constructing confidence intervals for our point forecasts, so as risk measures based on the percentiles of the simulated density (i.e. VaR and Tail-VaR).

2.6.1. Copula functions

Formally, a copula is a multivariate probability distribution such that $\mathrm{C}: [0,1]^{\mathrm{N}} \rightarrow [0,1]$ where C is the la copula and N is the number of mortality rates in our application. It is possible to use Sklar's Theorem (Sklar, 1959) to construct the copula function. The theorem establishes that if F is a joint distribution function with margins $F_1, \ldots, F_N$ there

exists a copula, such that for all real values $\underline{e}_1, \dots, \underline{e}_N$ $then$ $(\underline{e}_1, \dots, \underline{e}_N) = C\left(F_1(\underline{e}_1), \dots, F_N(\underline{e}_N)\right)$. If the margins are continuous, C is unique. Otherwise, it is uniquely determined by $rank(F_1) \times rank(F_2) \times \dots \times rank(F_N)$, where $rank(F_i) = F_i(\underline{e}_i)$ denotes the rank operator (McNeil, Frey and Embrechts, 2005).

Before proceeding to the parameter estimation, we need to construct a pseudo-sample defined as:

$$F_i(\underline{e}_i) = u_i \quad \forall \mathrm{i} = 1, .., \mathrm{N}, \tag{2.11}$$

where $\underline{e}_i = (e_1, \dots, e_T)'$ is a $(T \times 1)$ vector that contains the estimated idiosyncratic components for each individual i. We make use of the empirical cumulative distribution (ecd) of the idiosyncratic terms, estimated as the residuals of the GDFM, as an approximation to $F_i(\cdot)$ in (2.11). Once the pseudo-sample is constructed, checking its accuracy in describing the data is required through the Kolmogorov-Smirnov (KS) statistic.

One additional consideration must be highlighted at this stage. When there are several dimensions involved, as in our case, in which we have 110 mortality rates for each sex, the direct estimation of a N-dimensional copula is not recommended. Instead, the literature has developed an alternative estimation and simulation procedure, based on bivariate-conditional-copulae (i.e. pair copulae) as described by Aas, Czado, Frigessi and Bakken (2009).

2.6.2. Pair Copulae

Building on the work of Joe (1996) and Bedford and Cooke (2001, 2002), Aas et al. (2009) show that multivariate data, which exhibit complex patterns of dependence in the tails, can be modeled using a cascade of pair-copulae, acting on two variables at a time. This approach is particularly attractive in the present context, in which a very large cross-sectional dimension makes traditional high-dimensional copula methods unfeasible. The model construction is hierarchical and the various levels in the model correspond to the incorporation of more variables in the conditioning sets, using pair-copulae as simple building blocks.

Following Aas et al. (2009), consider a vector $\underline{e} = (\underline{e}_1, \dots, \underline{e}_N)'$ of random variables with a joint density function $f(\underline{e}_1, \dots, \underline{e}_N)$. This density can be factorized as:

$$f(\underline{e}_1, \dots, \underline{e}_N) = f_n(\underline{e}_N) \cdot f(\underline{e}_{N-1}|\underline{e}_N) \cdot f(\underline{e}_{N-2}|\underline{e}_{N-1}, \underline{e}_N) \cdots f(\underline{e}_1|\underline{e}_2, \dots, \underline{e}_N), \tag{2.12}$$

and each term in (2.12) can be decomposed into the appropriate pair-copula times a conditional marginal density, using the general formula:

$$f(\underline{e}|\boldsymbol{v}) = c_{ev_j|v_{-j}}\{F(\underline{e}|\boldsymbol{v}_{-j}), F(v_j|\boldsymbol{v}_{-j})\} \cdot f(\underline{e}|\boldsymbol{v}_{-j}), \tag{2.13}$$

for an N-dimensional vector v. Here each v_j is an arbitrary chosen component of v and v_{-j} denotes de vector v excluding the j-th component. Then, c_{ev_j} denotes a pair-copula between $\underline{e}$ and v_j. As it is noted by Aas et al. (2009), under appropriated regularity conditions a

multivariate density can be expressed as the product of pair-copulae, acting on several different conditional probability distributions. From their work it is also clear that the construction is iterative, and that given a specific factorization, there are still many different re-parameterizations, which in principle could be used to perform the estimation.

The different constructs of pair copulae available in the literature can be described following Bedford and Cooke (2001, 2002) by a graphical model named regular vines. The vine class is very large and it houses the graphical model used in this study, known as *d-vine* (Kurowicka and Cooke, 2005). A d-vine is a specific way to factorize the multivariate density and it implies the estimation of $N(N-1)/2$ bivariate copulae in ascending hierarchical order as showed in Figure 1.

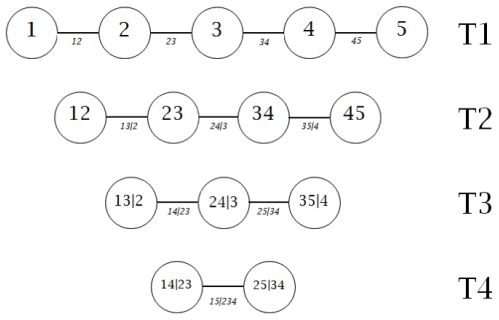

Figure 1. d-vine copula tree, taken from Aas et al. (2009), p. 184

Figure 1 shows the specification corresponding to a five-dimensional d-vine. It consists of four trees T_j, $j = 1, \dots, 4$ Tree T_j has $6-j$ nodes and $5-j$ edges. Each edge corresponds to a pair-copula density and the edge label corresponds to the subscript of the pair-copula density, e.g. edge $25|34$ corresponds to the copula $c_{25|34}(\cdot)$. In a d-vine, no node in any tree T_j is connected to more than two edges. This graphical model is well suited for the case in which no particular variable is known to be a key variable that governs all the interactions in the data set, as it is the case for the idiosyncratic components of mortality rates.

Bedford and Cooke (2001) provide the density of an N -dimensional distribution in terms of a regular vine, which Aas et al. (2009) further specialize to a d-vine. The density becomes in this case:

$$\prod_{k=1}^{N} f(\underline{e}_k) \prod_{j=1}^{N-1} \prod_{i=1}^{N-j} c_{i,i+j|i+1,\cdots,i+j-1} \left\{F\left(\underline{e}_i \middle| \underline{e}_{i+1}, \dots, \underline{e}_{i+j-1}\right), F\left(\underline{e}_{i+j} \middle| \underline{e}_{i+1}, \dots, \underline{e}_{i+j-1}\right)\right\}. \quad (2.14)$$

Here index j identifies the trees, while i runs over the edges in each tree. Given the specific d-vine decomposition it is possible to approach the problem of estimation and posterior simulation by a maximum pseudo-likelihood approach.

Aas et al. (2009) provide the necessary steps to perform simulations using the estimated cascade of copulae in the d-vine construct. This method allows us to estimate percentiles of the multivariate density of the idiosyncratic components in the model, and therefore, to construct risk measures based on specific percentiles of the forecasted error.

2.7. Risk measures

Given the copula simulation of different scenarios for the idiosyncratic terms and the point forecasts, we can chose from the family of distorted risk measures one that serves the purpose of estimating longevity risk. Several alternatives are available within this family, such as value-at-risk (VaR) or tail value-at-risk (TVaR). VaR at a level α is the α-quantile of a random variable x, that is, $VaR_\alpha(x) = inf\{x|F_x(x) \geq \alpha\}$, where F_x is the distribution function of x and α is the tolerance level $\alpha \in (0,1)$. A complementary measure, the Tail-VaR, or TVaR, corresponds to the mathematical expectation beyond VaR, and it is defined as $TVaR_\alpha(x) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_\lambda(x)d\lambda$. Additionally, GlueVaR measures, proposed by Belles-Sampera, Guillén and Santolino (2014), which are a combination of VaR and TVaR also can be implemented.

