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ADVANCED MATHEMATICS
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**STOCHASTIC DIFFERENTIAL EQUATIONS
DRIVEN BY A FRACTIONAL BROWNIAN MOTION**

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ABSTRACT

This project is a general study of Stochastic Differential equations driven by a fractional Brownian motion of Hurst parameter $H > 1/2$. Sections 3,4 and 5 follow the lines of [16] in order to define a stochastic integral with respect to the fractional Brownian motion and then, discussing the existence and uniqueness of solutions.

The sixth section is a general discussion about Malliavin calculus with respect to the fractional Brownian motion that will be useful in sections 7 and 8. Moreover, in section 6 we prove that by reinforcing the conditions on the coefficients, we obtain absolute continuity of the law of the solution in the same way as it is done in [14].

Section 7 is the application of the Malliavin calculus in order to bound the density function of the solution to a specific type of equations by using a general method constructed in [12].

Finally, section 8 is devoted to show all the work we weren't able to finish during the elaboration of this thesis. We decided to attack the problem of bounding the density of a general family of stochastic delay differential equations. The approach given in [12] turned out to be inefficient, so we decided to follow the same approach as in [1], [10] and [15].

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1. INTRODUCTION

In a broad variety of disciplines, such as physics and finance, one deals with the problem of having a system of objects (which can be particles in the physical context or stocks in the financial context) and trying to understand the evolution of a certain magnitude associated to this objects as time evolves. For instance, in finance, the people working in markets try to study the evolution of the price of a stock at time t , say S_t . The classical tool to capture continuous time dynamics is differential equations. However, when the dynamics of the magnitudes are contaminated with noise or can't be predicted, a probabilistic approach is needed in order to cover this problems.

In the beginning of the 20th century, authors like Albert Einstein, Marian Smoluchowski and Louis Bachelier started working with dynamical systems whose evolution law is driven by a standard Brownian motion $B = \{B_t; t \in [0, T]\}$. This studies motivated the mathematical community to study differential equations in which a term driven by a Brownian motion is added. This equations are the so-called Stochastic Differential Equations driven by a Brownian motion. From the 2000's on, the fractional Brownian motion B^H gained popularity. Several numerical studies show that this process is a better driving noise in some situations.

In this project, we study the differential equations driven by a fractional Brownian motion with Hurst parameter $H > 1/2$ from several points of view. In the first place, we consider equations of the form

$$(1.1) \quad X_t = x_0 + \int_0^t \sigma(s, X_s) dB_s^H + \int_0^t b(s, X_s) ds,$$

and the objective is giving sense to this new stochastic integral and, once this integral has a proper meaning, we study the existence and uniqueness of solutions. This first part does not require a lot of stochastic machinery, since we will use a deterministic approach. More specifically, we will use the generalized Stieltjes integral, introduced by Zähle in [20]. In order to define this generalization of the Stieltjes integral, we will make use of fractional integrals and derivatives. This is not the only possible approach to equations like (1.1). Indeed, the work of Terry Lyons in 1998 in the paper [11] started the so-called Rough Path Analysis. This new theory is useful to study differential equations driven by rough functions, understood as functions with finite p -variation for $p > 1$ or γ -Hölder continuous functions for $\gamma \in (0, 1)$. The second objective is studying the equations from a probabilistic point of view by exploring the law of the solution of this equations. To do so, we will use basic techniques of Malliavin calculus. The third objective goes in the same lines as the second objective. Under the same conditions for which the solution X_t is absolutely continuous with respect to the Lebesgue measure, we would like to go a bit further and give Gaussian bounds for the density function of X_t using more advanced Malliavin calculus techniques than in the study of the existence of the density function.

During the elaboration of this project, me and my advisor Carles were working on a research problem; we wanted to give Gaussian bounds for the density of solutions of Stochastic delay differential equations Driven by a fractional Brownian motion. In

the last chapter of this project, we present some of the work that has been done and the difficulties we faced. The problem of finding bounds is not solved yet due to some mistakes of the paper we were using as a reference.

2. MOTIVATION AND OBJECTIVES

As it has been explained in the introduction, stochastic differential equations are powerful tools in order to model phenomena evolving randomly in continuous time. From the 2000's on, the fractional Brownian motion has gained a lot of popularity, specially in finance, due to the numerous numerical studies giving evidence of how the models that use a fractional Brownian motion are a significant improvement.

In this thesis, however, we will discuss this equations from a theoretical point of view, rather than an applied point of view. In order to study the properties of this equations, we must be able to talk about existence and uniqueness of solutions. Then, we will use the Malliavin calculus to give properties of the law of the solutions to this equations. The main objectives of this project are:

1. Stablishing a result of existence and uniqueness of solutions under mild conditions of the coefficients.
2. Studying the Malliavin calculus in the fractional Brownian motion setting, relating the Malliavin differentiability of stochastic differential equations with the Fréchet differentiability of deterministic differential equations.
3. Understanding how the Malliavin derivative, the divergence operator, and the Ornstein-Uhlenbeck operator give us information about the law of the solution of the SDEs driven by a fractional Brownian motion. Moreover, we give a closed formula for the density function of the solution.
4. Since the formula for the density of the solution provided by the Malliavin calculus is not very accessible, we want to develop methods to bound the density function by Gaussian-type functions.
5. Extending the objective 4 to stochastic delay differential equations driven by a fractional Brownian motion.

3. BACKGROUND AND PRELIMINARIES

In this section, the objective is to state and define basic concepts of stochastic calculus that will be useful throughout this thesis. Most of this results are covered in a basic course about stochastic calculus or stochastic processes.

Definition 3.1. *Let I be an arbitrary set. A stochastic process X indexed by I is a collection of random variables $\{X_t; t \in I\}$.*

Definition 3.2. *Let $\{X_t; t \in I\}$ be a stochastic process. For each $\omega \in \Omega$, the map $t \in I \rightarrow X_t(\omega) \in \mathbb{R}$ is called a sample path of X .*

In this project, $I = [0, T]$ for some $T > 0$, so the sample paths are curves. In general, studying the regularity of the paths of a process is difficult. However, we got this result that helps us studying the sample paths

Theorem 3.3 (Kolmogorov's Continuity Criterion). *Let $\{X_t; t \in [0, T]\}$ be a stochastic process. If there exist $\alpha \geq 1$, $\beta > 0$ such that*

$$E(|X_t - X_s|^\alpha) \leq C|t - s|^{1+\beta},$$

then, almost surely, the paths of X are γ -Hölder continuous with $\gamma \in (0, \frac{\beta}{\alpha})$.

In the same way we can determine the law of a random variable or a random vector, we are interested in giving some sense to the "law" of a stochastic process.

Definition 3.4. *Given a process $\{X_t; t \in I\}$, we define its finite dimensional distributions as the family of probability laws of the random vectors $(X_{t_1}, \dots, X_{t_k})$ for all $t_1, \dots, t_k \in I$, $k \geq 1$.*

Given a process X , if we are optimistic we can find its finite dimensional distributions. An interesting question is if the reverse is true, i.e. given a family of probability laws $P_{t_1, \dots, t_k}$, can we find a process X such that the law of $(X_{t_1}, \dots, X_{t_k})$ coincides with $P_{t_1, \dots, t_k}$? The following result is an answer to this question

Theorem 3.5 (Kolmogorov's Extension Theorem). *Consider a family of probability laws*

$$\mathbb{P} = \{P_{t_1, \dots, t_k}; t_1, \dots, t_k \in [0, T]\}$$

such that

- $P_{t_1, \dots, t_k}$ is a probability in $\mathbb{R}^k$.
- If $\{t_{i_1}, \dots, t_{i_m}\} \subset \{t_1, \dots, t_k\}$ then, $P_{t_{i_1}, \dots, t_{i_m}}$ is the marginal law of $P_{t_1, \dots, t_k}$.

Then, there exists a process X such that its family of finite dimensional distributions is $\mathbb{P}$.

In this project we will focus mainly on one special kind of processes named Gaussian processes. We say that $X = \{X_t; t \in [0, T]\}$ is Gaussian if the family of finite dimensional distributions of X are Gaussian distributions. To justify the existence of such processes, we can apply Kolmogorov's Extension Theorem in the following way:

Let $K : I \times I \rightarrow \mathbb{R}$ be a symmetric and non-negative function. Fix $t_1, \dots, t_k \in I$. Denote by Γ the matrix $\Gamma = (K(t_i, t_j))_{1 \leq i, j \leq k}$ and $P_{t_1, \dots, t_k} \sim N(0, \Gamma)$. For each $\{t_{i_1}, \dots, t_{i_m}\}$ we have

$$A(X_{t_1}, \dots, X_{t_k})^T = (X_{t_{i_1}}, \dots, X_{t_{i_m}})^T,$$

where

$$A = \begin{pmatrix} \delta_{t_1, t_{i_1}} & \cdots & \delta_{t_k, t_{i_1}} \\ \cdots & \cdots & \cdots \\ \delta_{t_1, t_{i_m}} & \cdots & \delta_{t_k, t_{i_m}} \end{pmatrix}.$$

The vector $(X_{t_{i_1}}, \dots, X_{t_{i_m}})$ follows a Gaussian law with 0 mean and covariances matrix $A\Gamma A^T$. Moreover,

$$A\Gamma A^T = (K(t_{i_r}, t_{i_s}))_{1 \leq r, s \leq m}.$$

So Kolmogorov's Extension Theorem hypothesis are satisfied.

This result also shows us that every Gaussian process $X = \{X_t; t \in I\}$ is completely determined by a mean function $\mu(t) = E(X_t)$ and a covariance function $\Gamma(s, t) = \text{Cov}(X_s, X_t)$.

3.1. Fractional Brownian Motion. From all Gaussian processes, this project has a special focus on the fractional Brownian motion. This process is a natural generalization of the standard Brownian motion, and there exist evidences that it is indeed a better tool for some mathematical models.

