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AN APPROACH TO N-PERSON COOPERATIVE GAMES

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Abstract

This work is an overview on n-person cooperative games in Game Theory, the mathematical theory of interactive decision situations characterized by a group of agents, each of whom has to make a decision based on their own preferences on the set of outcomes. These situations are called *games*, agents are *players* and decisions are *strategies*. By focusing on Cooperative Game Theory, we analyze concepts such as coalition formation, equilibrium, stability, fairness and the most important proposed solution concepts.

Keywords: Cooperative Game Theory, Cooperative game, Shapley value, nucleolus, core, bankruptcy problem, airport problem, indices of power, voting games.

Resum

Aquest treball tracta sobre els jocs n-personals cooperatius en Teoria de Jocs, la branca de les matemàtiques que estudia i analitza les interaccions entre una sèrie d'agents que han de prendre una decisió, segons les seves preferències sobre el conjunt de possibles resultats dels jocs. En centrar-nos en la Teoria de Jocs Cooperatius, analitzarem conceptes com la formació de coalicions, l'equilibri, la justícia i les propostes de conceptes de solucions més importants.

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Contents

Chapter 1

Introduction

Game Theory is the mathematical theory of interactive decision situations characterized by a group of agents —each of whom has to make a decision—, the set of possible outcomes and the preferences that each agent has on that set of outcomes. These situations are called *games*, agents are *players* and decisions are *strategies*.

In Robert J. Aumann's —one of the most active Game Theory researchers— words, "Game Theory is optimal decision making in the presence of others with different objectives". One can argue that does not sound too far away from our everyday lives or that such a generic definition will have a broad range of applications.

Game Theory is concerned with both cooperative and noncooperative models, with the latter being the most studied of the two branches. Although Noncooperative Game Theory is able to include cooperation within its reach, the complexity of the description of some situations with a mathematical model has led game theorists to regard the necessity of building cooperative models as an imperative one.

Both of these fields study strategic aspects of cooperation and competition among the players. In Noncooperative Game Theory, players are assumed to choose their actions individually, selfishly seeking to realize their own goals and to maximize their own profit. While this does not mean that players are necessarily adversarial to other players, they are not interested in other players' welfare either. In contrast, in Cooperative Game Theory we deal with coalitions and allocations, since groups of players will be willing to join forces and to allocate the benefits derived from their cooperation.
Analytically, the real distinction between the two branches of Game Theory is that the first one specifies various actions that are available to the players while the second describes the outcomes that result when the players come together in different combinations.

This is what Aumann himself had to say about the idea hidden behind Game Theory:

"Cooperative theory starts with a formalization of games that abstracts away altogether from procedures and [...] concentrates, instead, on the possibilities for agreement. [...] There are several reasons that explain why cooperative games came to be treated separately. One is that when one does build negotiation and enforcement procedures explicitly into the model, then the results of a non-cooperative analysis depend very strongly on the precise form of the procedures, on the order of making offers and counter-offers and so on. This may be appropriate in voting situations in which precise rules of parliamentary order prevail, where a good strategist can indeed carry the day. But problems of negotiation are usually more amorphous; it is difficult to pin down just what the procedures are. More fundamentally, there is a feeling that procedures are not really all that relevant; that it is the possibilities for coalition forming, promising and threatening that are decisive, rather than whose turn it is to speak. [...] Detail distracts attention from essentials. Some things are seen better from a distance; the Roman camps around Metzada are indiscernible when one is in them, but easily visible from the top of the mountain."

Thus, following this line of thought, this work starts from the basics and gradually gains some perspective in order to observe the whole picture.

1.1 Motivation: a glance at the cooperative side of Game Theory

Although Game Theory often provides us with problems concerning conflict, there are plenty of situations in which cooperating is way more beneficial than fighting among ourselves. Besides, even in a me-first capitalist society like the one we live in nowadays, the objective of every interaction is not always to get the most money, and even when that would be the case, we may have social behaviour rules that prevent us from being completely selfish. Indeed, concepts like fairness or equality may arise when we have to decide how to divide the benefits obtained from cooperation. Those benefits are not always as obvious as when we are making coalitions while bargaining: evolution can also show us that cooperation within an individual's family to secure the survival of one's genetic particularities is what determines the sex ratios in some populations of insect species.

The question throughout this work is not why to cooperate as much as how to do so, i.e. we will ask ourselves questions such as: what are the agreements that rational players will reach to divide the benefits obtained from cooperation? How should we divide the costs of building a road or an airport between the agents that will use it? What are the stable solutions for those problems? We are talking about games known as general-sum games, since it is natural that for one to gain in utility in a zero-sum game, someone must be losing and that could never reinforce cooperation between rational players.

As we mentioned, in Game Theory, a *game* is a tool to model any situation in which there are *players* that interact (be it people, animals or computers), by taking decisions in order to attain a certain goal. This mathematical description of conflict began in the twentieth century thanks to the work of John Von Neumann, Oskar Morgenstern and John Nash and one of its first motivations was to help military officers design optimal war strategies. This first motivation clearly falls under Noncooperative Game Theory, therefore being the primary focus of Game Theory. In Noncooperative Game Theory, each player wants to maximize his own payoff and does not consider cooperating with other players. Nowadays, however, Game Theory is applied to a wide range of disciplines, like Biology or Political Science, but above all, to Economy. Many of those applications regard cooperation and therefore fall in the category of Cooperative Game Theory. We will see some of those applications throughout this thesis.

Nash argued that Cooperative and Noncooperative Game Theory are complementary ways of approaching the same problem. Furthermore, Nash defended that if a cooperative solution concept predicts the result of a rational agreement on how to play the game, then a noncooperative analysis of the enlarged negotiation game should yield the same answer. That is to say, noncooperative Game Theory provides a way of testing the predictions which Cooperative Game Theory produces —as easily applied predictions of rational agreements—.

Game Theory assumes that players have a completely rational behaviour. By rational, we mean that players know what is best for them, want to obtain it and think ahead of the game. Therefore, players will prefer some outcomes rather others and have to choose the strategies from their set of strategies that will lead to them. We can represent this by defining utility functions for each player, which could be obtained, according to the Theory of revealed preference —which relies on the assumption that players want to maximize utility—, by assuming that preferences are revealed by the players' habits.

1.2 What is utility?

1.2.1 A first example

Example 1.2.1. A couple decide to go to a new shop in town to buy some goods they need for the family Christmas dinner they are hosting that evening. When it is time for them to pay, both players (the seller and the couple) have to decide whether to trust each other or not: if the seller gives the couple the goods without having been paid, he faces the risk of them leaving without paying; and the same goes for the couple if they pay before having been handed the goods. To make matters worse, as the owner of the shop is new in town, the players cannot know if they can trust each other.

Although it may look like we drove ourselves into a dead end, we can work our way out of it. Naturally, both players want to get (or save) as much money as they can, but if they are not too short-sighted, they will know better than to swindle the other player, for if they do, they will face bigger losses than what they may win in this little game:

- If the owner does not give them the goods after being paid, he may earn a reputation as an dishonest shopkeeper and, hence, he will not make any profit of his business.
- If the couple do not pay for the goods they are given, they risk the owner telling the rest of the town shopkeepers not to trust them in the future, forcing them to go out of town to buy anything they need.

Then, in this example it is clear that money is not, in general, the only concept that comes into play to determine each player's utility in a game. △

1.2.2 Ordinal and linear utility

Since utility is not necessarily representing the monetary gain in every situation, we can characterize players' preferences by two relations between the events of the set of alternatives S: strict preference $\succ$ and indifference $\sim$.

We will find these two relations by defining the weak preference relation $\succcurlyeq \subset S \times S$, which is a transitive, reflexive, antisymmetric and complete binary relation.[1]

Since $A \succcurlyeq B \iff A \succ B \textit{ or } A \sim B$, we can equivalently say that, for each A, B $\in$ S, $A \succ B$ or $B \succ A$ or $A \sim B$.

Definition 1.2.1. In particular, we define the indifference relation $\sim \subset S \times S$ by: $A \sim B \iff A \succcurlyeq B$ and $B \succcurlyeq A$. We can see then that $\sim$ is a reflexive, symmetric and transitive binary relation.

Definition 1.2.2. On the other hand, we define $\succ$ by: $A \succ B \iff B \not\succcurlyeq A$ and see that it is an asymmetric and transitive binary relation.

Definition 1.2.3. A pair of a set of alternatives A and a weak preference $\succeq$ is called a decision problem.

Definition 1.2.4. Given a decision problem (A, $\succeq$), where A is a countable set of alternatives, we can define a utility function u representing $\succ$, $u : A \longrightarrow \mathbb{R}$ s.t., for each pair $a, b \in A, a \succeq b \iff u(a) \geq u(b)$.

[1]By complete, we mean that for each A, B $\in$ S, either $A \succcurlyeq B$ or $B \succcurlyeq A$.

Remark 1.2.1. In particular, $a \sim b \iff u(a) = u(b)$; and $a \succ b \iff u(a) > u(b)$.

This is called ordinal utility, since it only depicts which outcome a player prefers over another one. However, when possible, we can also define a linear utility function $\bar{u} : X \to \mathbb{R}$ representing $\succeq$ in $(X, \succeq)$ which tells us not only which are the preferred outcomes for each player but how much each player wants them.

If $\succeq$ is independent[2] and continuous[3] , there exists a utility function representing $\succeq$ in $(X, \succeq)$. Besides, it is unique up to positive affine transformation.

As we said, we can see in the following example that a player's utility may not equal the monetary value of his outcome:

Example 1.2.2. In a situation in which someone is asked to choose between getting 1€ with a probability of 100% or getting 2€ with 50% probability,

- a *risk averse* player (for example, with $u(x) = \sqrt{x}$) would choose the first option.
- a *risk neutral* player (for example, with $u(x) = x$) would be indifferent between both options.
- a *risk prone* player (for example, with $u(x) = x^2$) would choose the second option.

$\triangle$

We generally assume that players are risk neutral, but one could change this in order to model some situations in a more detailed manner.

1.3 Two is company, three is a crowd

Example 1.3.1. Imagine there are two people (players 1 and 2) who have been given 1$ to share if, and only if, they can reach an agreement on how to split them.

When bargaining, for example in this game (named the 2-player Divide-the-Dollar game, which we will see more of in the coming chapters, see Example 3.4.2), there is a set of all the outcomes of the possible agreements: in this case, it is

$$\{(x, y) \in \mathbb{R}^2 : x, y \geq 0, \text{ and } x + y \leq 1\},$$

where x and y are the payoffs for players 1 and 2, respectively.
Without getting into too much detail here, the Coarse Theorem states that a rational

[2]Given a convex decision problem $(X, \succeq)$, $\succeq$ is such that, for each $x, y, z \in X$ and each $\lambda \in (0, 1]$, $x \succeq y \iff \lambda x + (1 - \lambda)z \succeq \lambda y + (1 - \lambda)z$.

[3]Given a convex decision problem $(X, \succeq)$, $\succeq$ is such that, for each $x, y, z \in X$ such that $\succeq y \succeq z$, there is $\lambda \in (0, 1)$ satisfying $y \sim \lambda x + (1 - \lambda)z$.

agreement will be a pair (x^*, y^*) on the frontier of the set —i.e. a pair (x^*, y^*) such that $x^* + y* = 1$, naturally, since for every other point (x', y') not in the frontier (that is to say, $x' + y' < 1$) there is $\delta > 0$ such that $x' + y' = 1 - \delta$ and, thus, $(x' + \delta/2, y' + \delta/2)$ is a deal from the set that both players would prefer—.

The way negotiation is built —and each player's ability to negotiate— will determine in which way the 1$ will be split, but it is quite obvious that the most fair division would be for each player to get 0.5$. It is also quite clear that once the two players have agreed on that division, they will not move from it. Thus, (0.5, 0.5) is stable.

Now, a third player —named Charles— saw the previous two players —Alice and Bob— already playing Divide-the-Dollar and decided to join them. If we ask the three of them to reach an agreement, the only fair and stable outcome is $(1/3, 1/3, 1/3)$ —let's forget dollars can only be divided into cents here—.

But, what if we ask them to reach a majority of players that agree on a division? Here, the whole functioning of the game changes —as we will see in the following sections, the characteristic function that defines each game are different—. For a start, not all the players are needed for the dollar to be awarded, which puts them all in a very precarious situation. If a $(1/3, 1/3, 1/3)$ division was proposed by player 1, player 2 could offer him a $(1/2, 1/2, 0)$ which both would prefer. However, player 3 could then offer a preferable $(3/5, 0, 2/5)$ to player 1, which could be followed by $(4/5, 1/5, 0)$ and then players 2 and 3 could agree on a $(0, 1/2, 1/2)$ division, which would take us back to the beginning of the game —or an equivalent situation—, from where we could go on endlessly.

Thus, we can see that although there is, indeed, a fair solution to this game either two or three players are playing it, not always such a solution —or any other— is stable for $n = 3$. $\triangle$

Remark 1.3.1. Following that logic, we will dedicate Chapter 2 to two-player games before addressing n-player games in which $n \geq 3$ in Chapter 3.