We propose to estimate the longevity (or mortality) risk as:

$$LR = TVaR_\alpha(\hat{p}_{t+h}), \tag{2.15}$$

where α is associated to a suitable confidence level and $\hat{p}_{t+h}$ is the h-step ahead forecast of deaths. It is, after projecting N mortality rates, one need to aggregate these projections in order to get the total number of 'expected deaths' in a given portfolio. The number of expected deaths is a function of the exposed population by ages and sex in the portfolio and the projection associated to each one of the mortality rates. In this case α is a very low percentile (such as 0.5%) if we want to measure longevity risk and a very high percentile (i.e. 99.5%) if we want to measure mortality risk. We provide both quantities in our empirical implementation.

Given that we are interested in constructing a methodology useful for any firm, exposed to longevity risk, we provide estimations of LR for different configurations of the exposed population, as explained in section 4.

3. RELATED LITERATURE

Factor models have been extensively used in forecasting mortality rates; they were originally proposed to achieve this goal by Lee and Carter (1992). By the time that these authors introduced the model to the actuarial and demographic literature, factor models were well established in the fields of psychology and econometrics. The Lee-Carter model is a single-factor model, where the factor is a stochastic trend shared by all the specific mortality rates. The model has been subject to many criticisms, for example by Dushi, Friedberg and Webb (2010) or Mitchell, Brockett, Mendoza-Arriaga and Muthuraman (2013), but it still remains as a plausible alternative within the academia (See Bisetti and Favero (2014) for a recent implementation of the model). Indeed, it has become a 'workhorse' within the actuarial field and also have its extension to Poisson-log bilinear projections proposed by Brouhns and Denuit (2002) (See for example the works by Delwarde, Denuit, Guillén and Vidiella-i-Anguera (2006) or Lemoine (2014), who adds a switching component to the model).

The main criticism to the model is the fact that one single factor seems unable to capture all the common components subjacent to the mortality rates dynamic. Therefore, the model has been expanded to include more factors. These factors are generally additional stochastic

trends as in Yang, Yue and Huang (2010), Alai and Sherris (2014), Jevtic, Luciano and Vigna (2013) or Alonso (2008); but they also can be stationary factors subjacent to the differences (or the derivatives) of the log-mortality-rates, instead of the log-level-mortality rates (Cossette, Delwarde, Denuit, Guillot and Marceau, 2007; Haberman and Renshaw, 2012, 2013; Mitchell et al., 2013).

The last extension is particularly interesting both, from practical and theoretical points of view. In pragmatic terms, Mitchell et al. (2013) show that a stationary factor model can over-perform the results of a non-stationary model (with possibly more factors). Thus working with the differences of the log-mortality rates could increase the forecaster ability, with respect to working with the rates in log-levels as it is the Lee-Carter' style. From a theoretical perspective it is important because, from the econometrics literature it is well known, that linear regressions between integrated series of order greater than zero (for example I(1) series) can easily derive in spurious estimations (Granger and Newbold, 1974). To avoid spurious regressions the series must be differentiated as many times as needed to provide that all of them become stationary. After differentiating, traditional regressions can be performed. Actually, this is the intuition behind the main findings by Mitchell et al. (2013). Namely that using the series in differences seems a better empirical strategy that using the series in levels.

In other words, if log-mortality rates could be described by independent random-walks-trends and the Lee-Carter model imposes a common trend, it could lead to the imposition of a false cointegration relationship among all the log-mortality rates.

Another, potentially better alternative is available when the series at hand are indeed cointegrated and all of them shared the same stochastic trend or trends. In this case, to perform a regression analysis in levels (for example, when forecasting mortality rates using their non-stationary first singular values) is fully justified; it may be even necessary in order to preserve the correct specification of the model. Another more efficient alternative, in this case, is to use an Error Correction Model (ECM) (Engle and Granger, 1987) or their multivariate version, the Vector Error Correction (VEC) model. In both cases the estimation is super-consistent as shown for example in Stock and Watson (1988). To put this in different words, under the presence of cointegration, the differentiation of the series is not recommended, because this strategy could lead to biases in the estimated parameters and the forecasted quantities.

Thus, it is clear why pre-testing for unit roots and cointegration becomes a first-order necessity to avoid under- or over-differentiation of the series. Cointegration in multiple times series is generally based on tests constructed on the Vector-Autoregression (VAR) representation of the system (Johansen, 1988; Stock and Watson, 1988). Indeed, such econometrics-machinery: unit roots pretesting procedures, cointegration tests, VAR and VEC models, has been recently explored in some degree in the actuarial literature, specifically to perform the task of estimating and forecasting mortality rates (D'Amato, Haberman, Piscopo, Russolillo and Trapani, 2014; Gaille and Sherris, 2011; Njenga and Sherris, 2011; Torri, 2011). The mentioned works have shown evidence in favor of cointegration, using few variables. This finding seems to justify the use of factor models fitted over the levels of log-mortality rates (without differentiating the series). The relationship between factor models and cointegration, was explored more than two decades ago by Escribano and Peña (1994).

Unfortunately, the extension of this toolbox to the forecasting of mortality rates by age cohorts and sex is not straightforward. For example, when $N = 110$, which easily is the

case for mortality rates discriminated by age and sex, traditional cointegration test based on the VAR representation of the system are not well suited, given the extremely large number of parameters to be estimated in the reduced-form VAR. For example, in a VAR comprising 4 lags and 110 mortality rates, $(101 \wedge 2)4 = 48{,}400$ coefficients must be estimated, plus 110 variances and 5,995 covariances.

At this point the researcher faces a dichotomy: Either she assumes the cointegration of the mortality rates and estimates a factor model in levels, or she differentiates the series and fits a stationary factor model on the differences. We explore the empirical consequences of both alternatives and perform several comparisons, particularly in terms of the Mean Squared Forecasting Error for different horizons and for different modeling strategies.

3.1. Models M1-M8

In this section we compare theoretically GDFM and DFM with the models studied by Cairns, Blake, Dowd, Coughlan, Epstein, Ong and Balevich (2009) and we expand their typology with some important models proposed very recently. We also switch our notation to mimic the one in Cairns et al. (2009) in this section, hoping to provide further clarification to the experimented reader.

In general lines, the main models in the actuarial field used to forecast mortality rates can be resumed in Table 1. As can be noticed the models proposed here (M9, M11 and M12) belong to the family of the log-mortality rates models, and they differ from the logit-models, which use instead logit transformations of the mortality rates in the estimation. Nevertheless, following Mitchell et al. (2013) the latter do not seem to perform any better or worse than log models, so we concentrate in factor models of the log-rates or the differentiated log-rates.

Notice that M1 is a special case of M9, when $L = 0$ and $r = 1$. In the same vein M10 is a special case of M11 when $L = 0$ and $r = 1$, so M1 and M10 can be regarded as static factor models. The GDFM fitted over the differences of log-mortality rates (M12) is different from all the other models because it does not use principal components to estimate de factors, but instead it uses generalized principal components.

We do not consider any model with cohort effects in our empirical section, but Mitchell et al. (2013) provided comparisons of their model with M2 and their model seems to perform better in most of the cases. The extension of the M11 or M12 to incorporate cohort effects is straightforward and could be explored in future research following the proposals by Haberman and Renshaw (2012, 2013).

We instead have opted for modeling contemporaneous dependence trough copula functions. The copula approach to estimate the longevity or mortality risks is novel (see for example Lin et al. (2015) for a related implementation). It regards to the dependence structure of the idiosyncratic components of the model (which we obviated in Table 1). This modeling strategy only makes sense within the context of an approximate factor model as M11 and M12. Otherwise identification issues could arise in the estimation process, because static factor model such as M1 or M10 generally used the orthogonality condition between the idiosyncratic models to identify the factors.