Definition 3.6. A centered Gaussian process $B^H = \{B_t^H; t \in [0, T]\}$ is called a fractional Brownian motion (fBm) of Hurst parameter $H \in (0, 1)$ if it has the covariance function

$$(3.1) \quad \Gamma(s, t) = E(B_t^H B_s^H) = \frac{s^{2H} + t^{2H} - |t - s|^{2H}}{2}.$$

From this definition, we can deduce some properties:

Proposition 3.7. Let $\{B_t^H, t \in [0, T]\}$ be a fBm, the following properties hold:

- (1) $E(|B_t^H - B_s^H|^2) = |t - s|^{2H}$ and therefore B^H has stationary increments
- (2) if $H > 1/2$, the increments of B^H are positively correlated, whereas if $H < 1/2$, the increments of B^H are negatively correlated. As a consequence, if $H \neq 1/2$, the increments of B^H are not independent
- (3) B_{at}^H has the same law as $a^H B_t^H$.
- (4) B^H has continuous paths.

From this last proposition we can only make sure the continuity of the paths of B^H . As mentioned before, if we want to study the stochastic integral with respect to B^H as a Riemann-Stieltjes integral, continuity is not enough.

Lemma 3.8. *Let $p, \Psi : \mathbb{R} \rightarrow \mathbb{R}_+$ be continuous and strictly increasing functions vanishing at 0 and such that $\lim_{t \uparrow \infty} \Psi(t) = \infty$. Suppose that $\phi : \mathbb{R}^d \rightarrow E$ is a continuous function with values in a separable Banach space $(E, \|\cdot\|)$. Denote by $B_r(x_0)$ the ball of radius r centered at x_0 . If*

$$\Gamma = \int_{B_r(x_0)} \int_{B_r(x_0)} \Psi \left(\frac{\|\phi(t) - \phi(s)\|}{p(|t - s|)} \right) ds dt < \infty$$

holds, for all $t, s \in B_r(x_0)$, then

$$\|\phi(t) - \phi(s)\| \leq 8 \int_0^{|t-s|} \Psi^{-1} \left(\frac{4^{d+1} \Gamma}{\lambda_d u^{2d}} \right) p(du),$$

where λ_d is a universal constant depending only on d .

Now, we got a process $B^H = \{B_t^H; t \geq 0\}$ with values in $(\mathbb{R}, |\cdot|)$ (which is a separable Banach space) and $E(|B_t^H - B_s^H|^p) \leq C_H |t - s|^{pH}$. Therefore, we can take $\Psi(x) = x^{2/\varepsilon}$ and $p(x) = x^{\frac{m+2}{p}}$ with $0 < m < pH - 1$ and we obtain

$$|B_t^H - B_s^H|^p \leq C |t - s|^m \Gamma.$$

Moreover, the explicit expression of Γ is

$$\Gamma = \int_0^T \int_0^T \frac{|B_t^H - B_s^H|^p}{|t - s|^{m+2}} ds dt.$$

Since $\Gamma < \infty$ and last inequality holds for all $p \geq 2$, we conclude, by using $p = 2/\varepsilon$ and $m + 2 = 2H/\varepsilon$ that for all $\varepsilon > 0$, for all $T > 0$ there exists a random variable $\Gamma_{\varepsilon, T}$ such that

$$|B_t^H - B_s^H| \leq \Gamma_{\varepsilon, T} |t - s|^{H-\varepsilon}, \quad \forall s, t \in [0, T].$$

In fact, $\Gamma_{\varepsilon, T}$ has moments of all orders because

$$\Gamma_{\varepsilon, T} = \int_0^T \int_0^T \frac{|B_t^H - B_s^H|^{2/\varepsilon}}{|t - s|^{2H/\varepsilon}} ds dt.$$

This proves that the paths of B^H are γ -Hölder continuous of order $H - \varepsilon$ for all $0 < \varepsilon < H$.

4. FRAMEWORK

The standard Brownian motion has some properties that make studying the process or the differential equations driven by a Brownian motion a (relatively) comfortable experience. Even though the fractional Brownian motion may seem a slight generalization of the standard Brownian motion, we lose the martingale property and the independence of increments. These two properties, in case of the standard Brownian motion, are key to develop a stochastic integral in the sense of Itô, which is a well-known approach to define a stochastic integral. Facing the lack of these properties, the next most natural approach to define a stochastic integral with respect to the fractional Brownian motion is defining it pathwise. The classical Stieltjes integral turns out to be too restrictive for our case, so a generalization of this integral is needed. In this

project, we will use the *generalized Stieltjes integral* introduced by Zähle in [20]. To develop this integral for the fBm, we need that the paths of the fractional Brownian motion are regular enough (Hölder continuous) to be an integrator. This integration theory allows us to ensure that if $u = \{u_t; t \in [0, T]\}$ is a stochastic process with a.s.- γ -Hölder continuous paths where $\gamma > 1 - H$, then the integral in the generalized Stieltjes sense $\int_0^T u_t dB_t^H$ exists pathwise. A classical and pioneer version of this result can be found in [19]. We will consider this integral when we work stochastic differential equations when $H > 1/2$. In the case $H < 1/2$, existence and uniqueness of solutions of this equations cannot be ensured for any H in this range. A suitable approach in order to study the existence and uniqueness of solutions for this equations is using Rough Path theory (see, for example, [5]).

The generalized Stieltjes approach introduced in [20] relies on fractional integrals and derivatives. Therefore, a study of these operators on suitable function spaces is needed in order to proceed with the stochastic differential equations.

4.1. Fractional Integrals and Derivatives. Let $a, b \in \mathbb{R}$ with $a < b$. We denote by $L^p(a, b)$, $p \geq 1$ the space of Lebesgue measurable functions $f : [a, b] \rightarrow \mathbb{R}$ for which $\|f\|_{L^p(a, b)} < \infty$, where

$$\|f\|_{L^p(a, b)} = \begin{cases} \left(\int_a^b |f(x)|^p dx \right)^{1/p}, & 1 \leq p < \infty, \\ \text{ess sup}\{|f(x)|; x \in [a, b]\}, & p = \infty. \end{cases}$$

Definition 4.1. Let $f \in L^1(a, b)$ and $\alpha > 0$. The left-sided and right-sided fractional Riemann-Liouville integrals of f of order α are defined for almost all $x \in (a, b)$ by

$$I_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - y)^{\alpha-1} f(y) dy$$

and

$$I_{b-}^\alpha f(x) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_x^b (y - x)^{\alpha-1} f(y) dy.$$

respectively.

Notice that since (a, b) is an interval with finite measure, the previous definition also works for any $f \in L^p(a, b)$ with $p > 1$. We denote by $I_{a+}^\alpha(L^p)$ (resp. $I_{b-}^\alpha(L^p)$) the image of $L^p(a, b)$ by the operator I_{a+}^α (resp. I_{b-}^α).

Definition 4.2. Let $f \in L^p(a, b)$ and $\alpha \in (0, 1)$. We define the right-sided and the left-sided Weyl derivatives of f as

$$D_{a+}^\alpha f(x) = \frac{1}{\Gamma(1 - \alpha)} \left(\frac{f(x)}{(x - a)^\alpha} + \alpha \int_a^x \frac{f(x) - f(y)}{(x - y)^{\alpha+1}} dy \right) \mathbb{1}_{(a, b)}(x)$$

and

$$D_{b-}^\alpha f(x) = \frac{(-1)^\alpha}{\Gamma(1 - \alpha)} \left(\frac{f(x)}{(b - x)^\alpha} + \alpha \int_x^b \frac{f(x) - f(y)}{(y - x)^{\alpha+1}} dy \right) \mathbb{1}_{(a, b)}(x).$$

It is not obvious from the definitions that these operators behave as derivatives or integrals, but indeed they do. Despite the Weyl derivatives are global operators (unlike the classical derivatives, which are local) they satisfy the semigroup property and they are also the inverse of the fractional integrals. The main properties that these operators satisfy are the following:

Proposition 4.3. *This operators satisfy the following porperties:*

- (1) *The Riemann-Liouville integrals converge absolutely almost everywhere.*
- (2) *If $\alpha < 1/p$, and $q = \frac{p}{1-\alpha p}$ then*

$$I_{a+}^{\alpha}(L^p) = I_{b-}^{\alpha}(L^p) \subset L^q(a, b).$$

- (3) *If $\alpha > 1/p$ and $\alpha - 1/p \notin \mathbb{N}$ then*

$$I_{a+}^{\alpha}(L^p) \cup I_{b-}^{\alpha}(L^p) \subset \mathcal{C}^{\alpha-1/p}(a, b).$$

- (4) *The operators I_{a+}^{α} and I_{b-}^{α} have the semigroup property, i.e.*

$$I_{a+}^{\alpha} I_{a+}^{\beta} = I_{a+}^{\alpha+\beta} \quad (\text{resp. with } I_{b-}^{\alpha}).$$

- (5) *The following inversion formulas hold:*

$$I_{a+}^{\alpha} D_{a+}^{\alpha}(f) = f, \quad \forall f \in I_{a+}^{\alpha}(L^p),$$

and

$$D_{a+}^{\alpha} I_{a+}^{\alpha}(f) = f, \quad \forall f \in L^1(a, b).$$

(resp. with I_{b-}^{α} and D_{b-}^{α}).

- (6) *The operators D_{a+}^{α} and D_{b-}^{α} have the semigroup property, i.e.*

$$D_{a+}^{\alpha} D_{a+}^{\beta} = D_{a+}^{\alpha+\beta} \quad (\text{resp. with } D_{b-}^{\alpha}).$$

- (7) *The following integration by parts formula is satisfied*

$$\int_a^b D_{a+}^{\alpha} f(x) g(x) dx = (-1)^{\alpha} \int_a^b f(x) D_{b-}^{\alpha} g(x) dx,$$

for $f \in I_{a+}^{\alpha}(L^p)$, $g \in I_{b-}^{\alpha}(L^q)$ with $1/p + 1/q = 1$.