Chapter 2

A particular case of N-player games: two players

2.1 What are games in Game Theory?

First things first, we need to take our first step by defining games before we can start walking down this avenue. One can quickly see that we are all quite familiar with the idea of games —even if we never heard of them in a mathematical environment—, since they are not all that different from the parlor games we are used to play: there is a set of players who have to make choices (following some rules) which, together with the chance moves [1], determine the outcome. At the end of the game, a payoff is given to each player depending on the outcome.

Strategic games, or games in normal form, are characterized by the set of available strategies to the players and the corresponding payoff functions. In general, as we have discussed, players' payoffs need not be money but more general preferences according to personal inclinations, solidarity or even unselfishness. Whatever the origin from a player preferences may be, once they start playing, players are assumed to be totally rational —meaning that they try to maximize their own payoff and can think in advance of the game—.

Definition 2.1.1. An *n-player strategic game* with set of players $N := \{1, 2, ..., n\}$ is a pair $G := (A, u)$ where A and u are the sets of strategies and the payoff functions, respectively. In particular:

[1] A chance move is one which is not made by any of the players of the game. One clear example are the dealer's moves when dealing the cards in the games of Poker or Blackjack.

- $A := \prod_{i=1}^{n} A_i$, with A_i being the nonempty set of strategies for each player $i \in N$.
- For each $i \in N$, $u_i : A \to \mathbb{R}$ is the payoff function for player i.
  Then, $u := \prod_{i=1}^{n} u_i$ is the function that assigns, to each player, the payoff that they get for a given strategy profile $a \in A$.

Remark 2.1.1. The number of players in the set N, $|N|$, is denoted by a lower case n. In this chapter, $|N| = n = 2$.

We must also take into account that players are assumed to make their decisions simultaneously.

In practical terms, we can define a game in normal form by the n-dimensional array of n-vectors (a matrix with pairs of real numbers, in the case $n = 2$) representing the payoffs each player obtains depending on their strategies.

Remark 2.1.2. On the other side, it is quite useful to represent some games in extensive form, which consists in a game tree in which the players' decisions take the game to a terminal vertex and the payoff function assigns a payoff to each player depending on which terminal vertex the game ends in.
This representation of games is particularly helpful when players are not making their decisions simultaneously or when the information they have at the moment of making them is not complete.

Example 2.1.1. *The game of matching pennies*:
In this basic game, two players have a coin and decide to pick either heads or tails. In fact, it would not make a difference whether it is players themselves or chance who decides which side of the coin to show, but let's assume it is the players' decision so that we can consider heads or tails as their strategies. Then, if both players' coins show the same side (the pennies match), Player 1 wins. If, on the contrary, the pennies do not match, it is Player 2 who wins.

There are many alternative versions of this game —which we will refer to again—, such as *Odds and evens.*

$\triangle$

Remark 2.1.3. Example 2.1.1 is a *constant-sum* game, since the payoffs of both players always add up to the same. Moreover, since that constant value of the sum of both players is zero, we say that it is a *zero-sum* game.

Definition 2.1.2. Given an n-player game Γ, we say that a strategy n-tuple $(\sigma_1^*, \sigma_2^*, ..., \sigma_n^*)$

is in *equilibrium* if, for any player $i \in N = \{1, ..., n\}$ and any strategy $\hat{\sigma}_i$ belonging to the set of available strategies to player i, S_i, the payoff function satisfies:

$$\pi_i(\sigma_1^*, ..., \sigma_{i-1}^*, \sigma_i^*, \sigma_{i+1}^*, ..., \sigma_n^*) \geq \pi_i(\sigma_1^*, ..., \sigma_{i-1}^*, \hat{\sigma}_i, \sigma_{i+1}^*, ..., \sigma_n^*)$$

Therefore, given such a strategy, no player has any motivation to change his strategy since no change in his strategy will lead to a better payoff for him. In this case we say that such a strategy profile is a *Nash equilibrium*.

Remark 2.1.4. One must be conscious that this situation is not possible in every game, although they always exist in finite games of perfect information, in which nothing that has happened in the game so far is hidden from the players when they make a move (and there are no chance moves).

Example 2.1.2. In the game of matching pennies, no strategy is a Nash equilibrium.

△

2.2 Antagonistic games

Having laid the very foundations of the building, we can have a look at the simplest of games: two-person zero-sum games, also called *antagonistic games*.

Zero-sum games receive that name because the sum of the payoffs of all the players is always zero. In particular, when the number of players n is 2, one gains the same payoff as the other loses. Then, this is the last type of games that could foment cooperation —that is, unless one player wants the other to win, which would have to result in a change of the utilities and therefore, into a different game which would most likely not be zero-sum—. Thus, we will only use them to define a couple of essential concepts such as mixed strategies and maximin strategies.

Definition 2.2.1. We talk about *mixed strategies* when each player i can choose each strategy $a_{ik} \in A_i$ with a probability p_k, for $k = 1, \dots, m = |A|$. This probabilities must satisfy $\sum_{k=1}^{m} p_k = 1$.
In this case, the payoff is the expectation of the resulting outcome.

Although it is one of the most important theorems in Game Theory, The Minimax Theorem is not a central part of our work here. Thus, we are not going to prove it.

Theorem 2.2.1. *Every bimatrix game is strictly determined.*

For its proof, see Owen (1968) [4].

2.3 2-person general-sum games

Example 2.3.1. *Battle of the sexes*
A couple want to spend some leisure time together on the evening, but on the morning, when trying to decide where to go, they cannot reach an agreement and both leave home without having decided whether they are going to the theater or to the cinema and with no options to communicate with each other. Both of them want to meet each other, but each has different preferences over where they would rather be. The game is then represented by the following matrix:

	Cinema	Theatre
Cinema	(2, 4)	(0, 0)
Theatre	(0, 0)	(4,2)

Table 2.1: Values of the payoffs in the Battle of the Sexes game.

where (Cinema, Cinema) and (Theatre, Theatre) are the two equilibrium pairs. However, each player prefers a different equilibrium situation.

If player 1 plays a mixed strategy $X = (x, 1-x)$ and player 2 plays $Y = (y, 1-y)$, their expected payoffs will be:

$$P_1 = 2xy + 4(1-x)(1-y) = 6xy - 4x - 4y + 4$$
$$P_2 = 4xy + 2(1-x)(1-y) = 6xy - 2x - 2y + 2$$

However, if player 1 goes to the theatre twice as often as to the cinema (i.e. plays $X' = (1/3, 2/3)$, resulting in $x = 1/3$), the expected payoff for player 2 is independent of his strategy, making him indifferent between chossing one or the other:

$$x = 1/3 \Longrightarrow P_2 = 2 - 2/3 = 4/3$$

The same happens if player 2 plays $Y' = (2/3, 1/3)$

$$y = 2/3 \Longrightarrow P_1 = 4 - 8/3 = 4/3$$

Thus, we have found an equilibrium situation.

△

As we have just seen, a finite two-person general-sum game can be expressed as an $m \times n$ matrix (a_{ij}, b_{ij}), where a_{ij} and b_{ij} are the payoffs to players A and B, respectively, when they choose strategies i and j, respectively. In the case of two players, these are called *bimatrix games*.

If the game was zero-sum, the matrix for player 1 would already characterize the game since the payoffs for player 2 are always the opposite to those of player 1.

2.3.1 The prisoners dilemma

The prisoners dilemma (PD) is a landmark example of a game in Game Theory because of the seemingly contradictory payoffs obtained from its solution.

Example 2.3.2. Two criminals who are caught by the police while commiting a crime are immediately put into separate rooms so that they cannot communicate with each other. The police cannot convict them for all their crimes with the evidence they have at their disposal at the time, so they need some help from the criminals themselves. Then, this is what the police tells the criminals:

- If one of them confesses and the other does not, the one who confessed will be set free and the other will serve 10 years in prison.
- If none of them confesses, they will both serve 1 year in prison.
- If both of them confess, they will both serve 6 years in prison.

The matrix with the players' payoffs depending on their decisions looks like this:

	coop	defect
coop	(-1,-1)	(-10, 0)
defect	(0, -10)	(-6,-6)

Table 2.2: Values of the payoffs in the Prisoners Dilemma.

where the pairs represent each player's utility (the number of years each one will spend in prison) in every situation and we have labelled the strategies as coop (with your parter by not confessing) or defect (your partner by confessing).

Let's look at the game from the point of view of each player. Since the game is symmetric for both players, let's assume we are player 1 without loss of generality.
One should reason as follows:

- In case Player 2 plays coop, I should play defect, since 0 is a better payoff than -1.
- In case Player 2 plays defect, I should also play defect, because -6 is a better payoff than -10.

Then, it is clear that I should always play defect, because it is the best strategy in every situation —it is the best reply to any of my opponent's strategies. Therefore, a rational player will always play defect because defect *dominates* coop in this game.

This —which is perfectly fine from an individual perspective— is a recipe for disaster, because since both of them are going to make a decision independently and Player 2 is also rational, he will analogously find that defect is also his best reply to any of the strategies available to Player 1 and, therefore, he will always play defect. It becomes clear that (defect, defect) is an equilibrium situation, one none of the players has any incentive to unilaterally deviate from. But what is equally obvious that if they both defect, their utilities are much worse that if they both cooperated, as we can see in the matrix below.

$$\begin{array}{c} \\ \text{coop} \\ \text{defect} \end{array} \begin{array}{c} \begin{array}{cc} \text{coop} & \text{defect} \end{array} \\ \begin{pmatrix} -1,-1 & -10,0 \\ 0,-10 & -6,-6 \end{pmatrix} \end{array}$$

The question then is: would not the players be better off by cooperating between them? And the answer is —naturally— yes, indeed. This game is one of the most common sources of misinterpretations, in part because of the story used to introduce it and its moral implications. Nevertheless, it is a clear example that cooperation can be the best response from a selfish point of view. The problem in this case, however, is that we cannot argue that they would prefer the other to spend as least time in prison as possible, that they would be punished for betraying their partner or that both of them would choose the same strategy without changing the game, etc. because the two players make their decisions simultaneously and independently —or else we are talking about a different game like the twins game, in which both players are actually the same one, or a PD with different payoffs—.

Naturally, if both players could agree on a strategy before getting into those rooms —and stick to it no matter what— the result would have been much better for them. This would be an obvious example in which cooperation is the rational answer, but unfortunately for the players this game does not allow such a thing.

Remark 2.3.1. Although our goal in this essay is not about focusing on this issues, it is worth noting that there are other aspects of the game to take into account, such as:

- The particular values of each players' payoffs.

Namely, the number of years in prison each player will have to serve in each situation:

	coop	defect
coop	(R,R)	(S, T)
defect	(T, S)	(P, P)

Table 2.3: General values of the payoffs in the Prisoners Dilemma, with $T > R > P > S$.

- The number of times N that the players will play the game if this is iterated.

The iteration of a game like PD could enforce the urgency for players to cooperate, since they know that it is way more beneficial to find themselves in a (coop, coop)
Despite that, if N is finite and known by the players, in the last iteration of the game it is rational for both players to play defect since their actions cannot be punished any further. Therefore, since both players will know that, they will also play defect in the previous iteration and, repeating this argument, they will play defect in every iteration from the very first one.

This game is also applicable to many other situations, like:

- In sports, two athletes have to make a decision whether taking a performance-booster drug, which gives them an advantage A over athletes not taking the drug but also has legal and medical dangers D, or not.

$$\begin{array}{c} \\ \text{not take} \\ \\ \text{take} \end{array} \begin{array}{c} \begin{array}{cc} \text{not take} & \text{take} \end{array} \\ \begin{pmatrix} 0,\ 0 & -A,\ A-D \\ & \\ A-D,\ -A & -D,\ -D \end{pmatrix} \end{array}$$

  Unfortunately, their rival goes through exactly the same rational analysis. As a result, if the advantages are greater than the dangers, both of them take performance-enhancing drugs and neither gains an advantage over the other. Instead, they will only face the dangers.
  If they could just trust each other, they could refrain from taking the drugs and maintain the same non-advantage status, without any legal or physical danger.

- Another classic example, the nuclear weapons race, :

$$\begin{array}{c} \\ \text{disarm} \\ \\ \text{arm} \end{array} \begin{array}{c} \begin{array}{cc} \text{disarm} & \text{arm} \end{array} \\ \begin{pmatrix} 0,0 & -I, S-C \\ & \\ S-C, -I & -C, -C \end{pmatrix} \end{array}$$

  where S stands for superiority, I for inferiority and C are the costs of maintaining the weapons.

$$S > C, \text{ and } I > C > 0$$

  In this case, both countries would end up building their weapons just in fear of facing an inferiority situation if the other decides to arm.

△

Theorem 2.3.2. *Every bimatrix game has at least one equilibrium point.*

Proof. The interested reader is referred to Owen [4] for its proof. □

2.4 Allowing players to cooperate: The Bargaining Problem

In case the two players of a game are allowed to cooperate between them, meaning that they can make agreements before choosing their strategies, correlate their mixed strategies or transfer utility between them (although not necessarily linearly), we are facing a Bargaing Problem (BP), one in which the two players need to agree on which point from the set of obtainable outcomes satisfies both of them.