Our empirical illustration for the United Kingdom compares M1, M9, M11 and M12. And M10 is comprised as a special case of M11.

TABLE 1:

MORTALITY MODELS

Model	Formula
M1: Lee and Carter (1992)	$\log m(t,x) = \beta_x^{(1)} + \beta_x^{(2)} k_t$
M2: Renshaw and Haberman (2006)	$\log m(t,x) = \beta_x^{(1)} + \beta_x^{(2)} k_t + \beta^{(3)} \gamma_{t-x}^{(3)}$
M3: Currie (2006)	$\log m(t,x) = \beta_x^{(1)} + k_t + \gamma_{t-x}^{(3)}$
M4: Currie et al. (2004)	$\sum_{i,j} \theta_{ij} B_{ij}^{ay}(x,t)$
M5: Cairns, Blake and Dowd (2006)	$\text{logit}\, q(t,x) = k_t^{(1)} + k_t^{(2)}(x - \bar{x})$
M6: Cairns et al. (2009)	$\text{logit}\, q(t,x) = k_t^{(1)} + k_t^{(2)}(x - \bar{x}) + \gamma_{t-x}^{(3)}$
M7: Cairns et al. (2009)	$\text{logit}\, q(t,x) = k_t^{(1)} + k_t^{(2)}(x - \bar{x}) + k_t^{(3)}((x - \bar{x})^2 - \hat{\sigma}_x^2) + \gamma_{t-x}^{(4)}$
M8: Cairns et al. (2009)	$\text{logit}\, q(t,x) = k_t^{(1)} + k_t^{(2)}(x - \bar{x}) + \gamma_{t-x}^{(3)}(x_c - x)$
M9: Dynamic Factor Model in levels, DFM	$\log m(t,x) = \beta_x^{(1)} + \sum_{i=2}^{r+1} \beta_x^{(i)}(L) k_t^{(i-1)}$
M10: Mitchell et al. (2013)	$\Delta \log m(t,x) = \alpha_x^{(1)} + \beta_x^{(2)} \Delta k_t$
M11: Dynamic Factor Model in Differences, DDFM	$\Delta \log m(t,x) = \alpha_x^{(1)} + \sum_{i=2}^{r+1} \beta_x^{(i)}(L) \Delta k_t^{(i-1)}$
M12: Generalized Dynamic Factor Model in Differences, DGDFM	$\Delta \log m(t,x) = \alpha_x^{(1)} + \sum_{i=2}^{r+1} \beta_x^{(i)}(L) \Delta q_t^{(i-1)}$

NOTE: Mortality Models from M1 to M8 are taken from Cairns et al. (2009). Following them $\beta_x^{(i)}$, $k_t^{(i)}$, and $\gamma_{t-x}^{(i)}$ are age, period and cohort effects respectively. The $B_{ij}^{ay}(x,t)$ are B-spline basis functions and the θ_{ij} are weights attached to each basis function. $\bar{x}$ is the mean age over the range of ages being used in the analysis. $\hat{\sigma}_x^2$ is the mean value of $(x - \bar{x})^2$. L is the lag-operator and $q_t^{(i)}$ are generalized principal components.

In order to determine the number of static and dynamic factors, that is the number of principal components and their lags, we follow Bai and Ng (2002), who proposed the use of two statistics to achieve the goal. In particular to determine the number of static factors we make use of the IC information criterion given by:

$$IC(k) = \ln\big(S(k)\big) + kg(N,T), \tag{3.1}$$

where $S(k) = (NT)^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} \big(x_{it} - \hat{\lambda}_i^{k\prime} \hat{F}_t^k\big)^2$ is the Mean Squared Error divided by NT. k is the number of static factors and $g(N,T)$ is a penalty function, such that: $g(N,T) = \frac{N+T}{NT} ln\left(\frac{NT}{N+T}\right)$. The number of static factors, $\hat{k}_{IC}$ is such that:

$$\hat{k}_{IC} = \underset{0 \le k \le k_{max}}{\text{argmin}}\, IC(k), \tag{3.2}$$

where k_{max} is then maximum possible number of factors, $k_{max} = 15$ in our case. Once the number of static factors is selected, the number of dynamic factors is determined according by using the methodology proposed by Bai and Ng (2007). This methodology deals, in general lines, with the rank of the vector space spanned by the original dynamic

factors, which is expressed in the vectors containing the static factors. In general $q \leq r$, where q are the original, primitive, dynamic-factors, and r the number of static factors spanned by q.

4. DATA AND PRELIMINARY ANALYSIS

Our data comprise annual mortality rates for males and females in UK from 1950 to 2011. The data for 0 to 101 years were taken as they appear in the web page of Human Mortality Database. The data from 102 to 109+ years were extrapolated to make the mortality rate equal to 1 at 110 years. In this way we prevent implausible variability registered at older ages. We pretest for unit roots in the log-mortality rates and the differences of the log-mortality rates (See Table 4 in the Appendix). The series show evidence of unit root behavior in the great majority of cases in logs, and they seem stationary once we differentiate them.

We estimate the number of static and dynamic factors following equations (3.1) and (3.2) for different portfolio-populations. We consider six cases: An exposed population of 0-109+ years of males and females; another product designed to people between 18 and 64 years (male and females) and an exposed population ranging from 65 to 109+ years. The population was set in 30,000 people (either males or females), which is approximately 0.1% of the total population of the United Kingdom for the year 2011. The participation of each age in the total was set according to the participation in the population of the UK for 2011.

We also present the variability explained by the first 15 principal components of the series in differences, in Table 2. The percentage explained by the principal components selected for modeling is higher in the cases of males, ranging from 72.63% to 93.77% and lower for females between 18 and 64 years, for which the first principal component only explain the 11.46% of the total variation. In this case the copula function plays a central role determining the risk profile of the portfolio.

TABLE 2:

PERCENTAGE OF THE VARIABILITY EXPLAINED BY THE FIRST 10 PRINCIPAL COMPONENTS IN THE UNITED KINGDOM'S MORTALITY RATES AND OPTIMAL NUMBER OF FACTORS

	r	**1**	**2**	**3**	**4**	**5**
	0-109+	63,99%	69,34%	72,63%	75,03%	77,06%
	18-64	63,78%	69,75%	73,51%	76,34%	79,07%
	65-109+	75,91%	86,31%	88,92%	90,45%	91,77%
Males		**6**	**7**	**8**	**9**	**10**
	0-109+	79,01%	80,84%	82,38%	83,73%	84,98%
	18-64	81,45%	83,48%	85,25%	86,97%	88,49%
	65-109+	92,84%	93,77%	94,43%	95,00%	95,52%
	r	**1**	**2**	**3**	**4**	**5**
	0-109+	23,50%	30,88%	35,97%	40,47%	44,71%
	18-64	11,47%	19,83%	26,94%	33,54%	39,40%
Females	65-109+	48,79%	64,62%	67,91%	71,09%	73,97%
		6	**7**	**8**	**9**	**10**

0-109+	48,72%	52,38%	55,60%	58,69%	61,56%
18-64	44,77%	49,50%	53,89%	58,02%	61,92%
65-109+	76,32%	78,45%	80,51%	82,28%	84,00%

NOTE: series of the differences of log-mortality rates. Data of the United Kingdom from 1950 to 2011 taken from the Human Mortality Database. They correspond to males and females in different age populations. The highlighted numbers correspond to the number of factors, r, identified following the optimality criterion by Bai and Ng (2002).