In order to prove this proposition, we need the following result.

Theorem 4.4 (Hardy-Littlewood theorem). *If $0 < \alpha < 1$, $1 < p < 1/\alpha$ then the fractional integration operator I_{a+}^{α} is bounded from $L^p(a, b)$ to $L^q(a, b)$ with $q = p/(1 - \alpha p)$.*

Proof. Can be found in [18]. □

Now we can prove the proposition.

Proof.

- (1) We will only see the convergence for $I_{a+}^\alpha(f)$ because the $I_{b-}^\alpha(f)$ case is completely analogous. Notice that if we are able to prove that $\|I_{a+}^\alpha(f)\|_{L^1(a,b)} < \infty$ we are done, because this means that

$$m(\{x \in (a, b) \text{ such that } I_{a+}^\alpha(f)(x) = \infty\}) = 0,$$

where m denotes the Lebesgue measure in (a, b) . Notice that

$$|\Gamma(\alpha)| \|I_{a+}^\alpha(f)\| \leq \int_a^x (x-y)^{\alpha-1} |f(y)| dy,$$

so

$$\begin{aligned} |\Gamma(\alpha)| \cdot \|I_{a+}^\alpha(f)\|_{L^1(a,b)} &\leq \int_a^b \int_a^x (x-y)^{\alpha-1} |f(y)| dy dx \\ &= \int_a^b \int_y^b (x-y)^{\alpha-1} |f(y)| dx dy \\ &\leq \int_a^b |f(y)| \frac{(b-y)^\alpha}{\alpha} dy \\ &\leq C(a, b, \alpha) \|f\|_{L^1(a,b)}, \end{aligned}$$

where $C(a, b, \alpha) > 0$ is a constant depending on α, a and b . Not only does this show that $I_{a+}^\alpha(f)$ is finite almost everywhere, but also that I_{a+}^α is a bounded operator from $L^1(a, b)$ to $L^1(a, b)$.

- (2) In order to prove that $I_{a+}^\alpha(L^p) = I_{b-}^\alpha(L^p)$, it is enough to notice that $RI_{a+}^\alpha = I_{b-}^\alpha R$ (as operators), where R is the operator $Rf(x) = f(a+b-x)$. Now, $I_{a+}^\alpha(L^p) \subset L^q(a, b)$ is a consequence of the Hardy-Littlewood theorem 4.4.
- (3) First consider the case $\alpha - 1/p < 1$. Consider $x, x+h \in (a, b)$. Then, we got $I_{a+}^\alpha(f)(x+h) - I_{a+}^\alpha(f)(x) = I_1 + I_2$, where

$$I_1 = \frac{1}{\Gamma(\alpha)} \int_x^{x+h} (x+h-y)^{\alpha-1} f(y) dy,$$

$$I_2 = \frac{1}{\Gamma(\alpha)} \int_a^x ((x+h-y)^{\alpha-1} - (x-y)^{\alpha-1}) f(y) dy.$$

On the one hand,

$$|I_1| \leq Ch^{\alpha-\frac{1}{p}} \|f\|_{L^p(a,b)}.$$

On the other hand,

$$|I_2| \leq |\Gamma(\alpha)|^{-1} h^{\alpha-\frac{1}{p}} \|f\|_{L^p(a,b)} \left(\int_0^{\frac{x-a}{h}} |z^{\alpha-1} - (z+1)^{\alpha-1}|^{p'} dz \right)^{1/p'},$$

where p' is the conjugated exponent of p . If $x-a \leq h$, we can estimate the last expression by

$$|I_2| \leq C(\alpha, f, p) \|f\|_{L^p(a,b)} h^{\alpha-\frac{1}{p}}.$$

If $x - a > h$, using that $|x^\mu - y^\mu| \leq |\mu|(x - y)y^{\mu-1}$ when $x \geq y > 0$, $\mu \leq 1$ and $(A + B)^{1/p'} \leq A^{1/p'} + B^{1/p'}$ we got:

$$\begin{aligned}
& |I_2| \\
& \leq \frac{\|\varphi\|_{L^p(a,b)}}{\Gamma(\alpha)} h^{\alpha-1/p} \left[\int_0^1 |z^{\alpha-1} - (z+1)^{\alpha-1}|^{p'} dz + \int_1^{\frac{x-a}{h}} |z^{\alpha-1} - (z+1)^{\alpha-1}|^{p'} dz \right]^{1/p'} \\
& \leq \frac{\|\varphi\|_{L^p(a,b)}}{\Gamma(\alpha)} h^{\alpha-1/p} \left[c_1 + c_2 \int_1^{\frac{x-a}{h}} z^{(\alpha-2)p'} dz \right]^{1/p'} \\
& \leq \frac{\|\varphi\|_{L^p(a,b)}}{\Gamma(\alpha)} h^{\alpha-1/p} \left[c_3 + c_4 \left(\int_1^{\frac{x-a}{h}} z^{(\alpha-2)p'} dz \right)^{1/p'} \right] \\
& = \frac{\|\varphi\|_{L^p(a,b)}}{\Gamma(\alpha)} h^{\alpha-1/p} \left[c_3 + c_4 \left(\frac{x-a}{h} \right)^{\alpha-1-1/p} \right] \\
& = \frac{\|\varphi\|_{L^p(a,b)}}{\Gamma(\alpha)} h^{\alpha-1/p} \left[c_3 + c_4 \left(\frac{h}{x-a} \right)^{1-\alpha+1/p} \right] \\
& \leq c_\alpha \|\varphi\|_{L^p(a,b)} h^{\alpha-1/p},
\end{aligned}$$

because $\frac{h}{x-a} < 1$ and $1 - \alpha + 1/p > 0$. If $\alpha - 1/p > 1$ and it's not a natural number, then $\alpha - 1/p = k + \sigma$ where $k \in \mathbb{N}$, $\sigma \in (0, 1)$. In this case, notice that

$$\frac{d^k}{dx^k} I_{a+}^\alpha(\varphi)(x) = I_{a+}^{\alpha-k}(\varphi)(x),$$

and we can apply the work previously done. In the case of I_{b-}^α is analogous.

- (4) It can be proved by direct computations.
- (5) Can be found in [18].
- (6) To prove this, we will make use of the inversion formulas. Let $\varphi \in L^p(a, b)$ such that $I_{a+}^{\alpha+\beta}(\varphi) = f$. Let $\psi \in L^p(a, b)$ such that $I_{a+}^\beta(\psi) = f$. Then we got

$$D_{a+}^\alpha D_{a+}^\beta f = D_{a+}^\alpha D_{a+}^\beta I_{a+}^\beta(\psi) = D_{a+}^\alpha \psi.$$

Now, $I_{a+}^{\alpha+\beta}(\varphi) = I_{a+}^\beta I_{a+}^\alpha(\varphi)$, so $I_{a+}^\alpha(\varphi) = \psi$. Plugging this in the last equality, we got

$$D_{a+}^\alpha \psi = D_{a+}^\alpha I_{a+}^\alpha(\varphi) = \varphi = D_{a+}^{\alpha+\beta}(f).$$

- (7) If we suppose that both integrals are finite, then by defining $\varphi_f \in L^p(a, b)$ such that $I_{a+}^\alpha(\varphi_f) = f$ and $\varphi_g \in L^q(a, b)$ such that $I_{b-}^\alpha(\varphi_g) = g$, we need to prove that

$$\int_a^b \varphi_f(x) g(x) dx = (-1)^\alpha \int_a^b f(x) \varphi_g(x) dx.$$

This identity follows from direct computations using the definitions of I_{a+}^α and I_{b-}^α . The only thing left to do is justify that both sides of the equality are finite, but this is immediate using Hölder's inequality.

□

4.2. Generalized Stieltjes Integral. We have all the tools to define the generalized Stieltjes integral. The objective now is to define the integral of f with respect to g using fractional calculus. Let $f(a+) = \lim_{\varepsilon \downarrow 0} f(a + \varepsilon)$ and let $g(b-) = \lim_{\varepsilon \uparrow 0} g(b - \varepsilon)$ (supposing that this limits exists). We define the functions

$$\begin{aligned} f_{a+}(x) &= (f(x) - f(a+))\mathbb{1}_{(a,b)}(x), \\ g_{b-}(x) &= (g(x) - g(b-))\mathbb{1}_{(a,b)}(x). \end{aligned}$$

Definition 4.5. Suppose f, g are functions such that $f(a+), g(a+)$ and $g(b-)$ exist, $f_{a+} \in I_{a+}^\alpha(L^p)$ and $g_{b-} \in I_{b-}^{1-\alpha}(L^q)$ for some $p, q \geq 1$, $1/p + 1/q \leq 1$, $0 < \alpha < 1$. Then, the integral of f with respect to g is defined by

$$(4.1) \quad \int_a^b f dg = (-1)^\alpha \int_a^b D_{a+}^\alpha f_{a+}(x) D_{b-}^{1-\alpha} g_{b-}(x) dx + f(a+)(g(b-) - g(a+)).$$

First, let's check that this is a good definition.

Proposition 4.6. Under the same assumptions as the last definition, the generalized Stieltjes integral (4.1) is well defined in the sense that it takes finite values and does not depend on the value of α .