Since the case for $n = 2$ is a critical step to the BP in its general form, we will address this matter on Section 3.3.1.

Chapter 3

N-person games: cooperation and solutions

So far, we have studied games with only two players. However, in our general study of cooperative games, we will need to address the general n-person games, so it is time to shift our focus into n-person games, with $n \geq 3$.

We need to distinguish between n-person noncooperative games and n-person cooperative games, which are the type of games we will concentrate on.

3.1 N-person noncooperative games

Theorem 3.1.1. *Every finite n-person noncooperative game has at least one equilibrium n-tuple of mixed strategies.*

Proof. Proof can be extended from the proof of Theorem 2.3.2 to this case. □

Although the computation of equilibrium n-tuples is much more complex than the one for equilibrium pairs, there is no great difference, in general, between the theory of noncooperative n-person games and noncooperative two-personal general-sum games.

Hence, we will quickly move on to their cooperative counterparts.

3.2 N-person cooperative games

Definition 3.2.1. In an n-person game, we define the *set of players* $N := \{1, 2, \dots, n\}$.

Opposite to what we have just seen about noncooperative games, n-person cooperative games are not a generalization of the two-person case. Here, we find the concept of *coalitions*, which are basically the subsets of N. The essence of n-person games is not the randomization of the players' choices but the formation and stability of these aforementioned coalitions.

Definition 3.2.2. *Coalitions* are the nonempty subsets $S \subset N$, with $|S|$ the number of players in it.

Thus, the approach in cooperative games is different, since players can commit to behave in a socially optimal way. The question is then how to form coalitions and how to share the benefits obtained from this cooperation. Therefore, solutions are now rules to choose how to allocate utility according to some concepts like fairness or equity that we have not introduced thus far. However, stability will still be of vital importance to choose these allocations —vectors $x \in \mathbb{R}^n$, where $n = |N|$ is the number of players of the game—.

Depending on whether transferences of utility between players are restricted or not, we need to distinguish between *nontransferable utility games* (*NTU-games*) and *transferable utility games* (*TU-games*), respectively.

3.3 Nontransferable Utility games

As we mentioned, side payments are forbidden in this type of games, meaning that players cannot freely split the gains by means of their cooperation. However, this does by no means imply that players cannot reach an agreement.

Definition 3.3.1. We say that a set $A \subset \mathbb{R}^{|S|}$ is *comprehensive* if it satisfies that, for each pair $x, y \in \mathbb{R}^{|S|}$, if $x \in A$ and $y \leq x$, then we have $y \in A$.[1]
The *comprehensive hull* of a given set is the smallest comprehensive set that contains it.

Definition 3.3.2. An *n-player nontransferable utility game with set of players N* is defined as a pair (N, V), where V is a function that assigns a set $V(S) \subset \mathbb{R}^{|S|}$ to each coalition $S \subset N$. We assume that $V(\emptyset) = \{0\}$, by convention.[2]

[1] In some occasions (e.g. Def. 3.3.1, 3.3.7, 3.3.9), for the sake of brevity, we will write expressions such as $y \leq x$ for $x, y \in \mathbb{R}^m$ —which is an abuse of notation— to avoid writing $y_i \leq x_i$, for each $i = 1, 2, \dots, m$.

[2] In the same sense as in the previous footnote, we denote the point $(0, 0, ..., 0)$ by $\{0\}$ to simplify notation.

Remark 3.3.1. For any $S \subset N$ such that $S \neq \emptyset$, the set $V(S) \subset \mathbb{R}^{|S|}$ is nonempty, closed and comprehensive. Besides, there is $v_i \in \mathbb{R}$ s.t. $V(\{i\}) = (-\infty, v_i]$, for each $i \in N$. Moreover, the set $V(S) \cup \{x \in \mathbb{R}^{|S|} : \text{ for each } i \in S, x_i \geq v_i\}$ is bounded.

It is now time to formally define one of the most basic concepts of n-player games: allocations.

Definition 3.3.3. An *allocation* is the vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ of the payoffs x_j for every player $j \in N$.

Note that $\mathbb{R}^{|S|}$ is the set of outcomes that the players in coalitions S can obtain by themselves.

Definition 3.3.4. We say that an allocation $x = (x_1, \dots, x_n)$ is *feasible* —and, therefore, that it belongs to the *set of feasible allocations* F— if there is a partition $\{S_1, \dots, S_k\}$ of N such that, for each $j \in \{1, \dots, k\}$, there is $y \in V(S_j)$ satisfying that, for each $i \in S_j$, $y_i = x_i$.

3.3.1 Bargaining problems

In general, as we saw in Example 1.3.1, there is a set of outcomes available when the two players act together. Choosing a utility function for each player, this set of outcomes can be mapped into a subset of the Eulidian space $\mathbb{R}^2$. The image of this set under the mapping will be closed, bounded above and convex. The problem then becomes a question of how to choose a point from this set S that will satisfy both players.

Definition 3.3.5. A set $S \subset \mathbb{R}^2$ is *convex* if, for any $a, b \in S$ and λ, μ any real numbers in $[0, 1]$, we have $\lambda a + \mu b \in S$.

Remark 3.3.2. The set of all feasible allocations —the *feasible set*, denoted by F—, is the comprehensive hull of a compact and convex subset of $\mathbb{R}^n$.

Allocations in F represent the utilities players get from the outcomes of the available agreements. Then, for any $(u, v) \in F$, as we said, it is possible for players 1 and 2 to obtain utilities u and v, respectively.
In these games, players have to reach some type agreement on which allocation to choose. If the players do not reach an agreement, the allocation that will be obtained is the disagreement point d.

Definition 3.3.6. Thus, we define an n-player bargaining problem, with set of players N, by (F, d) for the F, d aforementioned above. We must note that the disagreement point d is an allocation in F and that (it is assumed) there exists $x \in F$ s.t. $x > d$.

Definition 3.3.7. Given two allocations $x, y \in F$, we say that x is *Pareto dominated* by y (or that y *Pareto dominates* x) if $x \leq y$ and $x \neq y$.
Equivalently, y Pareto dominates x if, for each $i \in N$, $x_i \leq y_i$ and at least one of these inequalities is strict.

This allows us to define efficient allocations:

Definition 3.3.8. We say that an allocation $x \in F$ is *efficient* or *Pareto efficient* if there is no allocation in F that Pareto dominates x.

Having defined this idea of efficiency, we may now define rules to find solutions for n-player bargaining problems.

Remark 3.3.3. The set of n-player bargaining problems is noted by B^N.

Definition 3.3.9. Given a game $(F, d) \in B^N$, we define the set $F_d := \{x \in F : x \geq d\}$.

Definition 3.3.10. *Allocation rules* are maps $\varphi : B^N \to \mathbb{R}^N$ s.t., for each $(F, d) \in B^N$, $\varphi(F, d) \in F_d$.

One can also define some appropriate properties that an allocation rule must satisfy. An important example is the Nash solution —which can be generalized to the n-player case—, the only allocation rule that satisfies Pareto Efficiency (EFF), Symmetry (SYM), Covariance with positive affine transformations (CAT) and Independence of irrelevant alternatives (IIA).
Let's take a look at these properties:

- **Pareto Efficiency (EFF)**:
  An allocation rule φ satisfies EFF if, for each $(F, d) \in B^N$, $\varphi(F, d)$ is a Pareto efficient allocation.
- **Symmetry (SYM):**
  An allocation rule φ satisfies SYM if, for each symmetric[3] bargaining problem $(F, d) \in B^N$ and each pair of players $i, j \in N$, we have $\varphi_i(F, d) = \varphi_j(F, d)$.

[3]A bargaining problem $(F, d) \in B^N$ is said to be *symmetric* if, for each permutation π of the elements of N, $d^\pi = d$ and, for each $x \in F$, $x^\pi \in F$.
Note that, given $x \in \mathbb{R}^n$ and a permutation π, $x_i^\pi := x_{\pi(i)}$, for each $i \in N$.

- **Covariance with positive affine transformations (CAT):** the choice of utility should not affect the allocation rule.
  An allocation rule φ satisfies CAT if, for each $(F, d) \in B^N$ and each positive affine transformation g [4], we have $\varphi(g(F), g(d)) = g(\varphi(F, d))$.
- **Independence of irrelevant alternatives (IIA):**
  An allocation rule φ satisfies IIA if, for each pair of bargaining problems $(F, d), (F^*, d) \in B^N$ such that $F^* \subset F$, if $\varphi(F, d) \in F^*$, then $\varphi(F^*, d) = \varphi(F, d)$.

Proposition 3.3.4. *Given $(F, d) \in B^N$, there is a unique $z \in F_d$ that maximizes the function g^d representing the product of the gains of all the players at x with respect to the disagreement point, if $x > d$,*

$$g^d(x) := \prod_{i \in N} (x_i - d_i)$$

Proof. There is a maximum of g^d in F_d for g^d is a continuous function and F_d is compact. The fact that it is unique can be proved by the convexity of F_d. □

Definition 3.3.11. The *Nash solution* (NA) selects the unique allocation in F_d that maximizes the product of the gains of the players with respect to the disagreement point. This means that, given a bargaining problem (F, d) such that $g^d(z) = \max_{x \in F_d} g^d(x) = \max_{x \in F_d} \prod_{i \in N}(x_i - d_i)$, the Nash solution is $NA(F, d) := z$.

Theorem 3.3.5. *The Nash solution is the only allocation rule for n-player bargaining problems that satisfies EFF, SYM, CAT and IIA.*

Proof. Checking that NA satisfies the four axioms is simple. Proof that it is the unique can be found on [5]. □

Remark 3.3.6. None of the axioms used to characterize NA were unnecessary.

Another interesting solution is the Kalai-Smorodinsky solution (KS), which is inspired by the definition of the utopia point:

Definition 3.3.12. Given a bargaining problem $(F, d) \in B^N$, the *utopia point* is the vector $b(F, d) \in \mathbb{R}^N$, which is defined by $b_i(F, d) = \max_{x \in F_d} x_i$. Therefore, the utopia point represents the aspirations of the players, the largest utility each player can get in F_d.

[4] $g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ s.t., for each $i \in N$, there are $a_i, b_i \in \mathbb{R}$ with $a_i > 0$ such that $g_i(x) = a_i x_i + b_i$, for each $x \in \mathbb{R}^n$.

Definition 3.3.13. Given a bargaining problem $(F, d) \in B^N$, we define the Kalai-Smorodinsky solution, KS, by:

$$\mathrm{KS}(F, d) := d + \bar{t}(b(F, d) - d)$$

where $\bar{t} := max\{t \in \mathbb{R} : d + t(b(F, d) - d) \in F_d\}$.

Remark 3.3.7. $\bar{t}$ is well defined by the compactness of F_d.

Besides the previous properties, we now also need to consider:

- **Individual Monotonicity (IM):**
  An allocation rule φ satisfies IM if, for each pairlie of bargaining problems $(F, d), (F^*, d) \in B^N$ such that $F_d^* \subset F_d$ and $b_j(F, d) = b_j(F^*, d)$ for each j different from a given $i \in N$, then $\varphi(F^*, d) \leq \varphi(F, d)$.

Proposition 3.3.8. *KS is the only allocation rule for 2-player bargaining problems that satisfies EFF, SYM, CAT and IM.*

Remark 3.3.9. For $n \geq 3$, there is no solution for n-player bargaining problems that satisfies EFF, SYM and IM at the same time.

Example 3.3.1. Let's consider the 3-player bargaining problem (F, d) where $d = (0, 0, 0)$ and F is the comprehensive hull of $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 \leq 1\}$ —the bargaining problem version of the Divide-a-Dollar game with 3 players. Here, since both NA and KS satisfy EFF and SYM, the obtained allocation will be (1/3, 1/3, 1/3) in both cases. △

Example 3.3.2. What if one of the players was told he could keep 1/2 if the three of them cannot reach an agreement —in which case, $d' = (0, 0, 1/2)$—?
We obtain $\mathrm{NA}(F, d') = (1/6, 1/6, 2/3) = \mathrm{KS}(F, d')$. △

Example 3.3.3. What if one of the players decided that under no circumstances did he want to get more than 0.3?
Now, F is the comprehensive hull of $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 \leq 1\} \cap \{x_3 \leq 0.3\}$.
Thus, we obtain: $\mathrm{NA}(F', d) = (7/20, 7/20, 6/20)$ and $\mathrm{KS}(F', d) = (10/23, 10/23, 3/23)$. △

From where we observe that the main difference between the two is that KS is more sensible to players' aspirations.

3.4 Transferable Utility games

All cooperation, such as correlated strategies and side payments, is permitted, which makes this type of games less general, but easier to analyze and, therefore, much more widely studied. This is due to the main difference between NTU and TU-games, which is the fact that in TU-games, given a coalition S and an allocation $x \in V(S) \subset \mathbb{R}^{|S|}$ enforced by the players in S, any allocation that can be obtained from x by transferring utility among the players in S is also in $V(S)$. Thus, we can characterize $v(S)$ simply by a single number, the *worth of coalition* S, $v(S)$. For each possible coalition S, we define the value of a coalition $v(S)$ as the largest payoff guaranteed to be available to be shared amongst the players that form the coalition.