5. RESULTS AND DISCUSSION

Our empirical section uses data for the United Kingdom. It compares M1, M9, M11 and M12 from Table 1 (M10 is comprised as a special case of M11). All factor models presented here were estimated using Matlab. In the estimations we used of some routines from the web page of Prof. Serena Ng (http://www.columbia.edu/~sn2294/) to estimate the DFM, and to select the optimal number of static and dynamic factors. To estimate the GDFM, both, one-side and two-sides filters we used codes from the web page of Prof. Mario Forni. (http://morgana.unimore.it/forni_mario/matlab.htm). To estimate the copula-functions instead we used R, specifically we used the package CDVine.

5.1. Forecasting

We define the h-step ahead forecast for mortality rate i, and its associated Mean Squared Forecasting Error as:

$$\hat{x}_{i,T+h|T} = x_{i,T} + \Delta\hat{x}_{i,T+1|T} + \cdots + \Delta\hat{x}_{i,T+h|T}, \tag{5.1}$$

$$MSFE_i^h = \frac{1}{T_1 - T_0 - h + 1}\sum_{T=T_0}^{T_1-h}\left(\hat{x}_{i,T+h|T} - x_{T+h}\right), \tag{5.2}$$

where T_0 is the last year within the sample (in our case 2001), T_1 is the last year out of sample (in our case 2011). Thus we use years from 1950 to 2001 to estimate the models and the last 10 years to measure their relative performance. The MSFE for several forecasting horizons is presented for males and females in Figure 2.

Given that we are primarily concerned with risk measures calculated in short periods of time (for example one year ahead, when a firm has to set a new capital buffer), we confined our forecasting analysis from 1 to 10 years. The MSFE, presented for each model, is a weighted average of the individual forecasting errors, for the individual changes in log-motility rates, the weights being the population in each category for the cases considered in the exercise.

	MALES	FEMALES

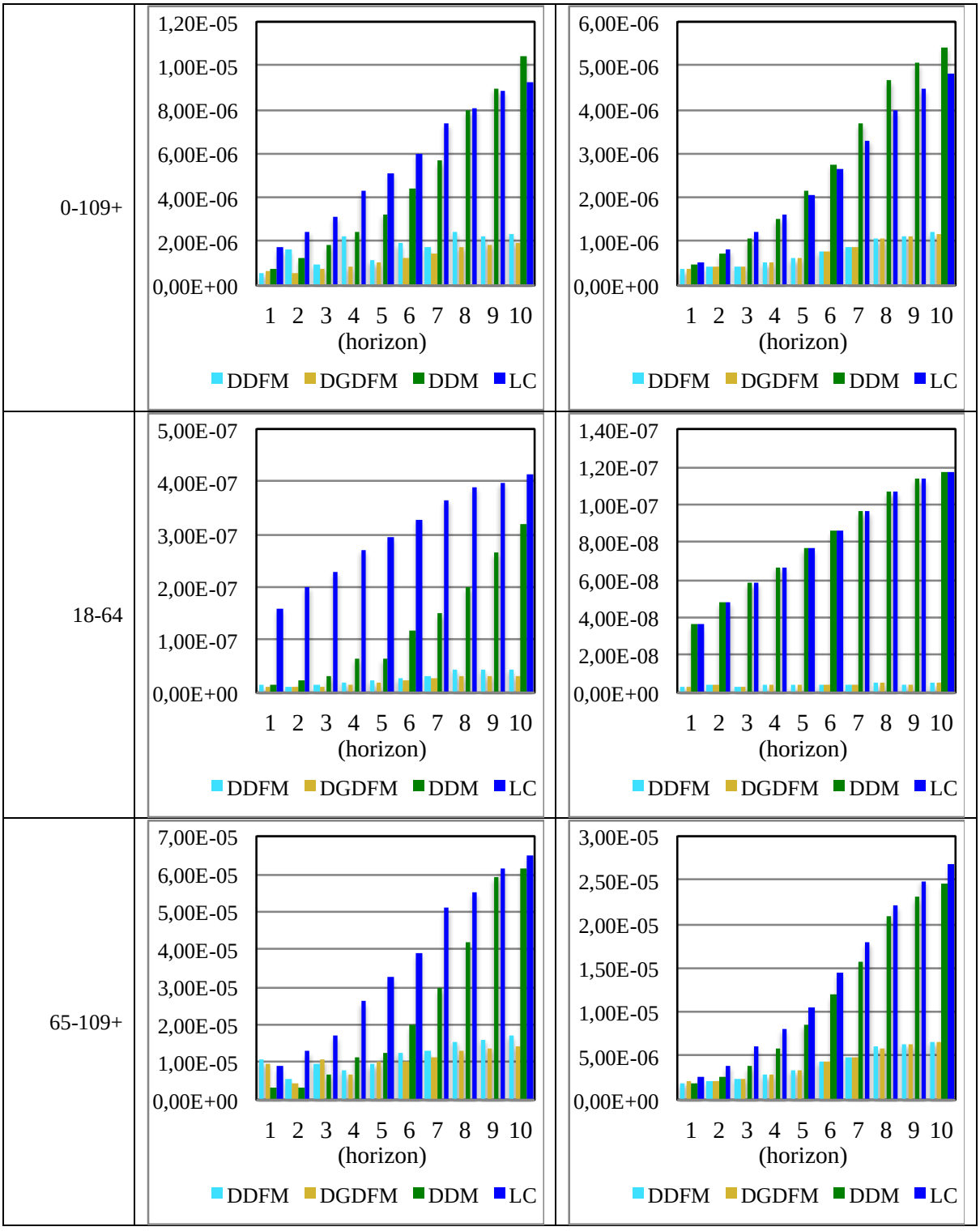

Figure 2: Mean Squared Forecasting Error by Forecasting Horizon. From left to right: DFM and GDFM on the log-differences, DFM and Lee-Carter on the log-levels. The models were estimated using the United Kingdom mortality data from 0 to 109+ years. The estimation period runs from 1950 to 2001, and the forecasting period comprises 10 years, from 2002 to 2011. The participation in the hypothetical populations of 30,000 individuals was set according to the UK population discriminated by ages in the year 2011. Source: Human Mortality Database

In general, the GDFM performs better than the other models in forecasting, especially for medium- and log-term horizons. The worst model is in general the Lee-Carter Model, followed by the Dynamic Model in log-rates. We confirmed the finding by Mitchell et al. (2013), namely that the models 'in differences' outperform the models in 'levels'. Our intuition for this finding is that the log-rates models impose a cointegration relationship between the mortality rates that can be false for the 110 rates as a whole. It is, even if some subsets of the variables are effectively cointegrated, i.e. they shared the same common trend, some others certainly do not and therefore, imposing a common trend can derive in spurious estimation of the factor loads and the forecasting projections. This in turn could deteriorate the forecasting exercise, especially for medium or long horizons. Additionally, we find that the adjustment, through the spectral density matrices applied in the GDFM improves the forecast, as expected, over de standard DFM, but the improvement is small in magnitude.

We also present information regarding the forecasting performance of the models by age-ranges (Figures 3 and 4 in the Appendix). We find that all the models perform better for ages between 0 and 95 years than for ages above 95. This poor performance is natural, given that higher ages are associated to bigger variability from year to year. Interestingly, the models in log levels seem to perform better than the models in differences at these ages. This could be related to the information loss in the process of differentiation and issues relating over-fitting of the models in differences for higher ages, which in turn, make them particularly weak dealing with outliers or huge variability. Thus, for the last ranges we conclude that the loss of information generated from the differentiation of the series, and because of a possible over-fitting, is bigger than the gain derived from avoiding the imposition of a cointegration relationship, at least for short-run horizons. Nevertheless, this 'outperforming' vanishes as the forecast is forwarded in time. It is, for 10 years ahead, the models in differences outperform the models in log-levels, even at ages above 95. The last result can be explained according to sections 2 and 3 because the models in differences do not impose an implausible cointegration relation among all the series in the system.