Proof. The only part that needs to be checked is $\int_a^b D_{a+}^\alpha f_{a+}(x) D_{b-}^{1-\alpha} g_{b-}(x) dx$ because we are assuming that $f(a+), g(b-)$ and $g(a+)$ exist (and they obviously don't depend on α). Let $\tilde{f} = D_{a+}^\alpha f_{a+}(x)$ and $\tilde{g} = D_{b-}^{1-\alpha} g_{b-}(x)$. By the inversion formulas, we got that $\tilde{f} \in L^p(a, b)$ and $\tilde{g} \in L^q(a, b)$. By Hölder's inequality,

$$\left| (-1)^\alpha \int_a^b D_{a+}^\alpha f_{a+}(x) D_{b-}^{1-\alpha} g_{b-}(x) dx \right| \leq \|\tilde{f}\|_{L^p(a,b)} \|\tilde{g}\|_{L^{p'}(a,b)},$$

where p' is the conjugated exponent of p . Since $1/p + 1/q \leq 1$, we got $q \geq p'$ and therefore $L^q(a, b) \subset L^{p'}(a, b)$. This implies $\|\tilde{g}\|_{L^{p'}(a,b)} < \infty$ and therefore, the integral is well defined (in the sense that it takes finite values).

To see that the definition does not depend on α , consider α, β two different indices. We can assume $\beta = \alpha + h$ and consider f and g satisfying the hypothesis of the definition for both indices. Then, since $f(a+), g(b-)$ and $g(a-)$ do not depend on α , we only need to check the property for the integral:

$$\begin{aligned} & (-1)^{\alpha+h} \int_a^b D_{a+}^{\alpha+h} f_{a+}(x) D_{b-}^{1-\alpha-h} g_{b-}(x) dx \\ &= (-1)^h (-1)^\alpha \int_a^b D_{a+}^h (D_{a+}^\alpha f_{a+})(x) D_{b-}^{1-\alpha-h} g_{b-}(x) dx \\ &= (-1)^\alpha \int_a^b D_{a+}^\alpha f_{a+}(x) D_{b-}^h (D_{b-}^{1-\alpha-h} g_{b-})(x) dx \\ &= (-1)^\alpha \int_a^b D_{a+}^\alpha f_{a+}(x) D_{b-}^{1-\alpha} g_{b-}(x) dx, \end{aligned}$$

as desired. $\square$

This definition can be simplified in some cases, as it can be seen in the following proposition.

Proposition 4.7. *If we consider the hypothesis of the definition 4.5 and we assume $\alpha p < 1$, we got $f \in I_{a+}^\alpha(L^p)$ and*

$$\int_a^b f dg = (-1)^\alpha \int_a^b D_{a+}^\alpha f(x) D_{b-}^{1-\alpha} g_{b-}(x) dx.$$

Proof. Since we are in the hypothesis of the definition, $f_{a+} \in I_{a+}^\alpha(L^p)$, so if we want to prove that $f \in I_{a+}^\alpha(L^p)$, it is enough to see that $f(a+) \in I_{a+}^\alpha(L^p)$ because $f_{a+}(x) + f(a+) = f(x)$. If we define

$$\varphi_{f(a+)}(x) = \frac{f(a+)}{\Gamma(1-\alpha)(x-a)^\alpha},$$

one can check that $\varphi_{f(a+)} \in L^p(a, b)$ (because $\alpha p < 1$ and $I_{a+}^\alpha(\varphi_{f(a+)}(x)) = f(a+)$). Now, we will define the following functions:

- We define φ_f as one of the functions in $L^p(a, b)$ such that $I_{a+}^\alpha(\varphi_f) = f$.
- We define $\varphi_{f_{a+}}$ as one of the functions in $L^p(a, b)$ such that $I_{a+}^\alpha(\varphi_{f_{a+}}) = f_{a+}$. Notice that

$$I_{a+}^\alpha(\varphi_{f_{a+}}) = I_{a+}^\alpha(\varphi_f) - I_{a+}^\alpha(\varphi_{f(a+)}).$$

- We define $\varphi_{g_{b-}}$ as one of the functions in $L^q(a, b)$ such that $I_{b-}^{1-\alpha}(\varphi_{g_{b-}}) = g_{b-}$.

Using that the Weyl derivatives are linear, we can write the Stieltjes integral as

$$\int_a^b f dg = (-1)^\alpha \int_a^b (D_{a+}^\alpha f(x) - D_{a+}^\alpha f(a+)(x)) D_{b-}^{1-\alpha} g_{b-}(x) dx.$$

The objective is, therefore, showing that

$$(-1)^\alpha \int_a^b D_{a+}^\alpha f(a+)(x) D_{b-}^{1-\alpha} g_{b-}(x) dx = -f(a+)(g(b-) - g(a+)).$$

Applying the inversion formulas, we got that the last expression is equal to

$$\begin{aligned} & \frac{(-1)^\alpha}{\Gamma(1-\alpha)} \int_a^b \frac{f(a+)}{(x-a)^\alpha} \varphi_{g_{b-}}(x) dx \\ &= f(a+) I_{b-}^{1-\alpha}(\varphi_{g_{b-}})(a) = f(a+) g_{b-}(a+). \end{aligned}$$

This last equation concludes the proof, because

$$g_{b-}(a+) = g(a+) - g(b-) = -(g(b-) - g(a+)).$$

$\square$

In [18] it is proved that, given $0 < \alpha < 1/p$ and $(c, d) \subset (a, b)$, the restriction of $f \in I_{a+}^\alpha(L^p)$ to the interval (c, d) belongs to $I_{c+}^\alpha(L^p(c, d))$. Reciprocally, if we got $f \in I_{c+}^\alpha(L^p)$ and we consider the extension of f as

$$\tilde{f}(x) = \begin{cases} f(x) & x \in (c, d), \\ 0 & \text{otherwise.} \end{cases}$$

we got that $\tilde{f} \in I_{a+}^\alpha(L^p(a, b))$ for all $(c, d) \subset (a, b)$. As a consequence, if $f \in I_{a+}^\alpha(L^p(a, b))$, $g \in I_{b-}^{1-\alpha}(L^q(a, b))$, $(c, d) \subset (a, b)$ then the following integrals coincide

$$\int_c^d f dg = \int_a^b \mathbf{1}_{(c,d)} f dg.$$

4.3. Fractional Sobolev spaces and useful estimates. The class of functions for which the generalized Stieltjes integral is well defined is very broad. In this section, we will introduce fractional sobolev spaces, which will be the suitable spaces of functions to study the differential equations. From now on, we will focus on the interval $[0, T]$ and $\alpha \in (0, 1/2)$. For a function $f : [0, T] \rightarrow \mathbb{R}$, we will use the following notation

$$\Delta_t^\alpha(f) = |f(t)| + \int_0^t \frac{|f(t) - f(s)|}{|t - s|^{\alpha+1}} ds.$$

Definition 4.8. We define the space $W_0^{\alpha,\infty}(0, T)$ as the space of measurable functions $f : [0, T] \rightarrow \mathbb{R}$ such that

$$\|f\|_{\alpha,\infty} := \sup_{t \in [0, T]} \Delta_t^\alpha(f) < \infty.$$

Let's recall the definition of Hölder spaces. We denote by $\mathcal{C}^\lambda(0, T)$ the space of measurable functions $f : [0, T] \rightarrow \mathbb{R}$ such that

$$\|f\|_\lambda := \|f\|_{L^\infty(0, T)} + \sup_{0 \leq s < t \leq T} \frac{|f(t) - f(s)|}{(t - s)^\lambda} < \infty.$$

We also introduce the following Sobolev spaces

Definition 4.9. We denote by $W_0^{\alpha,1}(0, T)$ the space of measurable functions $f : [0, T] \rightarrow \mathbb{R}$ such that

$$\|f\|_{\alpha,1} := \int_0^T \frac{|f(s)|}{s^\alpha} + \int_0^T \int_0^s \frac{|f(s) - f(y)|}{(s - y)^{\alpha+1}} dds < \infty.$$

From this definition, it is clear that $W_0^{\alpha,\infty}(0, T) \subset W_0^{\alpha,1}(0, T)$.

Definition 4.10. We denote by $W_T^{1-\alpha,\infty}(0, T)$ the space of measurable functions $g : [0, T] \rightarrow \mathbb{R}$ such that

$$\|g\|_{1-\alpha,\infty} := \sup_{0 \leq s < t \leq T} \left(\frac{|g(t) - g(s)|}{(t - s)^{1-\alpha}} + \int_s^t \frac{|g(y) - g(s)|}{(y - s)^{2-\alpha}} dy \right) < \infty.$$

It is important to know the relation between the fractional Sobolev spaces and the Hölder spaces. The relations are summarized in the following proposition.

Proposition 4.11. *For all $\varepsilon > 0$,*

$$\mathcal{C}^{1-\alpha+\varepsilon}(0, T) \subset W_T^{1-\alpha, \infty}(0, T) \subset \mathcal{C}^{1-\alpha}(0, T) \subset W_0^{\alpha, \infty}(0, T).$$

Proof.

- $\mathcal{C}^{1-\alpha+\varepsilon}(0, T) \subset W_T^{1-\alpha, \infty}(0, T)$: Consider $f \in \mathcal{C}^{1-\alpha+\varepsilon}(0, T)$. On the one hand,

$$\frac{|f(t) - f(s)|}{(t-s)^{1-\alpha}} = \frac{|f(t) - f(s)|}{(t-s)^{1-\alpha+\varepsilon}} (t-s)^\varepsilon \leq T^\varepsilon \sup_{0 < s < t < T} \frac{|f(t) - f(s)|}{(t-s)^{1-\alpha-\varepsilon}} < \infty.$$

This implies

$$\sup_{0 \leq s < t \leq T} \frac{|f(t) - f(s)|}{(t-s)^{1-\alpha}} < \infty.$$

On the other hand,

$$\begin{aligned} \int_s^t \frac{|f(y) - f(s)|}{(y-s)^{2-\alpha}} dy &= \int_s^t \frac{|f(y) - f(s)|}{(y-s)^{1-\alpha+\varepsilon}} (y-s)^{\varepsilon-1} dy \\ &\leq \sup_{0 \leq s < t \leq T} \frac{|f(t) - f(s)|}{(t-s)^{1-\alpha+\varepsilon}} \int_s^t (y-s)^{\varepsilon-1} dy \\ &\leq C_\varepsilon \sup_{0 \leq s < t \leq T} \frac{|f(t) - f(s)|}{(t-s)^{1-\alpha+\varepsilon}}. \end{aligned}$$

Taking supremums, we conclude that $\|f\|_{1-\alpha, \infty} < \infty$.