Definition 3.4.1. The *worth* of a coalition S, $v(S)$, is defined as $v(S) := \max_{x \in V(S)} \sum_{i \in S} x_i$. By convention, $v(\emptyset) = 0$.

Definition 3.4.2. The *characteristic function* of an n-person game with set of players N is a function $v : 2^N \longrightarrow \mathbb{R}$ which assigns, to each $S \subset N$, a real value corresponding to the maximin value for S of the 2-player game played by S and N\S. Thus, v is a function that assigns, to each coalition $S \subset N$, its worth $v(S)$.

Therefore, we shall study the characteristic function of games instead of their normal form. Besides, we will identify the game with their characteristic functions. Then, an n-person TU-game with set of players N is given by (N, v), being v the characteristic function of the game. We often refer to the game (N, v) simply by v.[5]

Once we assume —or choose— a utility such that it can be transferred with a 1:1 ratio, what is interesting to find is the utility that each coalition can obtain. However, let's see some properties which can characterize TU-games first.

3.4.1 Classifying games

Definition 3.4.3. If the characteristic function of an n-person game v is such that, for any S, $T \subset N$ with $S \cap T = \emptyset$, we have $v(S \cup T) \geq v(S) + v(T)$, we say that v is *superadditive*.

Remark 3.4.1. We denote the set of n-player games by G^N and the set of superadditive TU-games by SG^N.

[5] Note that a TU-game (N, v) can be seen as an NTU-game (N, v) if we define, for each coalition $S \subset N$ s.t. $S \neq \emptyset$, $V(S) := \{y \in \mathbb{R}^{|S|} : \sum_{i \in S} y_i \leq v(S)\}$.

Games whose characteristic function is superadditive are called *proper games*, opposite to improper games —whose characteristic function does not satisfy superadditivity—. This superadditivity property is what gives players a real incentive to cooperate. [6]

Example 3.4.1. Let a game (N, v) with $N = \{1, 2, 3\}$ and v such that $v(1) = v(2) = 0.1$, $v(3) = 0.2$, $v(1, 2) = 0.2$, $v(1, 3) = v(2, 3) = v(1, 2, 3) = 1$.
Since $v(1, 2, 3) < v(1, 3) + v(2)$, v is not superadditive. △

In proper games, it is then natural to assume that the grand coalition N will form, which makes us shift our focus from looking at which coalitions will form to finding out how to allocate $v(N)$ between the players. We must wait a little longer still, since we need to take a look at some more properties that define games before that:

Definition 3.4.4. If $v \in G^N$ is s.t., for each player $i \in N$ and each coalition $S \subset N \setminus \{i\}$, $v(S \cup \{i\}) \geq v(S) + v(\{i\})$, we say that v is *weakly superadditive.*

Definition 3.4.5. Let $v \in G^N$ s.t., for each player $i \in N$ and each coalition $S \subset N \setminus \{i\}$, $v(S \cup \{i\}) = v(S) + v(\{i\})$. In this case, we say that v is *additive.*
In particular, for each $S \subset N$, $v(S) = \sum_{i \in S} v(\{i\})$.

Definition 3.4.6. We say that a game $v \in G^N$ is *monotonic* if, for each pair $S, T \subset N$ s.t. $S \subset T$, we have that $v(S) \leq v(T)$.

Definition 3.4.7. A game $v \in G^N$ is *zero-normalized* if $v(\{i\}) = 0$, for each $i \in N$.

Definition 3.4.8. If a game $v \in G^N$ is zero-normalized and monotonic, we say that it is *zero-monotonic.*

Theorem 3.4.2. *A game $v \in G^N$ is weakly superadditive if and only if it is zero-monotonic.*

Proof. (i) v zero-monotonic $\Longrightarrow$ v weakly superadditive.
Let $S \subset N$ and $i \in N$ such that $i \notin S$. Then, since v is zero-monotonic, $v(i) = 0$, then:

$$v(S) + v(i) = v(S) \leq v(S \cup \{i\})$$

Thus, v is weakly superadditive.

(ii) v weakly superadditive $\Longrightarrow$ v zero-monotonic.

[6] That is the reason for us basically studying proper games in this work.

Let $S, T \subset N$ such that $S \subset T$. Then, there is $k \in N$ such that $k \in T \setminus S$. Thus,

$$v(T) \geq v(T \setminus \{j\}) + v(j)$$

Then, if $T \setminus \{j\} = S$, $v(S) \leq v(T)$ and if $S \subset T \setminus \{j\}$, in which case, by induction $v(S) \leq v(T \setminus \{i\}) \leq v(T)$. Whatever the case, we would already have that v is monotonic.
However, by iteration, given $S \subset T \subset N$ and $\{j_1, ..., j_m\}$ the set of players in $T \setminus S$, we obtain

$$v(T) \geq v(S) + \sum_{k=1}^{k=m} v(\{i_k\})$$

which is equivalent to the definiton of zero-monotonicity in this situation.

□

Definition 3.4.9. Let $v \in G^N$ be a game such that, for any coalition $S \subset N$, $v(S) + v(N \setminus S) = v(N)$. Then, we say that v is a *constant-sum* game.

Definition 3.4.10. Given an n-person TU-game $v \in G^N$, an *imputation* is a vector $x = (x_1, x_2, \ldots, x_n)$ such that:

- it is *efficient*: $\sum_{i \in N} x_i = v(N)$
- it is *individually rational*: $x_i \geq v(\{i\})$, for all $i \in N$.

Remark 3.4.3. We denote the set of all imputations of a game v by $I(v)$. As we have seen, this is the set of all the efficient and individually rational allocations.

Since for any imputation x, the grand coalition is formed, the question then becomes: which of all imputations from $I(v)$ will be obtained?

Proposition 3.4.4. *The set of imputations of any superadditive game is nonempty.*

Proof. Let v be a superadditive game and x an allocation such that $x_i = v(\{i\})$. It is obvious then that x satisfies the second condition from Definition 3.4.10. Then, since v is superadditive:

$$v(N) \geq v(N \setminus \{i\}) + v(\{i\}) \geq ... \geq \sum_{i \in N} v(\{i\})$$

Hence,

$$\frac{v(N)}{\sum_{i \in N} v(\{i\})} \geq 1$$

Let y be an allocation such that $y_i = \frac{v(N)}{\sum_{i\in N} v(\{i\})} x_i = \frac{v(N)}{\sum_{i\in N} v(\{i\})} v(\{i\})$. Then,

(i) $y_i = \frac{v(N)}{\sum_{i\in N} v(\{i\})} x_i \geq x_i = v(\{i\})$, for all $i \in N$.

(ii) $\displaystyle\sum_{i\in N} y_i = \sum_{i\in N} \frac{v(N)}{\sum_{i\in N} v(\{i\})} x_i = \frac{v(N)}{\sum_{i\in N} v(\{i\})} \sum_{i\in N} x_i = \frac{v(N)}{\sum_{i\in N} v(\{i\})} \sum_{i\in N} v(\{i\}) = v(N)$.

□

For games v with more than one allocation in $I(v)$, an allocation will need to be stable in order to be obtained as a solution of the game. Since allocations are already individually rational, no player will individually block an allocation in $I(v)$. But with players being capable of forming coalitions, stability requires an analogous concept regarding coalitions.

Definition 3.4.11. Given a game $v \in G^N$, we say that an imputation $x \in \mathbb{R}^N$ is *coalitionally rational* if, for each $S \subset N$, $\displaystyle\sum_{i\in S} x_i \geq v(S)$.

Remark 3.4.5. It was already obvious that no player would accept any payoff lower than the minimum each of them could obtain without cooperating. Now, also, we can see that stability also requires that no coalition can enforce a different imputation to the one obtained.
This leads us to look for those imputations that are also coalitionally rational and, hence, to the concept of the core, which we will study in the following section.

Definition 3.4.12. Now, we say that a game is *essential* if it satisfies: $v(N) > \displaystyle\sum_{i\in N} v(\{i\})$.

Otherwise, $v(N) = \displaystyle\sum_{i\in N} v(\{i\})$, since by superadditivity we had $v(N) \geq \displaystyle\sum_{i\in N} v(\{i\})$. In this case, we say the game is inessential and becomes of no interest to us here because there would only be one allocation in $I(v)$: $x \in \mathbb{R}^n$ s.t. $x_i = v(\{i\})$.

Definition 3.4.13. Two games u, v are *isomorphic* if there exists a 1-1 map f from $I(u)$ onto $I(v)$ such that, for $x, y \in I(u)$, x dominates y if, and only if, $f(x)$ dominates $f(y)$.

Since this is not a simple way of observing whether two games are isomorphic, we have an auxiliary result.

Definition 3.4.14. We say that two games $v, \bar{v}$ are *S-equivalent* if there are $k > 0$ and $a_1, \dots, a_n \in \mathbb{R}$ s.t., for each $S \subset N$, $\bar{v}(S) = k \cdot v(S) + \displaystyle\sum_{i\in S} a_i$.

Theorem 3.4.6. *Let u, v be two S-equivalent games. Then, u and v are isomorphic.*

Proof. See Von Neunmann & Morgenstern (1964). □

Remark 3.4.7. The converse of this theorem, while true, is not quite as interesting and is lengthy to prove, see Kuhn (1970).

Definition 3.4.15. We say that a TU-game $v \in G^N$ is in $(0,1)$–*normalization* if it satisfies that $v(\{i\}) = 0$, for all $i \in N$, and $v(N) = 1$.

Theorem 3.4.8. *For any essential game $v \in G^N$, there is only one (0,1)–normalized game v^* that is S-equivalent to v.*

Proof. Let v be a game such that $v(N) > \sum_{i \in N} v(\{i\})$. First, let's prove its unicity.
We are going to do so by assuming its existance: then, let v^* be a game such that $v^*(S) = kv(S) + \sum_{i \in S} a_i$.
Then, for each $i \in N$, $0 = v^*(\{i\}) = kv(S) + \sum_{j \in \{i\}} a_j = kv(\{i\}) + a_i$.
Thus, $\alpha_i = -kv(\{i\})$. Hence, the values for a_i are unique. Besides,

$$1 = v^*(N) = kv(N) + \sum_{j \in N} a_j = kv(N) - \sum_{j \in N} kv(\{j\}) = kv(N) - k \sum_{j \in N} v(\{j\}) \neq 0$$

Therefore, $k = \frac{v(N)}{\sum_{j \in N} v(\{j\})}$.
Therefore, we have proved the uniqueness and found the corresponding values. Thus, we have finished. □

Remark 3.4.9. Thus, we can choose a game in (0,1) normalization to represent the corresponding equivalence class of games. This way, the value $v(S)$ of a coalition tells us its strength (what the players inside it gain by forming it).

Definition 3.4.16. We say that a game v is *symmetric* if $v(S)$ only depends on the number of elements in S.

Example 3.4.2. *Divide a million*:
Three brothers are left one million dollars as inheritance from a distant relative, provided that the majority of them can agree on how to share them.
Thus, this situation can be described as a TU-game (N, v) with $N = \{1, 2, 3\}$ and v such that $v(S) = 0$ if S has only one player and $v(S) = 1$ if S has at least two players.

This is an example of a symmetric TU-game. Later on, we will be able to see in which ways can they divide the money. △

3.4.2 Simple games

Definition 3.4.17. We say that an n-player TU-game $v \in G^N$ is a *simple game* if

- v is monotonic
- $v(S) \in \{0, 1\}$, for all $S \subset N$
- $v(N) = 1$

Remark 3.4.10. We denote the set of n-player simple games by S^N.

Definition 3.4.18. The set $W := \{S \subset N : v(S) = 1\}$ contains the *winning coalitions* of a simple game v.

Definition 3.4.19. The *minimal winning coalitions* of a simple game v, $W^m := \{S \in W :$ for each $T \in W$, if $T \subset S$ then $T = S\}$.

Definition 3.4.20. Given a simple game $v \in S^N$, we say that j is a *veto player* in v if we have $v(N \setminus \{j\}) = 0$.

Example 3.4.3. *The Glove Game*:
In this situation, we have 3 players who have one glove out of the same pair: two of them have the left glove (let's say they are Player 2 and Player 3) and one of them has the right one (Player 1). They can only sell the gloves as a pair.
Thus, $v(1) = v(2) = v(3) = v(2, 3) = 0$ and $v(1, 3) = v(2, 3) = 1$, meaning that Player 3 is a veto player in this game.

△

Definition 3.4.21. A game v in $(0, 1)$ normalization is a *simple* game if, for each coalition $S \subset N$, $v(S)$ is either 0 or 1.

Definition 3.4.22. A *weighted majority game* is a simple game v in which each player has a nonnegative weight p_i and there is a quota q such that $\sum_{i \in N} p_i \geq q$.

In such a game,

$$v(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} p_i \geq q \\ 0 & \text{otherwise.} \end{cases}$$

We will analyze a couple of examples (4.1.2 and 4.1.3) of real-life weighted majority games in politics when we study voting games in subsection 4.1.1.

Now it is time to introduce one of the most important concepts of Cooperative Game Theory: the core of a game, which presents high requirements regarding stability.