5.2. Longevity and Mortality Risks

We calculate TVaR and VaR measures one year ahead, using copulae. The procedure is as follows:

- In order to estimate the copula functions, first we transformed the original data (i.e. the idiosyncratic components of the GDFM) using the empirical distribution function. This step provides us a pseudo-sample as described in (2.11), which must be distributed as a standard Uniform, if the empirical distribution is a good approximation of the marginal density. In order to check the accuracy of the procedure we simulated a Uniform random variable in the interval [0,1] and then, we compare the simulated variable with each series in the pseudo-sample, by the means of a Kolmogorov-Smirnov's statistic. In every case the null hypothesis is not rejected and therefore, the empirical distribution is a good approximation of the marginal, as needed for this empirical exercise (see results in tables 5 and 6 in the Appendix).

- The selection of each conditional copula in the estimation is an empirical concern. Therefore we considered different alternatives, which summarized different possible dependence structures in the data in a flexible way. We compare specifically copulae: Gaussian, Clayton, BB6, Survival-Joe, Rotated-Clayton (180 degrees and 270 degrees). We tried more than 40 copulae in a preliminary exercise and most of the times, these six copulae performed much better than other alternatives, so we concentrate our search for the best copula on these six functions. We selected the best copula among the candidates, through AIC criterion, and used it to construct the 'multivariate dependence tree' as shown in Figure 1.
- Lastly, we report the Tail-VaR and VaR at 0.5% (left tail) and the same statistics at 99.5 percentiles (right tail) in Table 3. This level of confidence is a standard practice in the insurance market. In this way, when we are located at the 0.5% VaR or TVaR, we are concerned with 'longevity risk', and conversely at the right tail we are regarding to 'mortality risk'. In both cases risk has to be understood as a significant dispersion from the expected number of deaths, forecasted with the GDFM.

TABLE 3:

LONGEVITY AND MORTALITY RISKS FOR THREE PORTFOLIOS OF SIZE 30,000 (FORECASTING ONE YEAR AHEAD)

		TVaR	VaR	Expected Deaths	VaR	TVaR
	Level	**0.5**	**0.5**		**99.5**	**99.5**
Males	0-109+	238	239	252	261	262
	18-64	78	78	80	82	82
	65-109+	1283	1288	1346	1394	1399
Females	0-109+	254	255	259	263	263
	18-64	52	52	54	56	56
	65-109+	1242	1244	1257	1269	1270

NOTE: The statistics were calculated using data for the United Kingdom from 1950 to 2011, taken from The Human Mortality Database. The forecasting horizon was set at one year. In the second column are labeled the different population-portfolios that we consider in the exercise. The hypothetical populations preserve the same age composition than the United Kingdom in the year 2011.

The results concerning the forecasting exercise can be read in Table 3 as the expected number of events (deaths) in one year. In the case of 30,000 males between 0-109+ years, in a portfolio that mimics the population structure of the UK males in 2011, we expect to see 252 events (deaths), while we expect to see 259 events in a portfolio composed only by woman, once again preserving the UK female's population structure for these ages.

Instead, in a population composed by 30,000 males between 18 and 64 years, we expect to observe 80 deaths in one year, and 54 events in the case of females at the same ages. Lastly, and more important, if we were to construct a population-portfolio with males between 65 and 109+ years, we would expect to observe in one year 1346 events of death; while in the same circumstances with female affiliates we would expect to observe 1257 events.

In terms of longevity risk we found that, as expected, longevity risk arises as the age advances. It is, the oldest part of the population not only shows higher mortality rates, which is obvious, but also presents greater variability and therefore, forecasting ages above 65 years is subject to considerably bigger uncertainty, which is to be understood as a greater risk. For example, the VaR (99.5%) for males between 0 and 109+ years is 261 and the VaR (0.5%) is 239. For woman these values are 263 and 255 respectively. These estimations provide useful insights for the operation of any insurance or pension's company. They tell us that it is possible to assert with a 99.5% of statistical-confidence, in one year, that no more than 261 persons will die, or conversely that no less than 239 will die, even when one expect 252 to die.

The calculations increase significantly for the upper ages, for males and females, but in bigger proportion for the former. In a portfolio of males between 65 and 109+ years, the one with the higher longevity (and mortality) risk, you can expect with a 99.5% of confidence to observe 1,346-1,288=58 persons surviving in one year above the projections provided by the best available model (the GDFM) or 1,394-1,346=48 people dying above such projection. If we instead use the TVaR's to make the same calculations, we would find that 63 people will survive above our expectations and 53 people could die above our expectations.

These results highlight an empirically important finding of our exercise, which is the asymmetric nature of mortality and longevity risks. By using the VaR at 99.5%, longevity risk is 20.8% higher than mortality risk for the portfolio of males (8% in females). Using the TVaR's these numbers are 15.8% and 15.3% respectively. In each case being bigger the longevity risk. Moreover the longevity risk of the older population, lets say only composed by males between 65 and 109+ years, is considerably higher than the mortality risk of a younger population between 18 and 64 years. In fact, in order to make those risks equivalent, you would have to affiliate 29 times more people between 18 and 64 than those between 65 and 109+, expecting some kind of 'cancellation' among the longevity and mortality risks of different population-portfolios.

Lastly, we compare longevity risk in males and females about the same ages (but with different population structures). We observe that it is bigger for males than females. It is not contradictory with the very well documented fact that woman tend to live longer than males, indeed it is because of the female greater longevity that male rate variability is greater in the upper ages and therefore more difficult to forecast.

6. CONCLUSIONS

Introducing dynamics by the means of adding lags (new factors) to forecast mortality rates generates a better fitting of the models to data, especially when modeling male populations; although it is the differentiation of the series, which increase considerably the forecasting capability of the factors models. We provide some intuition for these findings and we show that the gains in terms of forecasting are bigger as the forecast horizon increases. The intuition of this finding arises from the fact that traditional factor models on the log-mortality rates, as the Lee-Carter's model, impose a cointegration relationship among all the series in the system by assumption, which is very unlikely to be observed in the data, due to the great number of series in this kind of exercises, namely more than 200 series for males and females in our example.

Further improvements can be achieved by using (one sided) generalized principal components in the estimation stage. Generalized Factor Models allow for a richer dynamics in the data and use the information contained in the spectral density matrix to improve the fit of the model.

It has to be noticed however that the models in differences perform worse than the models in levels, for short forecasting horizons at ages above 95 years. This finding has to do with the fact that the greater fitting of the models in differences comes at the expense of some degree of over-fitting for the older population, especially for males. This over-performance of the models in levels disappears as the forecast horizon increases and, indeed, for 10 years ahead it completely reverses. The intuition is that the imposition of the shared-stochastic trend deteriorates the forecasting in a cumulative fashion, and it results worse than the over-fitting disadvantage of the models in differences, for ages above 95, in medium and long-term forecasting horizons.

Finally, we found that longevity risk is larger for older portfolios and particularly larger for men than for women. There is also an asymmetric relation between longevity and mortality risks, which makes it difficult to try to compensate one risk in one population with the other risk in a different exposed population.