- $W_T^{1-\alpha, \infty}(0, T) \subset \mathcal{C}^{1-\alpha}(0, T)$ The only thing that we need to prove is that

$$\|f\|_{L^\infty(0, T)} < \infty.$$

In fact, if we see that $\sup_{0 \leq s < t \leq T} |f(t) - f(s)| < \infty$ we are done. This follows from the following inequalities

$$|f(t) - f(s)| = \frac{|f(t) - f(s)|}{(t-s)^{1-\alpha}} (t-s)^{1-\alpha} \leq T^{1-\alpha} \sup_{0 \leq s < t \leq T} \frac{|f(t) - f(s)|}{(t-s)^{1-\alpha}}.$$

Then, taking supremums we got that

$$\sup_{0 \leq s < t \leq T} |f(t) - f(s)| < \infty \Rightarrow \|f\|_{L^\infty(0, T)} < \infty.$$

- $\mathcal{C}^{1-\alpha}(0, T) \subset W_0^{\alpha, \infty}(0, T)$: Take $f \in \mathcal{C}^{1-\alpha}(0, T)$. The only thing that we need to prove is that

$$\sup_{0 \leq t \leq T} \int_0^t \frac{|f(t) - f(s)|}{(t-s)^{\alpha+1}} ds < \infty.$$

By direct estimates, we got

$$\begin{aligned} \int_0^t \frac{|f(t)| - f(s)}{(t-s)^{\alpha+1}} ds &= \int_0^t \frac{|f(t)| - f(s)}{(t-s)^{1-\alpha}} (t-s)^{-2\alpha} ds \\ &\leq \sup_{0 \leq s < t \leq T} \frac{|f(t) - f(s)|}{(t-s)^{1-\alpha}} \int_0^t (t-s)^{-2\alpha} ds \\ &\leq C_{T,\alpha} \sup_{0 \leq s < t \leq T} \frac{|f(t) - f(s)|}{(t-s)^{1-\alpha}}. \end{aligned}$$

Taking supremums, we got the desired result. $\square$

The reason to define this spaces is the following. Notice that the norm of $f \in W^{\alpha,1}(0, T)$ is greater than the $L^1(0, T)$ norm of $D_{0+}^\alpha(f)$, so $W^{\alpha,1}(0, T) \subset I_{0+}^\alpha(L^1(0, T))$. Moreover, the restriction of f to $(0, t)$ belongs to $I_{0+}^\alpha(L^1(0, t))$ for all $0 < t \leq T$. In a similar way, if $g \in W_T^{1-\alpha,\infty}(0, T)$ then $g \in I_{T-}^{1-\alpha}(L^\infty(0, T))$ and the restriction to $(0, t)$ belongs to $I_{T-}^{1-\alpha}(L^\infty(0, t))$.

This means that for any $f \in W^{\alpha,1}(0, T)$, $g \in W_T^{1-\alpha,\infty}(0, T)$, $t \in [0, T]$ we got the following equality

$$\int_0^t f dg = \int_0^T f \mathbb{1}_{(0,t)} dg.$$

Moreover,

$$\int_s^t f dg = \int_0^t f dg - \int_0^s f dg.$$

For convenience, we will define the following operators:

$$F_t(f) = \int_0^t f(s) ds,$$

$$G_t(f) = \int_0^t f(s) dg(s) := \int_0^t D_{0+}^\alpha f(s) D_{t-}^{1-\alpha} g_{t-}(s) ds.$$

In order to prove the existence and uniqueness of solutions to differential equations, we will use a fixed point argument, which is a standard praxis. Hence, it is going to be very convenient to have good estimates of $F_t(f)$ and $G_t(f)$ in terms of some norm of f .

Proposition 4.12. *Let $f \in W_0^{\alpha,\infty}(0, T)$. The following properties hold:*

(1) *The function $G(f)$ belongs to $\mathcal{C}^{1-\alpha}(0, T)$ and we have*

$$\Delta_t^\alpha(G(f)) \leq c_{\alpha,T} \|g\|_{1-\alpha,\infty} \int_0^t ((t-r)^{-2\alpha} + t^{-\alpha}) \Delta_r^\alpha(f) dr,$$

$$\|G(f)\|_{1-\alpha} \leq c_{\alpha,T} \|g\|_{1-\alpha,\infty} \|f\|_{\alpha,\infty}.$$

(2) The function $F(f)$ belongs to $\mathcal{C}^1(0, T)$ and, moreover,

$$\begin{aligned}\Delta_t^\alpha(F(f)) &\leq c_{\alpha, T} \int_0^t \frac{|f(s)|}{(t-s)^\alpha} ds, \\ \|F(f)\|_{\alpha, \infty} &\leq C_{\alpha, T} \|f\|_{L^\infty(0, T)}.\end{aligned}$$

Proof.

(1) Let's define the quantity

$$\Lambda_\alpha(g) := \frac{1}{\Gamma(1-\alpha)} \sup_{0 \leq s < t \leq T} |D_{t-}^{1-\alpha} g_t(s)| \leq \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \|g\|_{1-\alpha, \infty}.$$

We got

$$\begin{aligned}|G_t(f) - G_s(f)| &= \left| \int_s^t f dg \right| \\ &= \left| \int_s^t D_{s+}^\alpha f(r) D_{t-}^{1-\alpha} g_t(r) dr \right| \\ &\leq \Lambda_\alpha(g) \left(\int_s^t \frac{|f(r)|}{r^\alpha} dr + \alpha \int_s^t \int_s^r \frac{|f(y) - f(r)|}{(y-r)^{\alpha+1}} ddr \right).\end{aligned}$$

In particular,

$$|G(f)| \leq \Lambda_\alpha(g) \|f\|_{\alpha, 1} \leq c_{\alpha, T}^{(1)} \|g\|_{1-\alpha, \infty} \|f\|_{\alpha, 1}.$$

On the other hand,

$$\begin{aligned}&\int_0^t \frac{|G_t(f) - G_s(f)|}{(t-s)^{\alpha+1}} \\ &\leq \Lambda_\alpha(g) \int_0^t (t-s)^{-\alpha-1} \left(\int_s^t \frac{|f(r)|}{r^\alpha} dr + \int_s^t \int_s^r \frac{|f(y) - f(r)|}{(r-y)^{\alpha+1}} ddr \right) ds \\ &\leq c_\alpha \Lambda_\alpha(g) \left[\int_0^t \frac{|f(r)|}{(t-r)^{2\alpha}} dr + \int_0^t \int_0^r \frac{|f(r) - f(y)|}{(r-y)^{\alpha+1}} (t-y)^{-\alpha} ddr \right] \\ &\leq c_{\alpha, T}^{(2)} \|g\|_{1-\alpha, \infty} \left(\int_0^t ((t-r)^{-2\alpha} + t^{-\alpha}) \Delta_r^\alpha(f) dr \right).\end{aligned}$$

Putting together both estimates and taking supremums we get the desired inequality. The only thing left to see is that $G(f) \in \mathcal{C}^{1-\alpha}(0, T)$. From the previous inequalities we got that

$$\begin{aligned}\left| \int_s^t f dg \right| &\leq \Lambda_\alpha(g) \int_s^t \frac{1}{r^\alpha} \left(|f(r)| + \alpha r^\alpha \int_s^r \frac{|f(r) - f(y)|}{(r-y)^{\alpha+1}} dy \right) \\ &\leq \Lambda_\alpha(g) \|f\|_{\alpha, \infty} \frac{(t-s)^{1-\alpha}}{1-\alpha} (1 \vee \alpha T^\alpha).\end{aligned}$$

this implies that $G(f)$ is Hölder continuous with parameter $1-\alpha$ and it is also bounded, so $G(f) \in \mathcal{C}^{1-\alpha}(0, T)$.

- (2) The fact that $F(f) \in \mathcal{C}^1(0, T)$ follows from the fundamental theorem of calculus. Now, doing some computations we deduce

$$|F_t(f)| \leq \int_0^t |f(s)| ds \leq T \|f\|_{L^\infty(0, T)},$$

and

$$\int_0^t \frac{|F_t(f) - F_s(f)|}{(t-s)^{\alpha+1}} ds \leq \int_0^t (t-s)^{-\alpha-1} \left(\int_s^t |f(r)| dr \right) ds.$$

By changing the limits of integration and integrating first with respect to s we have

$$\int_0^t (t-s)^{-\alpha-1} \left(\int_s^t |f(r)| dr \right) ds \leq c_{\alpha, T} \int_0^t \frac{|f(r)|}{(t-r)^\alpha} dr.$$

Joining this estimate with the bound of $|F_t(f)|$ we get the first inequality. For the second one,

$$\sup_{0 \leq t \leq T} \Delta_t^\alpha(F(f)) \leq \sup_{0 \leq t \leq T} c_{\alpha, T} \|f\|_{L^\infty(0, T)} \int_0^t (t-s)^{-\alpha} ds \leq C_{\alpha, T} \|f\|_{L^\infty(0, T)}.$$

□

We have the stochastic differential equations in mind, so it's time to add hypothesis on the coefficients of the equations. Consider the following hypothesis on the functions $\sigma, b : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$:

(H_1): $\sigma(t, x)$ is differentiable in x and there exist some constants $0 < \beta, \delta \leq 1$ and for every $N \geq 0$ there exists M_N satisfying

$$(H_\sigma) : \begin{cases} |\sigma(t, x) - \sigma(t, y)| \leq M_0 |x - y|, \\ |\partial_x \sigma(t, x) - \partial_x \sigma(t, y)| \leq M_N |x - y|^\delta \quad |x|, |y| \leq N, \forall t \in [0, T], \\ |\sigma(t, x) - \sigma(s, x)| + |\partial_x \sigma(t, x) - \partial_x \sigma(s, x)| \leq M_0 |t - s|^\beta \quad \forall x \in \mathbb{R}, \forall t, s \in [0, T]. \end{cases}$$

(H_2): There exists $b_0 \in L^\rho(0, T)$, $\rho \geq 2$, and for every $N \geq 0$ there exists $L_N > 0$ such that

$$(H_b) : \begin{cases} |b(t, x) - b(t, y)| \leq L_N |x - y|, \quad |x|, |y| \leq N, \forall t \in [0, T], \\ |b(t, x)| \leq L_0 |x| + b_0(t) \quad \forall x \in \mathbb{R}, \forall t \in [0, T]. \end{cases}$$

Definition 4.13. Let $\sigma, b : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying (H_1) and (H_2) respectively. Let $g \in W_T^{1-\alpha, \infty}(0, T)$ and let $f \in W_0^{\alpha, \infty}(0, T)$. We denote by G^σ and F^b de functionals

$$G_t^\sigma(f) = \int_0^t \sigma(s, f(s)) dg(s),$$

$$F_t^b(f) = \int_0^t b(s, f(s)) ds.$$

As we did for $G(f)$ and $F(f)$, we would like to find estimates for this new operators.