3.5 The Core

Definition 3.5.1. The *core* of a game v, $C(v)$, is the set of all n-vectors $x \in \mathbb{R}^N$ satisfying:

(i) $\sum_{i \in S} x_i \geq v(S)$, for all $S \subset N$.

(ii) $\sum_{i \in N} x_i = v(N)$

Equivalently, $C(v) := \left\{ x \in I(v) : \text{for each } S \subset N, \sum_{i \in S} x_i \geq v(S) \right\} \subset I(v)$.

That is to say, the core is the set of all efficient (ii) and coalitionally rational (i) allocations —known as *core allocations*—.
Now, we define a concept of unstability that will have a lot to do with the idea behind the core.

Definition 3.5.2. Given a game $v \in G^N$, a nonempty coalition $S \subset N$ and two imputations $x, y \in I(v)$, we say that y *dominates* x *through* S if:

- for each $i \in S$, $y_i > x_i$
- $\sum_{i \in S} y_i \leq v(S)$

This is, an allocation y dominates another allocation x through a certain coalition S if y is better than x for every member of S and those assigned payoffs are attainable for the members of S.

Definition 3.5.3. We say that x *dominates* y if there is a coalition $S \neq \emptyset$ such that x dominates y through S.

Definition 3.5.4. We call x an *undominated* imputation of v if there is no $y \in I(v)$ that dominates it.

Since a dominated imputation can be blocked by some coalition (and will be, in fact), a stable coalition should be undominated. As we are going to see, this is the case for core allocations.

The set of all payoff profiles of a game to which no objections can be made (i.e. the set of all payoff profiles for which there is no coalition that can enforce an alternative payoff that all its members prefer) is, as we are immediately going to see, the set of all undominated imputations of the game and is called the *core* of the game.

Proposition 3.5.1. *Given a game $v \in G^N$, any $x \in C(v)$ is undominated.*

Proof. Given any $x \in I(v)$, let's suppose there is an allocation $y \in I(v)$ that dominates x. This is, we are supposing there is a nonempty coalition $S \subset N$ such that y dominates x through S, satisfying the conditions we just saw in 3.5.2.
Then, we have:

$$v(S) \geq \sum_{i \in S} y_i > \sum_{i \in S} x_i \geq v(S),$$

which is a contradiction. Thus, we have finished. □

Remark 3.5.2. What's more, if v is superadditive, $C(v)$ is the set of undominated imputations of v.

Proposition 3.5.3. *Let $v \in SG^N$, then $C(v) = \{x \in I(v) : x \text{ is undominated}\} \subset I(v)$.*

Proof. We want to prove that any imputation outside the core is dominated: for any imputation $x \in I(v) \setminus C(v)$, there is $S \subset N$ such that $\sum_{i \in S} x_i < v(S)$. Let $y \in \mathbb{R}^N$ s.t.

$$\text{for each } i \in N, \quad y_i = \begin{cases} x_i + \dfrac{v(S) - \sum_{j \in S} x_j}{|S|} & \text{if } i \in S \\[2ex] v(i) + \dfrac{v(N) - v(S) - \sum_{j \in N \setminus S} x_j}{|N \setminus S|} & \text{if } i \notin S \end{cases}$$

We have that $v(S) - \sum_{j \in S} x_j > 0$, for v is superadditive. Hence, y belongs to the set of imputations $I(v)$ and dominates x through S. □

Remark 3.5.4. The previous proposition states that $C(v) = I(v) \cap U(v)$, where $U(v)$ is the set of undominated imputations of the game.

Example 3.5.1. Three players together can obtain 1$ to share, any two players can obtain 0.8$, and one player by herself can obtain zero.
Then, $N = \{1, 2, 3\}$ and $v(1) = v(2) = v(3) = 0$, $v(1, 2) = v(2, 3) = v(3, 1) = 0.8$, $v(1, 2, 3) = 1$. Which allocations form the core of this game?
From what we have just seen, in order for a coalition x to be in the core of v, the payoffs x_i of each player $i \in N$ need to satisfy:

- $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$
- $x_1 + x_2 \geq 0.8$
- $x_1 + x_3 \geq 0.8$
- $x_2 + x_3 \geq 0.8$,
- $x_1 + x_2 + x_3 = 1$

Which is not feasible, since combining the last three equations, we would obtain that $x_1 \leq 0.2$, $x_2 \leq 0.2$ and then $x_1 + x_2 \geq 0.8$ would not be satisfied. Thus, the core of this game is empty. △

Example 3.5.2. A similar case can be made for Example 3.4.2, where one would find the core of the *Divide a million* game to be empty as well. △

This comes to show that any situation the players may reach as an agreement in those games is highly unstable, which makes one wonder the question we ask ourselves in the next subsection.

3.5.1 When is the core nonempty?

In the case of the aforementioned simple games, the following result holds.

Theorem 3.5.5. *Given a simple game $v \in S^N$, $C(v) \neq \emptyset$ if, and only if, there is at least one veto player in v.*

Proof. Suppose there is no veto player for a game $v \in S^N$. Then, we would have that $v(N \setminus \{i\}) = 1$ for each $i \in N$. Thus, $v(N) = v(N \setminus \{i\})$ and, for any core allocation $x \in C(v)$,

$$0 = v(N) - v(N \setminus \{i\}) \geq \sum_{j \in N} x_j - \sum_{j \in N \setminus i} x_j = x_i \geq 0,$$

which leads to $x_i = 0$ for each $i \in N$ and, therefore, contradicts the efficiency of the allocation x. □

Corollary 3.5.6. *If the core of a simple game $v \in S^N$ is such that $C(v) \neq \emptyset$, then*

$$C(v) = \{x \in I(v) : \textit{for each nonveto player } j \in N,\, x_j = 0\}.$$

Example 3.5.3. *The Glove Game*:
In this situation, we have 3 players who have one glove out of the same pair: two of them have the left glove (let's say they are Player 2 and Player 3) and one of them has the right one (Player 1). They can only sell the gloves as a pair, so they need to agree on how to split the amount they will obtain (which we can normalize to 1 for the sake of simplicity) beforehand.
Since for every allocation $(1-\delta, \delta, 0)$ with $\delta > 0$ that Player 1 offers to Player 2, there is an allocation that dominates it —namely $(1-\delta/2, 0, \delta/2)$—, the only undominated allocation of this game (and thus, the only core allocation, for this is a simple game indeed) is $(1, 0, 0)$, which comes to tell us that the excess of one type of good completely depreciated it.

$$C(v) = (0, 0, 1)$$

This is also the result that we would obtain, by Corollary 3.5.6, if we observed that the player with the only right glove is also the only veto player of the simple 3-person game v characterized by $N = \{1, 2, 3\}$ and

$$v(S) = \begin{cases} 1 & \text{if } S \in \{N, (1,2), (1,3)\} \\ 0 & \text{otherwise.} \end{cases}$$

△

In order to answer this question for games in general, we need to define the concept of balanced games, which requires a previous definition.

Definition 3.5.5. A nonempty family of coalitions $\mathcal{F} \subset 2^n$ is *balanced* if there are $\{\alpha_s \in \mathbb{R}, \alpha_s > 0 : S \in \mathcal{F}\}$ such that

$$\text{for each } i \in N, \quad \sum_{\substack{S \in \mathcal{F} \\ i \in S}} \alpha_S = 1$$

Definition 3.5.6. Given a TU-game $v \in G^N$, we say that it is *balanced* if it satisfies that, for each balanced family $\mathcal{F}$ with balancing coefficients $\{\alpha_s \in \mathbb{R}, \alpha_s > 0 : S \in \mathcal{F}\}$,

$$\sum_{S \in \mathcal{F}} \alpha_S v(S) \leq v(N)$$

Theorem 3.5.7. *Let $v \in G^N$. Then, $C(v) \neq \emptyset$ if, and only if, v is balanced.* [7]

The proof for this theorem requires some linear programming results, which are not a subject of study in this thesis. Thus, we refer the reader to [5] for its proof.

[7] This is the Bondareva-Shapley theorem.

Proposition 3.5.8. *Let v_1, v_2 be S-equivalent games such that $v_i(N) > \sum_{j\in N} v_j(i)$, $i = 1, 2$. Then, v_1 is balanced if and only if v_2 is balanced.*

Proof. Let k and $a = (a_1, ..., a_n)$ be such that they satisfy the conditions from the definition of S-equivalence, Def. (3.4.14).
Let's assume that $v_2(S) = kv_1(S) + \sum_{i\in S} a_i$.
Therefore, if v_1 is balanced, with $\mathcal{F}$ its balanced family of coalitions and balancing coefficients $\{\beta_S : S \in \mathcal{F}\}$, we obtain:

$$\sum_{S\in\mathcal{F}} \beta_S\, v_2(S) = \sum_{S\in\mathcal{F}} \beta_S \Big(kv_1(S) + \sum_{i\in S} a_i\Big) = k \sum_{S\in\mathcal{F}} \beta_S v_1(S) + \sum_{S\in\mathcal{F}} \beta_S \Big(\sum_{i\in S} a_i\Big)$$

$$= k \sum_{S\in\mathcal{F}} \beta_S v_1(S) + \sum_{i\in S} \Big(a_i \sum_{\substack{S\in\mathcal{F}\\ i\in S}} \beta_S\Big) = k \sum_{S\in\mathcal{F}} \beta_S v_1(S) + \sum_{i\in S} a_i \le kv(N) + \sum_{j\in N} a_j = v_2(N)$$

Where we have used that v_1 is balanced and $k > 0$. Thus, we have that v_2 is also balanced. Note that the argument we used is completely analogous to the one we would need were we to prove the *only if* part of the theorem. □

Proposition 3.5.9. *Let v_1, v_2 be S-equivalent games such that $v_i(N) > \sum_{j\in N} v_j(i)$, $i = 1, 2$. Then, $C(v_1)$ is nonempty if and only if $C(v_2)$ is nonempty.*

Proof. The proof is quite immediate from the definition of core and S-equivalence. □

3.6 Alternative solution concepts

Despite these results, the core of a game is often empty and, therefore, although the core has a lot of importance as a solution of TU-games, it is of interest to study alternative types of stability such as those that arise when players can negotiate, threat and counterthreat each other or when we look for the allocation that would leave players as least dissatisfied as possible. First of all, however, we will study a less demanding concept.

3.6.1 Stable sets

Thus, we are interested in demanding less strong types of stability than the one required by the core, expecting to obtain less precise yet more general results.

Definition 3.6.1. Here, we say that a set $V \subset I(v)$ is a *stable set* for v if it satisfies:

(i) For any $x, y \in V$, There are no imputations x, y inside V such that dominates the other.[8]

(ii) For any $y \notin V$, there is $x \in V$ such that x dominates y. [9]

Remark 3.6.1. Neither the existence nor the uniqueness of stable sets is guaranteed. On the other hand, most games have a large number of stable sets and the stable set of some games are their cores.

In order to illustrate such uncertainty, we will use an example from Owen (1982) [4]. Due to this uncertainty of stable sets, we will quickly move on to the next proposed solution concept (the nucleolus) afterwards.

Example 3.6.1. Let v be the (0,1)-normalization of the constant-sum 3-person game given by

$$v(S) = \begin{cases} -2 & \text{if } |S| = 1 \\ 2 & \text{if } |S| = 2 \\ 0 & \text{if } |S| = 3 \end{cases}$$

Thus, $V = \{(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, 0, \frac{1}{2}), (0, \frac{1}{2}, \frac{1}{2})\}$ is a stable set, but so are the family of sets

$$V_c = \{(x_1, 1 - c - x_1, c) : 0 \leq x_1 \leq 1 - c\}$$

For further detail into this example —to see the proof that V_c is indeed a stable set—, see [4]. △

3.6.2 The nucleolus

The nucleolus is, like the core of a game, another essential allocation rule for TU-games based on a fairness idea, which we have to define first.

Definition 3.6.2. Given a game $v \in G^N$, let $x \in \mathbb{R}^N$ be an allocation. First, as a measure of the dissatisfaction of a coalition S with regards to an allocation x, we define the *excess of coalition* $S \subset N$ *with respect to* x as:

$$e(S, x) := v(S) - \sum_{i \in S} x_i$$

[8] Internal stability.

[9] External stability.

Remark 3.6.2. Note that, for any $x \in I(v)$, the excess of the grand coalition N with respect to x is $e(N, x) = 0$.

Remark 3.6.3. Besides, for any $x \in C(v)$ and each $S \subset N$, $e(S, x) \leq 0$.

Definition 3.6.3. Let us also define the 2^n-vector $\theta(x)$ called *vector of ordered excesses*, whose components are the excesses of the 2^n subsets $S \subset N$ in decreasing order.
That is to say, the elements $\theta_i(x)$ of the vector $\theta(x)$ are such that $\theta_i(x) \geq \theta_{i+1}(x)$.