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HELENA CHULIÁ
Universitat de Barcelona, Department of Econometrics and Riskcenter-IREA, Spain
E-Mail: hchulia@ub.edu

MONTSERRAT GUILLÉN

Universitat de Barcelona, Department of Econometrics and Riskcenter-IREA, Spain,
E-Mail: mgillen@ub.edu

JORGE M. URIBE (Corresponding author)
Universidad del Valle, Department of Economics, Colombia, and Universitat de Barcelona, Department of Economics, Spain,
E-Mail: jorge.uribe@correounivalle.edu.co

APPENDIX

TABLE 4:
NUMBER OF STATISTICS INDICATING A UNIT ROOT

		ADF	KPSS
Males	logs	97	101
	Δlogs	0	7
Females	logs	86	101
	Δlogs	0	1

NOTE: series from 0 to 100 years (101 series). We report the number of statistics showing evidence in favor of unit root behavior. The null-hypothesis of the ADF statistic is that there is a unit root, while the null of the KPSS statistic is stationarity. In the great majority of cases the series in 'levels' are shown to present a unit root, while the series in 'differences' are not. We use data for Males and Females in the United Kingdom for years 1950 to 2011 to perform the tests. The data were taken from the Human Mortality Database.

TABLE 5:
KOLMOGOROV-SMIRNOV STATISTICS AND P-VALUES. MALES 0-109+.

Age	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*
KS	0,08	0,06	0,09	0,14	0,11	0,04	0,08	0,11	0,06	0,09	0,09	0,11	0,09	0,08	0,18
P-value	0,98	0,99	0,93	0,52	0,82	0,99	0,98	0,82	0,99	0,93	0,93	0,82	0,93	0,98	0,27
Age	*15*	*16*	*17*	*18*	*19*	*20*	*21*	*22*	*23*	*24*	*25*	*26*	*27*	*28*	*29*
KS	0,24	0,08	0,09	0,09	0,16	0,08	0,11	0,09	0,08	0,08	0,14	0,09	0,13	0,09	0,11
P-value	0,04	0,98	0,93	0,93	0,38	0,98	0,82	0,93	0,98	0,98	0,52	0,93	0,67	0,93	0,82
Age	*30*	*31*	*32*	*33*	*34*	*35*	*36*	*37*	*38*	*39*	*40*	*41*	*42*	*43*	*44*
KS	0,13	0,11	0,22	0,08	0,16	0,08	0,08	0,16	0,09	0,14	0,08	0,13	0,09	0,09	0,11
P-value	0,67	0,82	0,08	0,98	0,38	0,98	0,98	0,38	0,93	0,52	0,98	0,67	0,93	0,93	0,82
Age	*45*	*46*	*47*	*48*	*49*	*50*	*51*	*52*	*53*	*54*	*55*	*56*	*57*	*58*	*59*
KS	0,16	0,13	0,09	0,24	0,09	0,08	0,11	0,11	0,11	0,08	0,14	0,06	0,14	0,19	0,09
P-value	0,38	0,67	0,93	0,04	0,93	0,98	0,82	0,82	0,82	0,98	0,52	0,99	0,52	0,18	0,93
Age	*60*	*61*	*62*	*63*	*64*	*65*	*66*	*67*	*68*	*69*	*70*	*71*	*72*	*73*	*74*
KS	0,16	0,11	0,11	0,16	0,14	0,13	0,09	0,08	0,08	0,11	0,09	0,06	0,08	0,09	0,16
P-value	0,38	0,82	0,82	0,38	0,52	0,67	0,93	0,98	0,98	0,82	0,93	0,99	0,98	0,93	0,38
Age	*75*	*76*	*77*	*78*	*79*	*80*	*81*	*82*	*83*	*84*	*85*	*86*	*87*	*88*	*89*
KS	0,06	0,11	0,16	0,13	0,08	0,19	0,08	0,11	0,08	0,14	0,13	0,09	0,18	0,11	0,13
P-value	0,99	0,82	0,38	0,67	0,98	0,18	0,98	0,82	0,98	0,52	0,67	0,93	0,27	0,82	0,67
Age	*90*	*91*	*92*	*93*	*94*	*95*	*96*	*97*	*98*	*99*	*100*	*101*	*102*	*103*	*104*
KS	0,11	0,08	0,14	0,11	0,09	0,14	0,09	0,09	0,13	0,11	0,08	0,08	0,09	0,11	0,11
P-value	0,82	0,98	0,52	0,82	0,93	0,52	0,93	0,93	0,67	0,82	0,98	0,98	0,93	0,82	0,82
Age	*105*	*106*	*107*	*108*	*109*										
KS	0,13	0,24	0,22	0,11	0,06										
P-value	0,67	0,04	0,08	0,82	0,99										

KOLMOGOROV-SMIRNOV STATISTICS AND P-VALUES. MALES 18-64.

Age	*18*	*19*	*20*	*21*	*22*	*23*	*24*	*25*	*26*	*27*	*28*	*29*	*30*	*31*	*32*
KS	0,13	0,13	0,08	0,09	0,09	0,09	0,09	0,09	0,06	0,06	0,09	0,08	0,09	0,06	0,11
P-value	0,67	0,67	0,98	0,93	0,93	0,93	0,93	0,93	0,99	0,99	0,93	0,98	0,93	0,99	0,82
Age	*33*	*34*	*35*	*36*	*37*	*38*	*39*	*40*	*41*	*42*	*43*	*44*	*45*	*46*	*47*
KS	0,11	0,06	0,06	0,06	0,08	0,13	0,06	0,13	0,09	0,13	0,08	0,06	0,09	0,13	0,09
P-value	0,82	0,99	0,99	0,99	0,98	0,67	0,99	0,67	0,93	0,67	0,98	0,99	0,93	0,67	0,93
Age	*48*	*49*	*50*	*51*	*52*	*53*	*54*	*55*	*56*	*57*	*58*	*59*	*60*	*61*	*62*
KS	0,16	0,09	0,14	0,08	0,19	0,13	0,08	0,09	0,11	0,14	0,08	0,11	0,21	0,09	0,14
P-value	0,38	0,93	0,52	0,98	0,18	0,67	0,98	0,93	0,82	0,52	0,98	0,82	0,12	0,93	0,52
Age	*63*	*64*													
KS	0,22	0,14													
P-value	0,08	0,52													

KOLMOGOROV-SMIRNOV STATISTICS AND P-VALUES. MALES 65-109+.

Age	*65*	*66*	*67*	*68*	*69*	*70*	*71*	*72*	*73*	*74*	*75*	*76*	*77*	*78*	*79*
KS	0,08	0,11	0,11	0,13	0,1	0,07	0,13	0,11	0,15	0,13	0,08	0,11	0,08	0,1	0,11
P-value	0,99	0,82	0,82	0,67	0,93	1	0,67	0,82	0,52	0,67	0,99	0,82	0,99	0,93	0,82
Age	*80*	*81*	*82*	*83*	*84*	*85*	*86*	*87*	*88*	*89*	*90*	*91*	*92*	*93*	*94*
KS	0,13	0,08	0,13	0,11	0,1	0,1	0,08	0,08	0,08	0,11	0,11	0,13	0,15	0,08	0,15
P-value	0,67	0,99	0,67	0,82	0,93	0,93	0,99	0,99	0,99	0,82	0,82	0,67	0,52	0,99	0,52
Age	*95*	*96*	*97*	*98*	*99*	*100*	*101*	*102*	*103*	*104*	*105*	*106*	*107*	*108*	*109*
KS	0,11	0,1	0,11	0,08	0,1	0,11	0,11	0,11	0,2	0,2	0,05	0,08	0,11	0,16	0,11
P-value	0,82	0,93	0,82	0,99	0,93	0,82	0,82	0,82	0,19	0,19	1	0,99	0,82	0,39	0,82

NOTE: The null of Kolmogorov-Smirnov test is that the pseudo-sample is Uniform [0,1]. We performed the same test for each portfolio-population: 0-109+, 18-64, 65-109+ years. The marginal of each mortality rate series from 1950 to 2011, was constructed using the data of the United Kingdom. We forecasted one-year ahead using the Generalized Dynamic Factor Model and we use the empirical cumulative distribution to construct the pseudo-sample. In no case the null is rejected, therefore we can rely on our estimations of the margins

TABLE 6:

KOLMOGOROV-SMIRNOV STATISTICS AND P-VALUES. MALES 0-109+.