Proposition 4.14. *Let $0 < \alpha < 1/2$, $f \in W_0^{\alpha,\infty}(0, T)$. Let $\sigma, b : [0, T] \times \mathbb{R}$ be functions satisfying hypothesis (H_1) and (H_2) respectively with $\beta > \alpha$ and $\rho = 1/\alpha$. Then, $G^\sigma(f)$ and $F^b(f)$ belong to $\mathcal{C}^{1-\alpha}(0, T)$. Moreover, we get the following estimates*

$$(i) \quad \|G^\sigma(f)\|_{1-\alpha} \leq \Lambda_\alpha(g)C(1 + \|f\|_{\alpha,\infty}).$$

$$(ii) \quad \|G^\sigma(f)\|_{\alpha,\lambda} \leq \frac{C\Lambda_\alpha(g)}{\lambda^{1-2\alpha}}(1 + \|f\|_{\alpha,\lambda}),$$

where

$$\|f\|_{\alpha,\lambda} = \sup_{t \in [0, T]} e^{-\lambda t} \Delta_t^\alpha(f).$$

(iii) *If $h \in W_0^{\alpha,\infty}(0, T)$ such that $\|f\|_{L^\infty(0, T)}, \|h\|_{L^\infty(0, T)} \leq N$, then there exists a constant C_N depending on N, f, g, h such that $\forall \lambda \geq 1$*

$$\|G_t^\sigma(f) - G_t^\sigma(h)\|_{\alpha,\lambda} \leq \frac{\Lambda_\alpha(g)C_N}{\lambda^{1-2\alpha}}(1 + \Delta(f) + \Delta(h))\|f - h\|_{\alpha,\lambda},$$

where

$$\Delta(f) = \sup_{r \in [0, T]} \int_0^r \frac{|f(r) - f(s)|^\delta}{(r - s)^{\alpha+1}} ds.$$

$$(iv) \quad \|F^b(f)\|_{1-\alpha} \leq c(1 + \|f\|_{L^\infty(0, T)}).$$

$$(v) \quad \|F^b(f)\|_{\alpha,\lambda} \leq \frac{c}{\lambda^{1-2\alpha}}(1 + \|f\|_{\alpha,\lambda}).$$

(vi) *If $h \in W_0^{\alpha,\infty}(0, T)$ is such that $\|f\|_{L^\infty(0, T)}, \|h\|_{L^\infty(0, T)} \leq N$, then there exists a constant C_N depending on α, T, L_N such that $\forall \lambda \geq 1$,*

$$\|F^b(f) - F^b(h)\|_{\alpha,\lambda} \leq \frac{C_N}{\lambda^{1-\alpha}}\|f - h\|_{\alpha,\lambda}.$$

Before proving the proposition, we need an auxiliary result.

Lemma 4.15. *Let $\sigma : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying (H_1) . then, for all $N > 0$ and $|x_1|, |x_2|, |x_3|, |x_4| \leq N$ we have*

$$\begin{aligned} & |\sigma(t_1, x_1) - \sigma(t_2, x_2) - \sigma(t_1, x_3) + \sigma(t_2, x_4)| \\ & \leq M_0|x_1 - x_2 - x_3 + x_4| + M_0|x_1 - x_3||t_2 - t_1|^\beta \\ & + M_N|x_1 - x_3|(|x_1 - x_2|^\delta + |x_3 - x_4|^\delta). \end{aligned}$$

Proof. Applying the mean value theorem we get

$$\begin{aligned} & \sigma(t_1, x_1) - \sigma(t_2, x_2) - \sigma(t_1, x_3) + \sigma(t_2, x_4) \\ & = \int_0^1 (x_1 - x_3) \partial_x \sigma(t_1, \xi) d\xi - \int_0^1 (x_2 - x_4) \partial_x \sigma(t_2, \xi) d\xi \\ & = \int_0^1 (x_1 - x_2 - x_3 + x_4) \partial_x \sigma(t_2, \xi) d\xi + \int_0^1 (x_1 - x_3) (\partial_x \sigma(t_1, \xi) - \partial_x \sigma(t_2, \xi)) d\xi. \end{aligned}$$

Bounding using (H_1) we get the desired estimate. $\square$

Now, we are ready to prove Proposition 4.14.

Proof. Let $f \in W_0^{\alpha,\infty}(0, T)$. First, notice that from the properties of σ we got that if $f \in W_0^{\alpha,\infty}(0, T)$ then $\sigma(\cdot, f(\cdot)) \in W_0^{\alpha,\infty}(0, T)$. Indeed $\sigma(t, f(t))$ is clearly bounded and

$$\int_0^t \frac{|\sigma(t, f(t)) - \sigma(s, f(s))|}{(t-s)^{\alpha+1}} \leq M_0 \int_0^t (t-s)^{\beta-\alpha-1} ds < \infty$$

because $\beta > \alpha$. Taking supremums we got that the last integral is bounded by a constant depending on T, α and β . By reproducing the estimates of last proposition, we get that $G^\sigma(f) \in \mathcal{C}^{1-\alpha}(0, T)$. Doing a similar procedure for $F^b(f)$ we got that $F^b(f) \in \mathcal{C}^{1-\alpha}(0, T)$. Let's prove all the items of the proposition.

- (i) Is direct because of the bounds of σ .
- (ii) Using the definition of $\|\cdot\|_{\alpha,\lambda}$ and the estimate of $\|G(f)\|_{\alpha,\lambda}$ given by proposition 4.12, we have

$$\begin{aligned} & \|G^\sigma(f)\|_{\alpha,\lambda} \\ & \leq c_\alpha \Lambda_\alpha(g) \sup_{t \in [0, T]} e^{-\lambda t} \int_0^t (t-r)^{-2\alpha} + r^{-\alpha} \left(|\sigma(r, f_r)| + \int_0^r \frac{|\sigma(r, f_r) - \sigma(s, f_s)|}{(r-s)^{\alpha+1}} ds \right) dr \\ & \lesssim c_\alpha \Lambda_\alpha(g) \sup_{t \in [0, T]} e^{-\lambda t} \int_0^t (t-r)^{-2\alpha} + r^{-\alpha} \left(1 + |f_r| + \int_0^r \frac{|f_r - f_s|}{(r-s)^{\alpha+1}} ds \right) dr. \end{aligned}$$

Now, we got

$$e^{-\lambda t} \int_0^t (t-r)^{-2\alpha} + r^{-\alpha} dr = \int_0^t e^{-\lambda r} e^{-\lambda(t-r)} ((t-r)^{-2\alpha} + r^{-\alpha}) dr.$$

This leads to

$$\begin{aligned} & c_\alpha \Lambda_\alpha(g) \sup_{t \in [0, T]} e^{-\lambda t} \int_0^t (t-r)^{-2\alpha} + r^{-\alpha} \left(1 + |f_r| + \int_0^r \frac{|f_r - f_s|}{(r-s)^{\alpha+1}} ds \right) dr \\ & = c_\alpha \Lambda_\alpha(g) \sup_{t \in [0, T]} \left(\int_0^t e^{-\lambda r} e^{-\lambda(t-r)} (r^{-2\alpha} + (t-r)^{-\alpha}) \left(1 + |f_r| + \int_0^r \frac{|f_r - f_s|}{(r-s)^{\alpha+1}} ds \right) dr \right) \\ & \leq c_\alpha \Lambda_\alpha(g) (1 + \|f\|_{\alpha,\lambda}) \sup_{t \in [0, T]} \int_0^t e^{-\lambda(t-r)} (r^{-2\alpha} + (t-r)^{-\alpha}) dr. \end{aligned}$$

Observe that

$$\begin{aligned} & \int_0^t e^{-\lambda(t-r)} ((t-r)^{-2\alpha} + r^{-\alpha}) dr = \int_0^t e^{-\lambda r} (r^{-2\alpha} + (t-r)^{-\alpha}) dr \\ & = \frac{1}{\lambda} \int_0^{\lambda t} e^{-r} (\lambda^{2\alpha} r^{-2\alpha} + \lambda^\alpha (\lambda t - r)^{-\alpha}) dr \\ & \leq \lambda^{2\alpha-1} \left(\int_0^\infty e^{-r} r^{-2\alpha} dr + \sup_{z>0} \int_0^z e^{-r} (z-r)^{-\alpha} dr \right) = c_\alpha \lambda^{2\alpha-1}. \end{aligned}$$