Definition 3.6.4. In addition, given two vectors $x, y \in \mathbb{R}^N$, we say that *y is larger than x in the lexicographic order* if there is $k \in \mathbb{N}, 1 \leq k \leq 2^n$ such that,

$$\text{for each } j \in \mathbb{N} \text{ with } j < k,\ \theta_j(y) = \theta_j(x) \text{ and } \theta_k(y) > \theta_k(x).$$

Remark 3.6.4. In this case, we write $\theta(y) \succ_L \theta(x)$.
We write $\theta(y) \succeq_L \theta(x)$ if either $\theta(y) \succ_L \theta(x)$ or $\theta(y) = \theta(x)$.

Example 3.6.2. Let's observe the Glove Game in Example 3.4.3 from this point of view. We had $N = \{1, 2, 3\}$ and v such that $v(1) = v(2) = v(3) = v(1, 2) = 0$ and $v(1, 3) = v(2, 3) = v(1, 2, 3) = 1$.

If we propose the following allocations:

$x = (1/3, 1/3, 1/3),\ y = (1/6, 1/6, 2/6),\ z = (0, 0, 1),\ x' = (1/3, 0, 2/3),\ x'' = (0, 1/3, 2/3),$

we obtain:

	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	0	0	1	1	1
$e(S,x)$	0	-1/3	-1/3	-1/3	-2/3	1/3	1/3	0
$e(S,y)$	0	-1/6	-1/6	-2/6	-2/6	3/6	3/6	0
$e(S,z)$	0	0	0	-1	0	0	0	0
$e(S,x')$	0	-1/3	0	-2/3	-1/3	1/3	0	0
$e(S,x'')$	0	0	-1/3	-2/3	-1/3	0	1/3	0

Table 3.1: Values of the payoffs and the excess of all the possible coalitions with regards to each proposed allocation for the Glove Game.

Therefore, we get:

$$\theta(x) = (1/3, 1/3, 0, 0, -1/3, -1/3, -1/3, -2/3)$$

$$\theta(y) = (1/2, 1/2, 0, 0, -1/6, -1/6, -1/3, -1/3)$$

$$\theta(z) = (0, 0, 0, 0, 0, 0, 0, -1)$$

and

$$\theta(x') = \theta(x'') = (1/3, 0, 0, 0, 0, -1/3, -1/3, -2/3)$$

Thus,

$$\theta(y) \succeq_L \theta(x) \succeq_L \theta(x') = \theta(x'') \succeq_L \theta(z)$$

△

Definition 3.6.5. Let $v \in G^N$ be a game such that $I(v) \neq \emptyset$. The *nucleolus* of v, $\eta(v)$, is the set:

$$\eta(v) := \{x \in I(v) : \text{ for each } y \in I(v), \theta(y) \succeq_L \theta(x)\}$$

Remark 3.6.5. It is obvious, then, that the nucleolus of a game v, $\eta(v)$, is a subset of the set of imputations of v, $I(v)$. Thus, it can only be defined if $I(v)$ is nonempty.
The idea behind the nucleolus is to minimize the vector of decreasingly ordered excesses, therefore seeking the allocations that generate the least dissatisfaction among the coalitions.

Let's see an important result on the nature of the nucleolus before looking back at the previous example. In order to prove it, we will use a lemma whose proof can be found in [5].

Lemma 3.6.6. Let $v \in G^N$, $x, y \in \mathbb{R}^N$ be such that $x \neq y$ and $\theta(x) = \theta(y)$ and let $\alpha \in (0, 1)$. Then,

$$\theta(x) \succ_L \theta(\alpha x + (1 - \alpha)y)$$

Theorem 3.6.7. *Let $v \in G^N$ be such that $I(v) \neq \emptyset$. Then, $\eta(v)$ contains only one allocation.*

Proof. First, let's see that $\eta(v) \neq \emptyset$. Let us define $I^0 := I(v)$ and, recursively, the sets

$$I^k := \left\{x \in I^{k-1} : \text{ for each } y \in I^{k-1}, \theta_k(y) \geq \theta_k(x)\right\},$$

for each $k \in \{1, 2, \dots, 2^n\}$. Since $\theta_k(x)$ is a continuous function of x, for each $k \in \{1, 2, \dots, 2^n\}$, and $I^0 = I(v)$ is a nonempty and compact set, so is I^1. By induction, all the I^k sets are compact and nonempty. So is, in particular, I^{2^n}, which we claim to be $\eta(v)$.

By reductio ad absurdum, let's suppose the opposite, that $I^{2^n} \neq \eta(v)$. Then, there are $x \in I^{2^n}$ and $y \in I(v) \backslash I^{2^n}$ such that $\theta(y) \leq \theta(x)$.

Let k be the lowest integer such that $y \notin I^k$. Then, $\theta_k(y) > \theta_k(x)$. However, since for any $j \leq k-1$, we have $x, y \in I^j$, then $\theta_j(y) = \theta_j(x)$ and, consequently, $\theta(y) \succ_L \theta(x)$. Thus, $\eta(v)$ is nonempty.

Now, we want to see that the nucleolus is a set with only one element. We will prove so, again by reductio ad absurdum, by supposing there are different elements x, y that belong to $\eta(v)$.
Since we have assumed that they both belong to the nucleolus, by definition, $\theta(y) \succeq_L \theta(x)$ and $\theta(x) \succeq_L \theta(y)$, which leads to $\theta(x) = \theta(y)$.
Hence, for $I(v)$ is a convex set, $\alpha x + (1-\alpha)y \in I(v)$, for each $\alpha \in (0,1)$. However, from the previous lemma, we have that $\theta(x) \succ_L \theta\big(\alpha x + (1-\alpha)y\big)$ and, therefore, x does not belong to $\eta(v)$, contradicting our hypothesis. □

Remark 3.6.8. As a consequence of this result, we refer to the unique element of the set $\eta(v)$ as the nucleolus itself.

Example 3.6.3. Following from the previous example, we can propose $z = (0,0,1)$ as a nucleolus allocation —as *the* nucleolus, from what we have just learnt—.

△

Remark 3.6.9. Note that, given a TU-game v with a nonempty core $C(v)$:

(i) As have seen in Remark 3.6.3, the maximum excess for a core allocation is never positive.

(ii) By the definition of the core, for any allocation x outside the core, either

- $\sum_{i \in S} x_i \leq v(S)$, for some $S \subset N$.
- $\sum_{i \in N} x_i \neq v(N)$

But we saw that $\eta(v) \subset I(v)$, and imputations are efficient and individually rational. Then, this means that, for any allocation x outside the core, at least one coalition has a positive excess.

Then, the vector of ordered excesses cannot be lexicographically minimized outside the core. Hence, we have just proved the following result.

Theorem 3.6.10. *If a TU-game v has a nonempty core, the nucleolus is necessarily an element of the core.*

Proof. See the reasoning above. □

Example 3.6.4. Looking back for the last time at Example 3.6.2, for which $z = (0, 0, 1)$ was proposed as the nucleolus, we can now observe, from what we know on the core of a simple game —as is the case the example we are referring to—, the only core allocation of this game is $(0, 0, 1)$, which coincides with the one we proposed as the nucleolus. △

3.6.3 The kernel

Like the nucleolus, the kernel is a concept which is very related to the idea of excess. Now, additionally, we need to define the concept of *surplus*.

Definition 3.6.6. Given two players $i \neq j$ of an n-person game v and an allocation $x = (x_1, \ldots, x_n)$, the *surplus* of player i against player j is

$$s_{ij}(x) := \max\{e(S, x) : i \in S, j \notin S\}$$

This way, $s_{ij}(x)$ represents the most player i could hope to obtain without the help from player j.

Definition 3.6.7. Let $\langle x, \mathcal{T} \rangle$ an individually rational payoff configuration [10] and i, j different members of some $T_k \in \mathcal{T}$. We say that i *outweighs* j (and denote it by $i \gg j$) if, and only if, $s_{ij}(x) > s_{ji}(x)$ and $x_j > v(\{j\})$.

Remark 3.6.11. Thus, if $i \gg j$ there is a certain instability, since i can make a demand on j that the latter cannot contest. The definition of the kernel is such that such instabilities do not occur inside it.

Definition 3.6.8. The *kernel* of a game is the set $\mathcal{K}$ of all the individually rational payoff configurations $\langle x, \mathcal{T} \rangle$ such that

$$\text{for } T_k \in \mathcal{T}, \text{ there are no } i, j \in T_k \text{ with } i \gg j.$$

Let's observe the solutions proposed by the kernel of some games we have already covered.

Example 3.6.5. For the 3-person Divide the Dollar game (the 3-person simple majority game in which any coalition containing at least two player wins), we get $\langle x, N \rangle \in \mathcal{K}$ only when $x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. △

[10] A payoff configuration is a pair $\langle x, \mathcal{T} \rangle = (x_1, ..., x_n; T_1, ..., T_n)$ such that $\mathcal{T}$ is a partition of N and x is a vector satisfying $\sum_{i \in T_k} x_i = v(T_k)$.

Example 3.6.6. In the 3-person Glove game (the 3-person simple game in which any coalition containing Player 3 and any other player wins), we get $\langle x, N\rangle \in \mathcal{K}$ only when $x = (0, 0, 1)$, which coincides with the only core allocation and the the nucleolus of this game.

△

Theorem 3.6.12. *For any game* v*,* $\mathcal{K}(v) \neq \emptyset$*.*

Since its proof relies on the analogous one for the bargaining set, which we have not defined in this work, we refer the reader to Owen (1982) [4].

3.7 The Shapley Value

Opposite to the core or the nucleolus, the Shapley Value is an allocation rule for n-player Transferable Utility Games which approaches its value axiomatically.

Definition 3.7.1. We say that $i \in N$ is a *null player* in a game $v \in G^N$ if, for each $S \subset N$,

$$v(S) = v(S \cup \{i\})$$

Definition 3.7.2. We say that $i, j \in N$ are *symmetric players* in a game $v \in G^N$ if, for each $S \subset N \backslash \{i, j\}$, $v(S \cup \{i\}) = v(S \cup \{j\})$.

Shapley proposed a set of assumptions (axioms) that define a unique prediction for the outcome of not only bargaining 2-player games but any TU cooperative game that has become the most important allocation rule for TU-games:

- **Efficiency (EFF):** the sum of every player's payoff equals $v(N)$.

  $$\text{For each } v \in G^N, \sum_{i \in N} \varphi_i(v) = v(N)$$

- **Null player (NPP)**: players who do not contribute should receive nothing.
  An allocation rule ψ satisfies NPP if, for each $v \in G^N$ and each null player $i \in N$, $\varphi_i(v) = 0$.

- **Symmetry (SYM):** symmetric players should receive the same (i.e. treat equal players equally).
  An allocation rule ψ satisfies SYM if, for each pair of symmetric players $i, j \in N$, $\varphi_i(v) = \varphi_j(v)$

- **Additivity (ADD):** players receive the same payoff no matter if two games are played simultaneously or separately.

An allocation rule ψ satisfies ADD if, for each pair of games $v, w \in G^N$, $\varphi(v+w) = \varphi(v) + \varphi(w)$.

This last property is the only one that does not have a fairness idea to sustain it, but it is a natural requirement. The Shapley value is the only allocation rule in G^N satisfying these four properties.

Theorem 3.7.1. *The Shapley value is the only allocation rule in G^N satisfying EFF, NPP, SYM and ADD.*

Proof. The fact that Φ satisfies all four properties is quite straightforward. The proof for the uniqueness can be found in [5]. □

What's more, all four axioms used to characterize it are unexpendable, since removing any of them leaves room for other allocation rules such as

- the *equal division rule* $\psi_i(v) = v(N)/n$, which satisfies all axioms but NPP.
- or $\psi_i(v) = \lambda\Phi(v)$, with $\lambda \neq 1$, which satisfies all axioms but EFF.

Averaging the marginal contributions for each player over all the ways in which the grand coalition N can be formed, starting from the empty coalitions and adding players until we obtain the grand coalition, we obtain the expected payoff for each player and, thus, the Shapley value.

Definition 3.7.3. For each game $v \in G^N$, the *Shapley value* assigns a payoff Φ_j to each player j equal to the average of their marginal contributions when the player j is added to each possible coalition $S \subset N\backslash\{j\}$.
In particular,

$$\Phi_i(v) := \sum_{S \subset N\backslash\{i\}} \frac{|S|!(n-|S|-1)!}{n!} \Big(v(S \cup \{i\} - v(S) \Big)$$

Definition 3.7.4. We define the *marginal contribution* of a player j to a coalition S as

$$M(j, S) := v(S \cup \{j\}) - v(S)$$

Remark 3.7.2. We denote the set of all permutations of the elements in N by $\Pi(N)$ and, by $P^\pi(j)$ the set of all predecessors of j given the order determined by the permutation $\pi \in \Pi(N)$. That is to say, $k \in P^\pi(j)$ if, and only if, $\pi(k) < \pi(j)$.

Definition 3.7.5. Given an n-person game $v \in G^N$ and a permutation $\pi \in \Pi(N)$, the *vector of marginal contributions* associated with π is $m^\pi(v) \in \mathbb{R}^n$, defined, for each $i \in N$, as:

$$m_i^\pi(v) := v(P^\pi(i) \cup \{i\}) - v(P^\pi(i))$$

Then, equivalently to its previous expression, the Shapley value can also be computed as:

$$\Phi_i(v) := \frac{1}{n!} \sum_{\pi \in \Pi(N)} m_i^\pi(v)$$

Remark 3.7.3. The *Weber set* is the convex hull of the set of vectors of marginal contributions.