Age	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*
KS	0,13	0,14	0,06	0,14	0,08	0,09	0,16	0,08	0,06	0,11	0,11	0,11	0,11	0,08	0,06
P-value	0,67	0,52	0,99	0,52	0,98	0,93	0,38	0,98	0,99	0,82	0,82	0,82	0,82	0,98	0,99
Age	*15*	*16*	*17*	*18*	*19*	*20*	*21*	*22*	*23*	*24*	*25*	*26*	*27*	*28*	*29*
KS	0,11	0,08	0,13	0,09	0,08	0,11	0,11	0,13	0,14	0,16	0,04	0,11	0,06	0,09	0,13
P-value	0,82	0,98	0,67	0,93	0,98	0,82	0,82	0,67	0,52	0,38	0,99	0,82	0,99	0,93	0,67
Age	*30*	*31*	*32*	*33*	*34*	*35*	*36*	*37*	*38*	*39*	*40*	*41*	*42*	*43*	*44*
KS	0,13	0,09	0,09	0,06	0,09	0,06	0,08	0,09	0,09	0,13	0,08	0,09	0,16	0,16	0,14
P-value	0,67	0,93	0,93	0,99	0,93	0,99	0,98	0,93	0,93	0,67	0,98	0,93	0,38	0,38	0,52
Age	*45*	*46*	*47*	*48*	*49*	*50*	*51*	*52*	*53*	*54*	*55*	*56*	*57*	*58*	*59*

KS	0,13	0,08	0,18	0,09	0,11	0,11	0,13	0,21	0,11	0,18	0,13	0,09	0,11	0,08	0,08
P-value	0,67	0,98	0,27	0,93	0,82	0,82	0,67	0,12	0,82	0,27	0,67	0,93	0,82	0,98	0,98
Age	*60*	*61*	*62*	*63*	*64*	*65*	*66*	*67*	*68*	*69*	*70*	*71*	*72*	*73*	*74*
KS	0,09	0,11	0,13	0,08	0,13	0,18	0,13	0,09	0,11	0,11	0,11	0,04	0,08	0,13	0,18
P-value	0,93	0,82	0,67	0,98	0,67	0,27	0,67	0,93	0,82	0,82	0,82	0,99	0,98	0,67	0,27
Age	*75*	*76*	*77*	*78*	*79*	*80*	*81*	*82*	*83*	*84*	*85*	*86*	*87*	*88*	*89*
KS	0,08	0,09	0,16	0,08	0,09	0,06	0,09	0,11	0,11	0,08	0,18	0,13	0,11	0,08	0,09
P-value	0,98	0,93	0,38	0,98	0,93	0,99	0,93	0,82	0,82	0,98	0,27	0,67	0,82	0,98	0,93
Age	*90*	*91*	*92*	*93*	*94*	*95*	*96*	*97*	*98*	*99*	*100*	*101*	*102*	*103*	*104*
KS	0,11	0,09	0,06	0,08	0,14	0,08	0,13	0,14	0,13	0,06	0,08	0,08	0,08	0,14	0,14
P-value	0,82	0,93	0,99	0,98	0,52	0,98	0,67	0,52	0,67	0,99	0,98	0,98	0,98	0,52	0,52
Age	*105*	*106*	*107*	*108*	*109*										
KS	0,08	0,16	0,14	0,09	0,09										
P-value	0,98	0,38	0,52	0,93	0,93										

KOLMOGOROV-SMIRNOV STATISTICS AND P-VALUES. MALES 18-64.

Age	*18*	*19*	*20*	*21*	*22*	*23*	*24*	*25*	*26*	*27*	*28*	*29*	*30*	*31*	*32*
KS	0,09	0,06	0,18	0,14	0,11	0,11	0,09	0,11	0,04	0,06	0,14	0,08	0,11	0,13	0,09
P-value	0,93	0,99	0,27	0,52	0,82	0,82	0,93	0,82	0,99	0,99	0,52	0,98	0,82	0,67	0,93
Age	*33*	*34*	*35*	*36*	*37*	*38*	*39*	*40*	*41*	*42*	*43*	*44*	*45*	*46*	*47*
KS	0,06	0,11	0,16	0,14	0,06	0,13	0,09	0,11	0,16	0,06	0,13	0,09	0,09	0,06	0,11
P-value	0,99	0,82	0,38	0,52	0,99	0,67	0,93	0,82	0,38	0,99	0,67	0,93	0,93	0,99	0,82
Age	*48*	*49*	*50*	*51*	*52*	*53*	*54*	*55*	*56*	*57*	*58*	*59*	*60*	*61*	*62*
KS	0,21	0,06	0,13	0,13	0,11	0,18	0,16	0,06	0,11	0,16	0,09	0,09	0,09	0,19	0,14
P-value	0,12	0,99	0,67	0,67	0,82	0,27	0,38	0,99	0,82	0,38	0,93	0,93	0,93	0,18	0,52
Age	*63*	*64*													
KS	0,08	0,09													
P-value	0,98	0,93													

KOLMOGOROV-SMIRNOV STATISTICS AND P-VALUES. MALES 65-109+.

Age	*65*	*66*	*67*	*68*	*69*	*70*	*71*	*72*	*73*	*74*	*75*	*76*	*77*	*78*	*79*
KS	0,09	0,09	0,09	0,09	0,13	0,09	0,11	0,09	0,09	0,18	0,11	0,08	0,11	0,06	0,14
P-value	0,93	0,93	0,93	0,93	0,67	0,93	0,82	0,93	0,93	0,27	0,82	0,98	0,82	0,99	0,52
Age	*80*	*81*	*82*	*83*	*84*	*85*	*86*	*87*	*88*	*89*	*90*	*91*	*92*	*93*	*94*
KS	0,11	0,13	0,06	0,13	0,13	0,11	0,09	0,08	0,09	0,13	0,06	0,09	0,13	0,21	0,11
P-value	0,82	0,67	0,99	0,67	0,67	0,82	0,93	0,98	0,93	0,67	0,99	0,93	0,67	0,12	0,82
Age	*95*	*96*	*97*	*98*	*99*	*100*	*101*	*102*	*103*	*104*	*105*	*106*	*107*	*108*	*109*
KS	0,16	0,13	0,08	0,11	0,06	0,13	0,14	0,13	0,08	0,09	0,08	0,14	0,16	0,11	0,09
P-value	0,38	0,67	0,98	0,82	0,99	0,67	0,52	0,67	0,98	0,93	0,98	0,52	0,38	0,82	0,93

NOTE: The null of Kolmogorov-Smirnov test is that the pseudo-sample is Uniform [0,1]. We performed the same test for each portfolio-population: 0-109+, 18-64, 65-109+ years. The marginal of each mortality rate series from 1950 to 2011, was constructed using the data of the United Kingdom. We forecasted one-year ahead using the Generalized Dynamic Factor Model and we use the empirical cumulative distribution to

construct the pseudo-sample. In no case the null is rejected, therefore we can rely on our estimations of the margins.