So, if we write all together, we end up with

$$\|G^\sigma(f)\|_{\alpha,\lambda} \leq C\Lambda_\alpha(g)\lambda^{2\alpha-1}(1 + \|f\|_{\alpha,\lambda}).$$

(iii)

$$\begin{aligned} \|G^\sigma(f) - G^\sigma(h)\|_{\alpha,\lambda} &\leq \Lambda_\alpha(g)c_{\alpha,T} \sup_{t \in [0,T]} e^{-\lambda t} \int_0^t ((t-r)^{-2\alpha} + r^{-\alpha}) \times \\ &\times \left(|\sigma(r, f(r)) - \sigma(r, h(r))| + \right. \\ &\left. + \int_0^r \frac{|\sigma(r, f(r)) - \sigma(s, f(s)) - \sigma(r, h(r)) + \sigma(s, h(s))|}{|r-s|^{\alpha+1}} ds \right) dr. \end{aligned}$$

Applying the lemma 4.15, we can show

$$\begin{aligned} &|\sigma(r, f(r)) - \sigma(s, f(s)) - \sigma(r, h(r)) - \sigma(s, h(s))| \\ &\leq M_0|f(r) - f(s) - h(r) + h(s)| + M_0|f(r) - f(s)|(r-s)^\beta \\ &+ M_N|f(r) - h(r)|(|f(r) - f(s)|^\delta + |h(r) - h(s)|^\delta). \end{aligned}$$

Therefore,

$$\begin{aligned} &\int_0^r \frac{|\sigma(r, f(r)) - \sigma(s, f(s)) - \sigma(r, h(r)) + \sigma(s, h(s))|}{|r-s|^{\alpha+1}} ds \\ &\leq M_0 \int_0^r \frac{|f(r) - f(s) - h(r) + h(s)|}{(r-s)^{\alpha+1}} + \frac{M_0}{\beta-\alpha} |f(r) - h(r)| r^{\beta-\alpha} \\ &+ M_N |f(r) - h(r)| (\Delta(f) + \Delta(h)). \end{aligned}$$

Joining everything, we deduce

$$\|G^\sigma(f) - G^\sigma(h)\|_{\alpha,\lambda} \leq \frac{\Lambda_\alpha(g)C_N}{\lambda^{1-2\alpha}} (1 + \Delta(f) + \Delta(h)) \|f - h\|_{\alpha,\lambda}.$$

(iv) We got

$$\begin{aligned} |F_t^b(f) - F_s^b(f)| &\leq \int_s^t |b(r, f(r))| dr \leq \int_s^t L_0 |f(r)| + b_0(r) dr \\ &\leq (t-s)L_0 \|f\|_{L^\infty(0,T)} + (t-s)^{1-\alpha} \|b_0\|_{L^{1/\alpha}(0,T)} \\ &\leq (T^\alpha L_0 \|f\|_{L^\infty(0,T)} + \|b_0\|_{L^{1/\alpha}(0,T)}) (t-s)^{1-\alpha}, \end{aligned}$$

and

$$\|F^b(f)\|_{L^\infty(0,T)} \leq L_0 \|f\|_{L^\infty(0,T)} + T^{1-\alpha} \|b_0\|_{L^{1/\alpha}(0,T)}.$$

This proves that there exist a constant c depending on T and α such that

$$\|F^b(f)\|_{1-\alpha} \leq c(1 + \|f\|_{L^\infty(0,T)}).$$

(v) Because of proposition 4.12 we got

$$\begin{aligned}
& |F_t^b(f)| + \int_0^t \frac{|F_t^b(f) - F_s^b(f)|}{(t-s)^{\alpha+1}} ds \leq c_{\alpha,T} \int_0^t \frac{|b(s, f(s))|}{(t-s)^{\alpha+1}} ds \\
& \leq c_{\alpha,T} \int_0^t \frac{L_0 |f(s)| + b_0(s)}{(t-s)^\alpha} ds \\
& \leq c_{\alpha,T} \left(L_0 \int_0^t \frac{|f(s)|}{(t-s)^\alpha} ds + \left(\int_0^t (t-s)^{-\alpha/(1-\alpha)} ds \right)^\alpha \|b_0\|_{L^{1/\alpha}(0,T)} \right) \\
& \leq c_{\alpha,T} L_0 \int_0^t \frac{|f(s)|}{(t-s)^\alpha} ds + c_{\alpha,T} \left(\frac{1-\alpha}{1-2\alpha} \right)^\alpha \|b_0\|_{L^{1/\alpha}(0,T)} t^{1-2\alpha}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|F^b(f)\|_{\alpha,\lambda} & \leq c_{\alpha,T} L_0 \sup_{t \in [0,T]} e^{-\lambda t} \int_0^t \frac{|f(s)|}{(t-s)^\alpha} ds \\
& \quad + c_{\alpha,T} \left(\frac{1-\alpha}{1-2\alpha} \right)^\alpha \|b_0\|_{L^{1/\alpha}(0,T)} \sup_{t \in [0,T]} e^{-\lambda t} t^{1-2\alpha} \\
& \leq c_{\alpha,T} L_0 \Gamma(1-\alpha) \sup_{s \in [0,T]} e^{-\lambda s} |f(s)| + \lambda^{2\alpha-1} C_{\alpha,T,b_0}.
\end{aligned}$$

Now, $\lambda^{\alpha-1} \leq \lambda^{2\alpha-1}$ because $\lambda \geq 1$, so we can take common factor of the last estimate to get

$$\|F^b(f)\|_{\alpha,\lambda} \leq c_{\alpha,T,b} \lambda^{2\alpha-1} (1 + \|f\|_{\alpha,\lambda}).$$

(vi) Again, by proposition 4.12 we are lead to

$$\|F^b(f) - F^b(h)\|_{\alpha,\lambda} \leq c_{\alpha,T} \sup_{t \in [0,T]} e^{-\lambda t} \int_0^t \frac{|b(s, f(s)) - b(s, h(s))|}{(t-s)^\alpha} ds.$$

By the Lipschitz property of b we can easily deduce an estimate:

$$\begin{aligned}
& c_{\alpha,T} \sup_{t \in [0,T]} e^{-\lambda t} \int_0^t \frac{|b(s, f(s)) - b(s, h(s))|}{(t-s)^\alpha} ds \\
& \leq c_{\alpha,T} L_N \sup_{t \in [0,T]} \int_0^t \frac{|f(s) - h(s)|}{(t-s)^\alpha} ds \\
& \leq \lambda^{\alpha-1} c_{\alpha,T} L_N \Gamma(1-\alpha) \|f - h\|_{\alpha,\lambda}.
\end{aligned}$$

□

5. STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY A FRACTIONAL BROWNIAN MOTION

The study of the differential equations driven by a fractional Brownian motion that is presented in this project follows closely the same lines as [16], that is, we will first establish an existence and uniqueness theorem for deterministic equation and we will translate the result to the stochastic case.

5.1. Deterministic differential equations. The study of the generalized Riemann-Stieltjes integral allow us to study (deterministic) differential equations driven by Hölder continuous functions. We have in mind studying differential equations with respect to the fractional Brownian motion, so thanks to the fact that the sample paths of the fractional Brownian motion with Hurst parameter $H > 1/2$ are Hölder continuous of order $H - \varepsilon$ for all $\varepsilon > 0$, a result of existence and uniqueness of solutions to deterministic equations will immediately give us a result of existence and uniqueness of equations driven by a fractional Brownian motion. From now on, we will use the notation $f_s = f(s)$ for any deterministic function f . The setup is the following:

Let $0 < \alpha < 1/2$ and let $g \in W_T^{1-\alpha, \infty}(0, T)$. We want to study the following differential equation:

$$(5.1) \quad x_t = x_0 + \int_0^t \sigma(s, x_s) dg_s + \int_0^t b(s, x_s) ds, \quad t \in [0, T].$$

As mentioned, we want to study the existence and uniqueness of solutions to this equations. In order to accomplish our objective, we assume that σ and b satisfy (H_1) and (H_2) respectively, with $\rho = 1/\alpha$, $0 < \beta, \delta \leq 1$. Moreover, we will restrict the parameter α to

$$0 < \alpha < \alpha_0 = \frac{1}{2} \wedge \beta \wedge \frac{\delta}{1 + \delta}.$$

The general strategy to prove existence and uniqueness of solutions to differential equations goes through a fixed point argument. The difficulty resides on how to approach in each case the fixed point problem. In our case, we will make use of the following result:

Lemma 5.1. *Let (X, d) be a metric space and d_0, d_1, d_2 some metrics in X equivalent to d . If $\mathcal{L} : X \rightarrow X$ satisfies:*

- (i) *There exists $r_0 > 0$, $x_0 \in X$ such that if $B_0 = \{x \in X; d_0(x_0, x) \leq r_0\}$ then*

$$\mathcal{L}(B_0) \subset B_0.$$
- (ii) *There exists $\varphi : (X, d) \rightarrow [0, \infty]$ lower semicontinuous function and some positive constants C_0, K_0 such that denoting $N_\varphi(a) = \{x \in X; \varphi(x) \leq a\}$*
 - (a) $\mathcal{L}(B_0) \subset N_\varphi(C_0)$.
 - (b) $d_1(\mathcal{L}(x), \mathcal{L}(y)) \leq K_0 d_1(x, y), \forall x, y \in N_\varphi(C_0) \cap B_0$,
- (iii) *There exists $a \in (0, 1)$ such that*

$$d_2(\mathcal{L}(x), \mathcal{L}(y)) \leq a d_2(x, y), \quad \forall x, y \in \mathcal{L}(B_0).$$

Then there exists x^* such that $\mathcal{L}(x^*) = x^*$.