Example 3.7.1. Going back to the previously mentioned *Divide a million* game (Example 3.4.2), which is also known as the *Odd-Man-Out*, in which at least 2 of the 3 players have to agree on how to divide a dollar —or a million dollars, what difference does that make?—, the value of any coalition S is 1\$ if the coalition S has at least 2 players and is 0 if S has 1 player or is empty.
For the sake of clarity, we will now identify players with names instead of numbers: players 1, 2 and 3 will now be Alice, Bob and Charles, respectively —A, B and C for brevity—. Now, we can find out what Alice's marginal value in each form of coalition would be:

- $ABC \to 0$
- $ACB \to 0$
- $BAC \to 1$
- $BCA \to 0$
- $CAB \to 1$
- $CBA \to 0$

Therefore, Alice (or any other player, since the game is symmetric) is assigned a payoff

$$\frac{0+0+1+0+1+0}{6} = \frac{1}{3}$$

by the Shapley value. Thus,

$$\Phi(v) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

As we saw in Example 3.5.2, the core of this game was empty. △

Therefore, we can observe that the Shapley value of a game does not necessarily belong to its core —this can happen even when the core is nonempty—. However, we will see that the Shapley value of a convex game always v falls inside the core.

Theorem 3.7.4. *The Shapley value of a superadditive game belongs to the set of imputations.*

Proof. For each player $i \in N$ of a superadditive game $v \in SG^N$,

$$\Phi_i(v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} {m_i}^{\pi}(v) \geq \frac{1}{n!} \sum_{\pi \in \Pi(N)} v(i) = v(i)$$

since we have that ${m_i}^{\pi}(v) \geq v(i)$ for each $i \in N$ and each $\pi \in \Pi(N)$. □

3.8 Convex games

Definition 3.8.1. We say that a TU-game $v \in G^N$ is *convex* if, for each $i \in N$, and each pair $S, T \subset N \setminus \{i\}$ with $S \subset T$,

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$$

As we will see in this section, for any convex game v, the Shapley value $\Phi(v)$ is an element of the core $C(v)$, which is nonempty. This property is indeed important enough for us to study this type of games, but it is not the only one.

From their definition, we can see that any convex game is superadditive. Therefore, the core of any convex game is nonempty.

Theorem 3.8.1. *Let $v \in G^N$. Then:*

(i) v is a convex game

(ii) For each $\pi \in \Pi(N)$, $m^{\pi}(v) \in C(v)$

(iii) Its core and Weber set coincide.

Proof. The first implication $\big((\text{i}) \Longrightarrow (\text{ii})\big)$ was proved by Shapley (1971), the second one $\big((\text{ii}) \Longrightarrow (\text{iii})\big)$ by Weber (1988) and the last one $\big((\text{iii}) \Longrightarrow (\text{i})\big)$, by Ichiisi (1981). □

Corollary 3.8.2. *The core $C(v)$ of any convex game $v \in G^N$ contains its Shapley value $\Phi(v)$.*

Chapter 4

Applications of Cooperative Game Theory

4.1 Indices of power

Simple games often model voting situations and are useful to determine the power each player has in each game —by power, we refer to the strength each player has in the sense of being able to change the outcome of the game—. That is the reason why allocation rules are usually called power indices in simple games, when changing from the outcome for your own coalition from 0 to 1 is critical.
In fact, we have already seen one of these indices of power, the Shapley value in section 3.7. These are the definition of the most common power indices:

Definition 4.1.1. The Shapley-Shubik power index (S.S.P.Y) is just the restriction of the Shapley value to the case of simple games. Thus, the power index it assigns to each player is also noted by Φ_i.

Another important example is the Banzhaf power index, of which there are two important versions. We need to define the concept of *swings* first:

Definition 4.1.2. Given a game $v \in G^N$, a *swing* for a player $i \in N$ is a coalition $S \subset N \setminus \{i\}$ such that $S \notin W$ and $S \cup \{i\} \in W$. We say that i is a *pivot* for S.

Remark 4.1.1. We denote the number of swings for player i by $\mu_i(v)$.

Definition 4.1.3. The "raw" Banzhaf index (R.B.I.), or Banzhaf-Coleman index, is defined as:

$$\beta_i(v) := \frac{\mu_i(v)}{\sum_{i \in N} \mu_i(v)}$$

Definition 4.1.4. Last but not least, we define the Banzhaf index for player i as:

$$B_{zi}(v) := \frac{\mu_i(v)}{2^{n-1}}$$

For each player i of a game v, this last index represents the probability that, if we choose a random coalition $S \subset N \setminus \{i\}$, the player i is a pivot for S. Hence, it is often given as a percentage.

One well-known game in which we can easily see who is the most powerful player is the Glove Game, so let's see how the players would do in this scenario:

Example 4.1.1. Remember that $v(S) = 1$ if and only if $S = \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$. Otherwise, $v(S) = 0$.

Player	Hand	Φ_i	μ_i	β_i	B_{zi}
1	Right	1/6	1	0.2	0.25
2	Right	1/6	1	0.2	0.25
3	Left	4/6	3	0.6	0.75

Table 4.1: Values each division rule proposes for the Glove Game.

△

4.1.1 Voting games

As we have just seen, allocation rules for simple games are commonly known as power indices and measure the power each player has in a game. Thus, they become very useful when it comes to political situations we may want to analyze. Now, as we mentioned in section 3.4.2, we will see a couple of examples of real-life weighted majority games.

Example 4.1.2. After the recent 2015 Spanish general elections, in the current state of the Congress of Deputies of Spain, the 350 total deputies are distributed among 10 parties in the following way: PP (123), PSOE (90), Podemos (69), Cs (40), ERC (9), DiL (8), PNV (6), IU (2), EHB (2), CC (1).
There are two types of games being played by these parties, the first one being the approval of laws or the decision of who will be the next president which follows the simple majority

rule. For the sake of simplicity, we assume that each player has to vote either in favour or against each proposed candidate. Hence, a winning coalition needs more than half of the votes (at least 176) in their favor.
Then, the minimal winning coalitions are: {PP, PSOE}, {PP, Podemos}, {PP, Cs, ERC, DiL}, {PP, Cs, ERC, PNV}, {PP, Cs, ERC, IU, EHB}, {PP, Cs, DiL, PNV}, {PP, Cs, DiL, IU, EHB, CC}, {PSOE, Podemos, Cs}, {PSOE, Podemos, ERC, DiL}, {PSOE, Podemos, ERC, PNV, IU}, {PSOE, Podemos, ERC, PNV, EHB}, {PSOE, Podemos, DiL, PNV, IU}, {PSOE, Podemos, DiL, PNV, EHB}.
Since {PP, PSOE} is a winning coalition and both PP and PSOE can also form winning coalitions in which the other is not included, one can see that there is no veto player in this game. Thus, by the result we saw in 3.5.5, the core of this game is empty.

Computing the number of swings for each party, we get:

Party	% of votes	Seats	% of seats	Φ_i	μ_i	β_i	B_{zi}
PP	28.72%	123	35.14%	40.24 %	329	37.73 %	64.26 %
PSOE	22.01%	90	25.71%	21.98 %	183	20.99 %	35.74 %
Podemos	20.66%	69	19.71%	21.98 %	183	20.99 %	35.74 %
Cs	13.93%	40	11.43%	6.90 %	73	8.37 %	14.26 %
ERC	2.39%	9	2.57%	3.02 %	35	4.01 %	6.84 %
DiL	2.25%	8	2.29%	2.54 %	29	3.33 %	5.66 %
PNV	1.20%	6	1.71%	1.98 %	23	2.64 %	4.49 %
IU	3.67%	2	0.57%	0.56 %	7	0.80 %	1.37 %
EHB	0.87%	2	0.57%	0.56 %	7	0.80 %	1.37 %
CC	0.33%	1	0.29%	0.24 %	3	0.34 %	0.59 %

Table 4.2: Values each division rule proposes for the first example (real) of the Spanish Congress of Deputies Game.

Barring the obvious biases that the electoral system presents —which tend to benefit bigger parties— one can see that:

- As is natural, the power indices are monotonic with weight. However, such monotony is not strict: quite interestingly, the second and third parties are exactly equally powerful according to all power indices although their weights are not nearly as close as the ones between smaller parties.
- Since the smaller parties are needed by very few coalitions to become winning coalitions, their power is, as expected, very small.
- However, there are no null players (dummies).
- The computation checks that all of them satisfy SYM.

△

Example 4.1.3. In the same situation, in order to change some fundamental laws like the Spanish Constitution, two thirds of the Congress need to vote in favor of those changes. In this case, that number of votes is 234, representing the quota of this game. As in the previous example, the weights are also the number of seats each party has.
Thus, any coalition not containing PP cannot win. This means that PP is a veto player, a fact that —besides the obvious political consequences for their political rivals asking for major changes— implies that the core of this game is nonempty and that the only core allocation is such that $v(\text{PP}) = v(N) = 1$ and the rest are 0. △

Remark 4.1.2. Even if we diverge slightly from the point of the rest of this thesis, let's see what would happen if we modified Example 4.1.2. Here, instead of the current provinced-based D'Hont system, we took apart the 3 autonomous communities in which regional parties got seats and we applied the D'Hont system to a unique circumscription which contained the rest of the voters:

Party	% of votes	Seats	% of seats	Φ_i	μ_i	β_i	B_{zi}
PP	28.72 %	105	30.00 %	38.45 %	309	35.44 %	60.35 %
PSOE	22.01 %	83	23.71 %	23.77 %	203	23.28 %	39.65 %
Podemos	20.66 %	75	21.43 %	23.77 %	203	23.28 %	39.65 %
Cs	13.93 %	49	14.00 %	5.12 %	53	6.08 %	10.35 %
ERC	2.39 %	9	2.57 %	2.14 %	25	2.87 %	4.88 %
DiL	2.25 %	8	2.23 %	1.79 %	21	2.41 %	4.10 %
PNV	1.20 %	6	1.71 %	1.11 %	13	1.49 %	2.54 %
IU	3.67 %	12	3.43 %	3.21 %	37	4.24 %	7.23 %
EHB	0.87 %	2	0.57 %	0.40 %	5	0.57 %	0.98 %
CC	0.33 %	1	0.29 %	0.24 %	3	0.34 %	0.59 %

Table 4.3: Values each division rule proposes for the second example (not real) of the Spanish Congress of Deputies Game.

This system seems to express in a much more fair way what the voters want without having to change the D'Hont rule —which, traditionally, big parties are quite fond of—. However, leaving aside the obvious changes that take place, this is a good way for us to see that power indices for a given player are not necessarily higher when the weight of the player in question grows —observe what happened for Cs, who went from 40 to 49 deputees yet lost power according to all power indices—.
Although the power indices have shifted quite significantly, PSOE and Podemos would still be equally powerful according to all power indices.

4.2 Bankruptcy problems

A company with important debts with various creditors goes bankrupt. Each of those creditors claims a share of what is left, adding up to an amount larger than what is available. How should those creditors share the amount of money that was left or obtained by selling the estate? It seems natural that, since the estate left is not enough to cover every creditor's claim, they should reach some type of agreement on how to divide it.
We are now going to see how does Cooperative Game Theory handle this type of problems.

Definition 4.2.1. A bankruptcy problem with a set of claimants N is a pair (E, d), where $E \in \mathbb{R}$ and $d \in \mathbb{R}^N$ are such that $d_i \geq 0$, for each $i \in N$; and $0 \leq E \leq \sum_{i=1}^{n} d_i$.

We define the associated bankruptcy game as the TU-game (N, v) defined, for each $S \subset N$, by

$$v(S) := \max\left\{0, E - \sum_{i \notin S} d_i\right\}$$

which looks at the game from a pessimistic perspective, since it determines the worth of a coalition as the amount that is left once the rest of the players have received their own claims.

Remark 4.2.1. Since bankruptcy games are convex —by definition—, from the results we saw in section 3.8, their core is nonempty and contains both the Shapley value and the nucleolus of the game in question.

Definition 4.2.2. Let f be a function that assigns a division $f(E, d)$ to each bankruptcy problem (E,d) such that $\sum_{i \in N} f_i(E, d) = E$ and, for each $i \in N$, $0 \leq f_i(E, d) \leq d_i$. Then, we say that f is a *division rule*.

Obviously, there are many alternative proposals for division rules, although not all of them seem equally fair. Thus, before getting into these possible division rules, one could argue that it would be optimal if a division rule satisfies some stability properties such as:

- **Consistency (CONS):**

Given two bankruptcy problems (E, d), with set of claimants N; and (E_S, d_S), with set of claimants $S \subset N$; with $E_S = E - \sum_{j \notin S} f_j(E, d)$ and $d_S = d$, we say that a division rule satisfies CONS if, for each $i \in S$, we have that $f_i(E_S, d_S) = f_i(E, d)$.

The idea behind it is that the order in which agents make their claims should not determine the final divison.