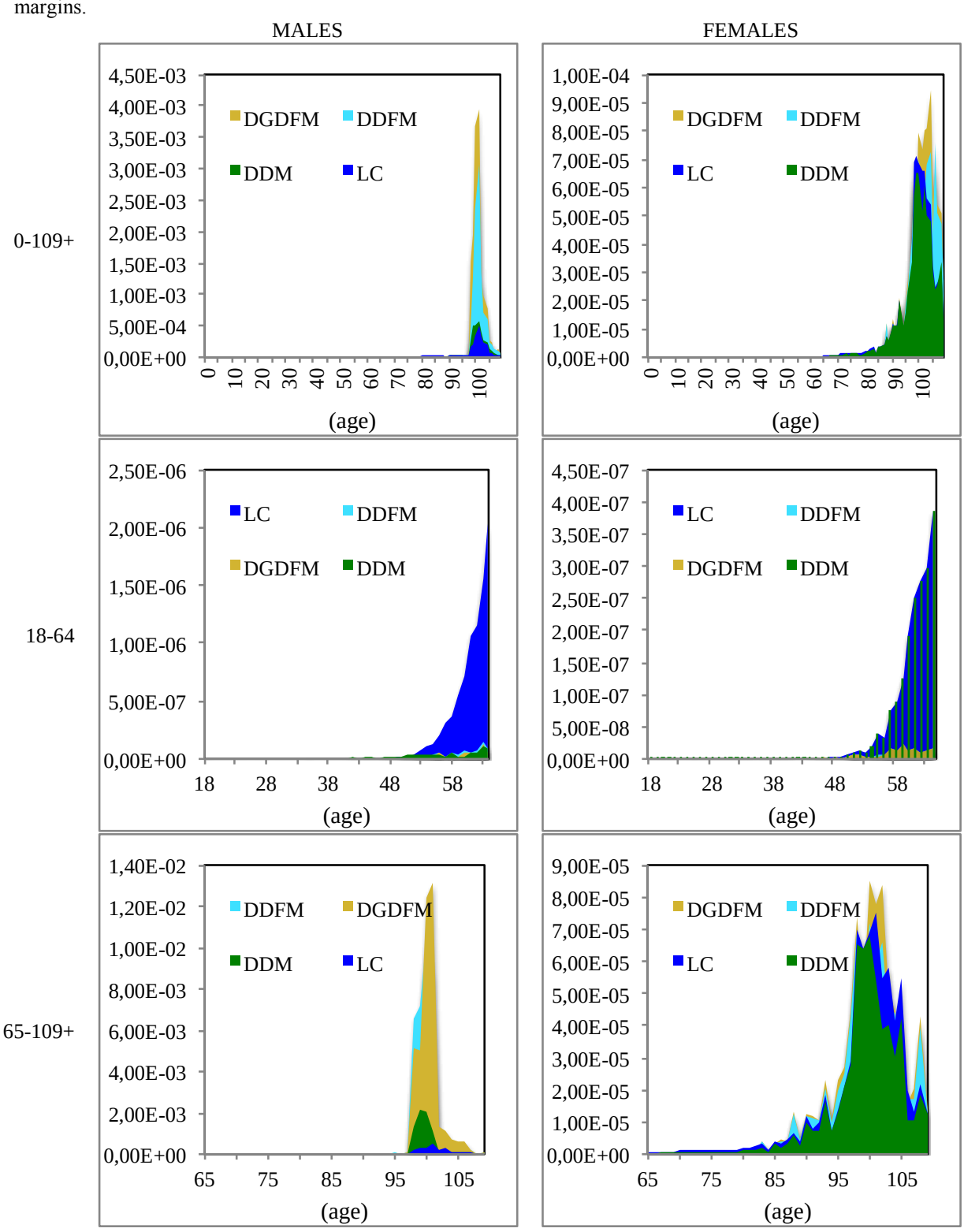

Figure 3: Mean Squared Forecasting Error (MSFE) by Age: One-year-ahead. The models were estimated using the United Kingdom mortality data from 0 to 109+ years. The estimation period runs from 1950 to 2010, and the forecasting is for the year 2011. The participation in the hypothetical populations of 30,000 males or females, was set according to the UK population discriminated by ages in the year 2011.

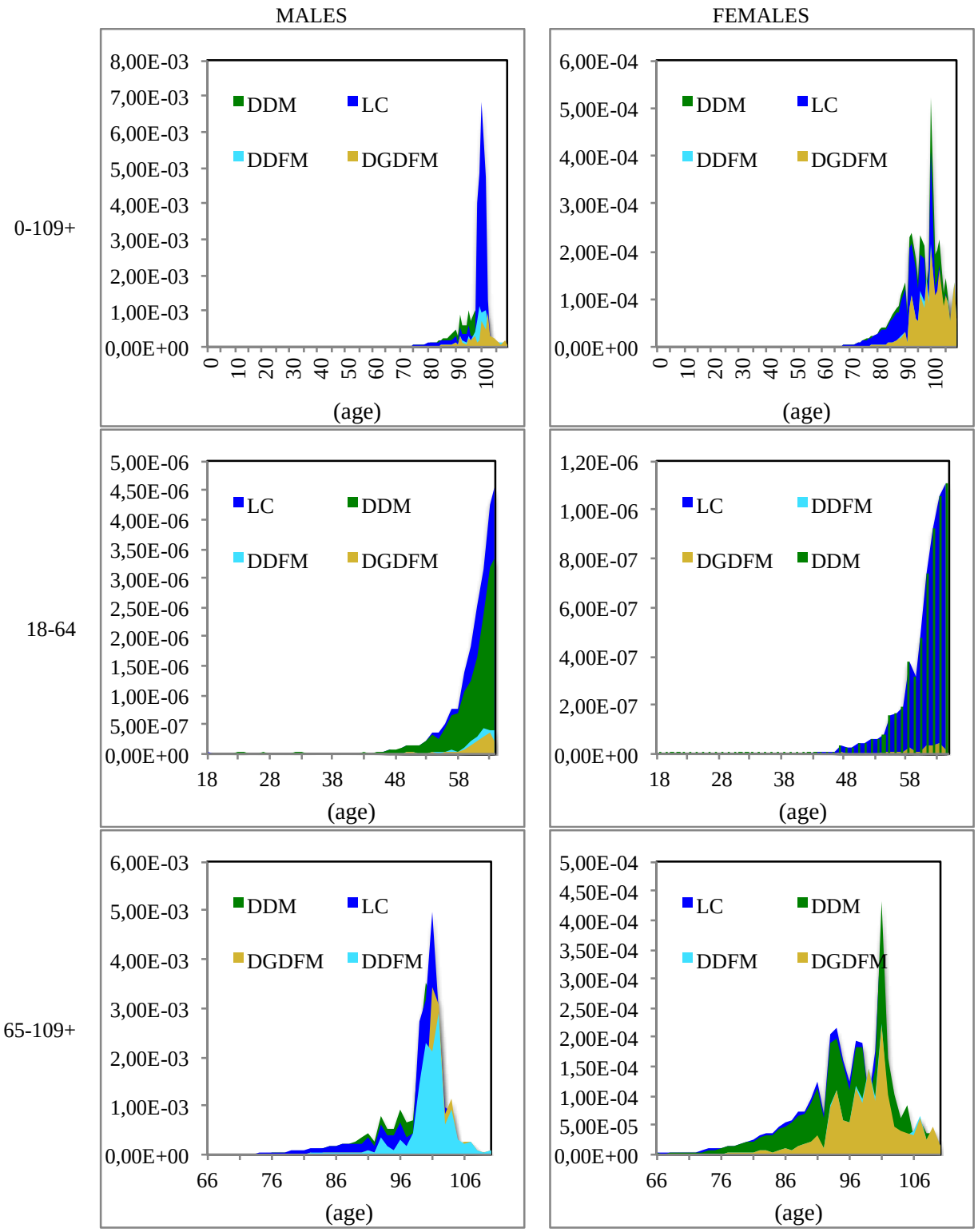

Figure 4: Mean Squared Forecasting Error (MSFE) by Age: Ten-years-ahead. The models were estimated using the United Kingdom mortality data from 0 to 109+ years. The estimation period runs from 1950 to 2001, and the forecasting is for the year 2011. The participation in the hypothetical populations of 30,000 males or females, was set according to the UK population discriminated by ages in the year 2011.

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