- **No advantegeous merging or splitting (NAMS):**

Let (E, d), with set of claimants N, and (E', d'), with set of claimants $S \subset N$, be two bankruptcy problems such that $E' = E$ and with d' such that $\exists i \in S$ s.t. $d'_i = d_i + \sum_{k \notin S} d_k$ and, for each $j \neq i$, $d'_j = d_j$. Then, we say that a division rule satisfies NAMS if:

$$\text{for each } k \in S, \quad f_k(E', d') = f_k(E, d) + \sum_{j \notin S} f_j(E, d)$$

This second property guarantees that no agent can gain by splitting or joining forces with others.

Let's take a look at some of these division rules:

Definition 4.2.3. Given a bankrupcy problem (E, d), the constrained equal awards (CEA) rule is defined as the division rule that assigns

$$f_i^{CEA}(E, d) := min\{d_i, \lambda\}$$

to each player $i \in N$, with λ chosen so that $\sum_{i \in N} min\{d_i, \lambda\} = E$. This ensures that every player gets the same —as long as that amount is not greater than their claim—.

Definition 4.2.4. Similarly, for a given a bankrupcy problem (E, d), the constrained equal losses (CEL) rule, which is defined as

$$f_i^{CEL}(E, d) := max\{0, d_i - \lambda\}$$

with λ s.t. $\sum_{i \in N} max\{0, d_i - \lambda\} = E$, provides a division such that every player is equally far from obtaining his claim—as long as that does not make them get less than 0—.

Definition 4.2.5. Given a bankrupcy problem (E, d), the random arrival rule assigns

$$f_i^{RA}(E, d) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} min\Big\{d_i, max\{0,\ E - \sum_{j \in P^{\pi}(i)} d_j\}\Big\}$$

to each claimant $i \in N$. [1]

[1] As we saw in the previous chapter, $\Pi(N)$ represents the set of all the permutations of the elements in N, while $P^{\pi}(i)$ is the set of all the predecessors of the element i under the permutation π.

The random arrival rule assigns, to each claimant, their expected share of the division if each player arrives at a different time and receives the minimum between his or her claim and what is left —assuming that all the orders in which the players can make their claims are equally likely—.

Remark 4.2.2. The random arrival rule of a bankruptcy problem (E, d) coincides with the Shapley value of the associated bankruptcy game v.

Definition 4.2.6. The Talmud rule, proposed to match with the proposed divisions from the Talmud [2], is defined as

$$f_i{}^{TR}(E,d) := \begin{cases} \min\Big\{\dfrac{d_i}{2}, \lambda\Big\} & \text{when } E \leq \displaystyle\sum_{i\in N} \dfrac{d_i}{2} \\ \dfrac{d_i}{2} + \max\Big\{0, \dfrac{d_i}{2} - \lambda\Big\} & \text{when } E \geq \displaystyle\sum_{i\in N} \dfrac{d_i}{2} \end{cases}$$

with λ s.t. $\displaystyle\sum_{i\in N} f_i{}^{TR}(E,d) = E$.

Remark 4.2.3. The Talmud rule of a bankruptcy problem (E, d) coincides with the nucleolus of the associated bankruptcy game v.

Definition 4.2.7. The proportional rule is defined as

$$f_i^{PR}(E,d) := \frac{d_i}{\displaystyle\sum_{j\in N} d_j} E$$

unless $d_i = 0$ for each $i \in N$, in which case $f_i^{PR}(E,d) = 0$ for all players $i \in N$.

Example 4.2.1. Let v be a bankruptcy problem characterized by $E = 500$ and claims $d_1 = 250, d_2 = 200, d_3 = 100$.

	Player 1	Player 2	Player 3
Constrained equal awards rule	200,00	200,00	100,00
Constrained equal losses rule	233,33	183,33	83,33
Random arrival rule	233,33	183,33	83,33
Talmud rule	233,33	183,33	83,33
Proportional Rule	227,27	181,82	90,91

Table 4.4: Values each division rule proposes for the bankruptcy problem in Ex. 4.2.1.

$\triangle$

[2] The Talmud is a central text of Rabbinic Judaism which consisting on a record of discussions on Jewish law, ethics, tradition, etc.

Remark 4.2.4. From this example, it is easy to find that the only division rule of these 5 that satisfies both properties is the proportional rule, since the neither the Talmud rule, the CEA nor the CEL satisfy NAMS and the random arrival rule satisfies none.

Remark 4.2.5. Since Aumann and Maschler (1985) showed that when the bankruptcy problem is represented by a TU-game, it is impossible to obtain either the proportional or the constrained equal award rules out of any symmetric, Pareto efficient, and invariant to strategic equivalence solution concept we may apply to it, the bankruptcy problem was also studied as an NTU-game by Dagan and Volij (1992), when they associated each bankruptcy problem with a bargaining problem to which they applied well known bargaining solutions —instead of associating it to a TU-game and observing how it relates to known solutions, as done here—.

Remark 4.2.6. This type of problems can also be applied to the allocation of finite resources moving towards sustainability. One example of such problems is to establish fishing quotas to each of the countries with access to such resource —each of whom has a claim of their own— to guarantee its sustainability.

4.3 Airport problems

Example 4.3.1. A group of neighbours who live on the outskirts of a town are hosting a meeting to decide how to divide the costs of restoring the road to their houses. Since the cost of the work depends on the length of the road, we group them according to which of the four exits leads to their houses. Thus, we have 4 players for this cost-sharing game.

The cost of restoring the section of the road needed by Player 1 is the lowest: 7 (i.e. $v(\{1\}) = -7$). The cost of the works needed by players 2 and 3 are 11 and 16, respectively, while it costs 25 to restore the whole road, which is what player 4 needs.

$$v(S) := \begin{cases} -7 & \text{if } S = \{1\} \\ -11 & \text{if } 2 \in S \text{ and } 3, 4 \notin S \\ -16 & \text{if } 3 \in S \text{ and } 4 \notin S \\ -25 & \text{if } 4 \in S \end{cases}$$

Now, if each player pays only for the cost of restoring the part of the road that he needs —which seems pretty fair— and each excess is divided only between the players that need it:

- Player 1 only needs to pay for the first section, whose cost is divided between the four players. Thus, $x_1 = -\frac{7}{4}$.

- Player 2 has to pay his quarter of the cost of the first section plus a third of the cost of the second one. Then, $x_2 = -\frac{7}{4} - \frac{4}{3} = -\frac{37}{12}$.
- Similarly, for player 3, $x_3 = -\frac{7}{4} - \frac{4}{3} - \frac{5}{2} = -\frac{67}{12}$.
- Last, player 4 has to pay his share of all sections. Thus, $x_4 = -\frac{7}{4} - \frac{4}{3} - \frac{5}{2} - 9 = -\frac{175}{12}$.

△

The situation from the previous example is very similar to the one we could face in an airport, in which we have m different types of players (different types of airplanes) $i \in \{1, 2, \ldots, m\}$ who make a number of movements $n_i > 0$. Each of these movements has a cost c_i, which are assumed —without loss of generality— to be already placed in increasing order, i.e. $0 < c_1 < \ldots < c_m$.

Remark 4.3.1. Let $n = \sum_{i=1}^{m} n_i$ and $N = \bigcup_{i=1}^{m} N_i$.

Remark 4.3.2. The elements described so far characterize the cost allocation problems known as *airport problems.*

It was Littlechild and Owen (1973) [6] who came up with this simple yet important application of the Shapley value *"whenever the characteristic function is a "cost" function with the property that the cost of any subset of players is equal to the cost of the "largest" player in that subset"*, prompted by some previous work from Baker (1965) and Thompson (1971). The calculation of the Shapley value matched precisely with the previously proposed rule for calculating airport landing —or building— charges.
The cost function of the airport problem is defined as

$$c(S) = max\{c_i \,,\ i : N_i \cap S \neq \emptyset\}, \text{ for each nonempty } S \subset N.$$

Remark 4.3.3. The cost function is not superadditive but its negative clearly is so.

Remark 4.3.4. The Shapley value of a game based on the negative of a cost function is exactly minus the Shapley value of the game based on the cost function itself.

Definition 4.3.1. The *airport game* associated with the airport problem described above is the TU-game (N, v), characterized by $v(S) := -c(S)$.

Remark 4.3.5. Not only are airport games (N, v) superadditive, but also convex. Then, as we saw in section 3.8, their core is nonempty and contains both the Shapley value and the nucleolus of the game in question.

Theorem 4.3.6. *Given an airport game* (N, v) *such that* $c_0 := 0$*,*

$$\Phi_i(v) = -\sum_{k=1}^{k_i} \frac{c_k - c_{k-1}}{r_k}$$

where $r_k := \sum_{j=k}^{m} n_j$ *is the total number of movements of aircrafts equal or bigger than Player* i*'s type.*

Proof. Since the Shapley value satisfies ADD, the Shapley value for the sum of two (or more) games is equal to the sum of the Shapley values of the games. In this case, as we will see, the game —its characteristic function— can be represented as the sum of m games v_1, ..., v_m. Hence, we will have: $\Phi_i(v) = \sum_{k=1}^{m} \Phi_i(v_k)$.

Let $R_k = \bigcup_{i=k}^{m} N_i$, $k \in \{1,\ 2,\ ...,\ m\}$ and $S \subset N$ such that $\max\{j : N_j \cap S \neq \emptyset\} = k$. Then, $R_j \cap S = \emptyset$ if $j > k$ and $R_j \cap S \neq \emptyset$ if $j \leq k$. Now, if for each $k = 1, 2, ..., m$, we define the characteristic function —the TU-game— v_k on N as:

$$v_k(S) := \begin{cases} 0 & \text{if} \quad S \cap R_k = \emptyset \\ -(c_k - c_{k-1}) & \text{if} \quad S \cap R_k \neq \emptyset \end{cases}$$

we get that $\sum_{j=1}^{m} v_k(S) = \sum_{j=1}^{k} -(c_j - c_{j-1}) + \sum_{j=k+1}^{m} 0 = -(c_k - c_0) = v(S)$. Thus, $v(S) = \sum_{j=1}^{m} v_k(S)$, for each $S \subset N$.

It is quite obvious that, for any $k \in \{1, 2, ..., m\}$, v_k is a symmetric game for the r_k players in R_k. Since the Shapley value satisfies SYM, they will all get the same. The rest —the $n - r_k$ players that do not belong to R_k— are null players. Hence, since the Shapley value satisfies NPP, none of them will pay anything. Besides, as the Shapley value also satisfies EFF, this means that the r_k symmetric players get to share the whole cost. Hence, we have:

$$\Phi_j(v_k) = \begin{cases} 0 & \text{if} \quad j \notin R_k \\ -\frac{(c_k - c_{k-1})}{r_k} & \text{if} \quad j \in R_k \end{cases} \qquad \text{for each } j \in N.$$

For each $j \in N$, then, let $k_i = \max\{j : R_j \ni i\}$. Then, $\Phi_i(v) = -\sum_{k=1}^{k_i} \frac{c_k - c_{k-1}}{r_k}$.

□

Remark 4.3.7. (i) Let $\Phi_i(v)$ be the proposed ,$\Phi_i(v) = \Phi_{i-1}(v) - \frac{c_i - c_{i-1}}{r_i}$.

(ii) If $n_i = 1$ for all $i \in \{1, 2, \ldots, m\}$, $\Phi_i(v) = \Phi_{i-1}(v) - \frac{c_i - c_{i-1}}{n-i+1}$.

Remark 4.3.8. Littlechild (1974) also proposed the nucleolus of the airport game as the option to allocate the fees, by solving several linear programming problems.

Conclusions

The aim of this undergraduate thesis was to gain insight into the field of Game Theory inside Mathematics, which I was mostly unaware of when all this began. By centering our attention on the branch of Cooperative Game Theory, although not many grandiloquent results have been proved, we have been able to define and analyze a wide range of concepts regarding equilibrium, fairness and stability, which I believe could prove to be increasingly important as Game Theory earns its spot as a tool to analyze any situation involving interaction, be it conflict or cooperation. This work also covered a wide variety of problems which Game Theory can solve or at least propose solutions to. Therefore, I hope to have been able to transmit my enthusiasm to the reader, who has hopefully also gained some perspective into the usefulness of Game Theory to analyze and solve situations as diverse as cost-sharing road building problems, the division of inheritances —the claims problem—, or computing the power each political party has in a parliament.

Besides, I must say that Cooperative Game Theory was attractive to me because I believe a fairer society is possible, even at this seemingly point of no return, and I yearned for having a look at the mathematical tools that allowed us to model such interaction. Even though such ambition is probably out of reach in an overview work, I believe that this may be just the first step in a long journey.

Bearing in mind that there are further aspects of Cooperative Game Theory, such as non-atomic games, that could not find their place in this thesis —not to mention the endless proposed solution concepts one can find when reading different authors—, nevertheless, what I relish the most is the chance to study more realistic and tangible applications of Game Theory to economy, politics, global environmental sustainability, management of finite resources or wealth distribution and the possible implication this could have on such fields —fields which are utterly strategic for our future—. One can never stop learning, so although I wish I had had more time to get more out of the theory developed with broader and more exotic types of applications and problems, on the other side, this thesis may act as an inflection point in my studies whether it means keeping my academic career related with this field or following it on my own.

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