Proof. Let's consider the iterative sequence $\{x_n; n \geq 1\}$ defined by

$$x_{n+1} = \mathcal{L}(x_n), \quad n \in \mathbb{N}.$$

If we start with $x_0 \in B_0$, then $x_n \in \mathcal{L}(B_0)$ for all $n \geq 0$. Moreover, since $\mathcal{L}(B_0) \subset N_\varphi(C_0)$ we also got that $x_n \in N_\varphi(C_0)$. By definition of the sequence and item (iii) we

got

$$d_2(x_{n+1}, x_n) \leq a d_2(x_n, x_{n+1}) \leq \cdots \leq a^n d_2(x_1, x_0).$$

This implies

$$d_2(x_{n+p}, x_n) \leq \frac{a^n}{1-a} d_2(x_1, x_2) \rightarrow 0, \quad n \rightarrow \infty.$$

Since X is complete and B_0 is closed, $x_n \rightarrow x^*$ in all metrics d_i because they are all equivalent by assumption. Moreover, using that φ is lower semicontinuous, we got $\varphi(x^*) \leq C_0$ by taking limits. This implies, by item (ii) that

$$d_1(\mathcal{L}(x_n), \mathcal{L}(x^*)) \leq K_0 d_1(x_n, x^*) \rightarrow 0, \quad n \rightarrow \infty,$$

meaning that $\mathcal{L}(x^*) = x^*$. □

Now we are ready to state and prove the existence and uniqueness result.

Theorem 5.2. *Equation(5.1) has a unique solution $x \in W_0^{\alpha, \infty}(0, T)$. Moreover, this solution is $(1 - \alpha)$ -Hölder continuous.*

Proof.

Existence: We will use the previous lemma to prove it. Consider the operator $\mathcal{L} : W_0^{\alpha, \infty}(0, T) \rightarrow W_0^{\alpha, \infty}(0, T)$ given by

$$\begin{aligned} x = \mathcal{L}(u) &= x_0 + \int_0^\cdot \sigma(s, u_s) dg_s + \int_0^\cdot b(s, u_s) ds \\ &=: x_0 + G^\sigma(u) + F^b(u). \end{aligned}$$

Observe that $\mathcal{L}(W_0^{\alpha, \infty}(0, T)) \subset \mathcal{C}^{1-\alpha}(0, T)$. Moreover, since $\mathcal{C}^{1-\alpha}(0, T) \subset W_0^{\alpha, \infty}(0, T)$ we can think, without loss of generality, that the operator $\mathcal{L}$ is defined from $W_0^{\alpha, \infty}(0, T)$ to itself and has $\mathcal{C}^{1-\alpha}(0, T)$ as an invariant subspace.

We have

$$\|\mathcal{L}(u)\|_{\alpha, \lambda} \leq |x_0| + \|G^\sigma(u)\|_{\alpha, \lambda} + \|F^b(u)\|_{\alpha, \lambda}.$$

By proposition 4.14,

$$\begin{aligned} \|\mathcal{L}(u)\|_{\alpha, \lambda} &\leq |x_0| + C\Lambda_\alpha(g)\lambda^{2\alpha-1}(1 + \|u\|_{\alpha, \lambda}) + c_{\alpha, T, b}\lambda^{2\alpha-1}(1 + \|u\|_{\alpha, \lambda}) \\ &\leq |x_0| + (1 + \|u\|_{\alpha, \lambda}) \frac{C}{\lambda^{1-2\alpha}}. \end{aligned}$$

If we let λ big enough such that $C\lambda^{2\alpha-1} \leq 1/2$ then, if $\|u\|_{\alpha, \lambda} \leq 2|x_0| + 1$ implies $\|\mathcal{L}(u)\|_{\alpha, \lambda}$. Hence, if we define

$$B_0 = \{u \in W_0^{\alpha, \infty}(0, T) \text{ such that } \|u\|_{\alpha, \lambda} \leq 2|x_0| + 1\},$$

then $\mathcal{L}(B_0) \subset B_0$ and the first hypothesis of the fixed point lemma is satisfied (we used the metric given by the norm $\|\cdot\|_{\alpha, \lambda}$). In order to check the second hypothesis of the fixed point lemma, we need a function φ lower semicontinuous such that $\mathcal{L}(B_0) \subset N_\varphi(C_0)$. We have

$$\|\mathcal{L}(u)\|_{1-\alpha} \leq |x_0| + c(1 + \|u\|_{L^\infty(0, T)}) + \Lambda_\alpha(g)C(1 + \|u\|_{\alpha, \infty}).$$

Since

$$\|u\|_{\alpha,\infty} \leq 2(1 + |x_0|)e^{\lambda T}$$

and $\|u\|_{L^\infty(0,T)} \leq \|u\|_{\alpha,\infty}$, we have

$$\|\mathcal{L}(u)\|_{1-\alpha} \leq K$$

for some constant K . With this,

$$\begin{aligned} \Delta(\mathcal{L}(u)) &= \sup_{r \in [0,T]} \int_0^r \frac{|\mathcal{L}(u)_r - \mathcal{L}(u)_s|^\delta}{(r-s)^{\alpha+1}} ds \\ &\leq \sup_{r \in [0,T]} \int_0^r \frac{K(t-s)^{(1-\alpha)\delta}}{(r-s)^{\alpha+1}} ds \\ &\leq K \frac{T^{\delta-\alpha(1+\delta)}}{\delta-\alpha(1+\delta)} \\ &= \tilde{K}. \end{aligned}$$

We also have

$$\|\mathcal{L}(u) - \mathcal{L}(v)\|_{\alpha,\lambda} \leq \|G^\sigma(u) - G^\sigma(v)\|_{\alpha,\lambda} + \|F^b(u) - F^b(v)\|_{\alpha,\lambda}.$$

Since $1 + \Delta(u) + \Delta(v) \geq 1$ and $\lambda^{\alpha-1} \leq \lambda^{2\alpha-1}$ because $\lambda \geq 1$, one has

$$\|\mathcal{L}(u) - \mathcal{L}(v)\|_{\alpha,\lambda} \leq \frac{C_1}{\lambda^{1-2\alpha}} (1 + \Delta(u) + \Delta(v)) \|u - v\|_{\alpha,\lambda}.$$

Now, consider $\varphi(u) = C_1(1/2 + \Delta(u))$. We claim that φ is lower semicontinuous. To see it, it is enough to see that Δ is lower semicontinuous, and this is direct from the fact that for every $u \in W_0^{\alpha,\infty}(0, T)$ and for every $\{u_n; n \geq 1\} \rightarrow u$ in $W_0^{\alpha,\infty}(0, T)$ then the convergence also holds uniformly and by Fatou's lemma, proving that $\liminf_n \Delta(u_n) \geq \Delta(u)$ is immediate.

Define $C_0 = C_1(1/2 + \tilde{K})$. By previous computations, $\varphi(\mathcal{L}(u)) \in N_\varphi(C_0)$. Moreover,

$$1 + \Delta(u) + \Delta(v) = \varphi(u) + \varphi(v) = \frac{\varphi(u) + \varphi(v)}{C_1},$$

so

$$\|\mathcal{L}(u) - \mathcal{L}(v)\|_{\alpha,\lambda} \leq \frac{2C_0}{\lambda^{1-2\alpha}} \|u - v\|_{\alpha,\lambda}.$$

In particular, for $\lambda = 1$,

$$\|\mathcal{L}(u) - \mathcal{L}(v)\|_{\alpha,\lambda=1} \leq 2C_0 \|u - v\|_{\alpha,\lambda=1},$$

so the second hypothesis of the fixed point lemma holds for the metric given by $\|\cdot\|_{\alpha,\lambda=1}$. We can choose also λ' so that

$$\frac{2C_0}{\lambda'^{1-2\alpha}} \leq \frac{1}{2}$$

and hence,

$$\|\mathcal{L}(u) - \mathcal{L}(v)\|_{\alpha,\lambda'} \leq \frac{1}{2} \|u - v\|_{\alpha,\lambda'}.$$

so the third hypothesis of the fixed point lemma holds for the metric given by $\|\cdot\|_{\alpha,\lambda'}$. In conclusion, there exists a function $x \in W_0^{\alpha,\infty}(0,T)$ such that

$$x_t = x_0 + \int_0^t \sigma(s, x_s) dg_s + \int_0^t b(s, x_s) ds.$$

Uniqueness: Let x, y be solutions, and let $N = \max\{\|x\|_{1-\alpha}, \|y\|_{1-\alpha}\}$. We have $\|x\|_{L^\infty(0,T)}, \|y\|_{L^\infty(0,T)} \leq N$. Moreover

$$\|x - y\|_{\alpha,\lambda} \leq \|G^\sigma(x) - G^\sigma(y)\|_{\alpha,\lambda} + \|F^b(x) - F^b(y)\|_{\alpha,\lambda}.$$

We can bound the last quantity by

$$\|x - y\|_{\alpha,\lambda} \leq \left(\frac{\Lambda_\alpha(g)C_N}{\lambda^{1-2\alpha}}(1 + \Delta(x) + \Delta(y)) + \frac{C_N}{\lambda^{1-\alpha}} \right) \|x - y\|_{\alpha,\lambda}.$$

In the proof of the existence, we have seen that we can bound $\Delta(x)$ and $\Delta(y)$ by constants that do not depend on x nor y , so making $\lambda \rightarrow \infty$ we deduce that $\|x - y\|_{\alpha,\lambda} = 0$ and, hence, the solution is unique. $\square$

Even though this is a good result because it gives us the existence and uniqueness of the equation (5.1), we can't directly extract information about the behaviour of the solution as it evolves with time. In order to get a notion about the growth, we shall include an extra hypothesis on σ :

(H_3) : There exists $\gamma \in [0, 1]$ and $K_0 > 0$ such that

$$|\sigma(t, x)| \leq K_0(1 + |x|^\gamma), \quad \forall x \in \mathbb{R}, t \in [0, T].$$

To do so, we need the following Gronwall-type lemma.

Lemma 5.3. *Fix $0 \leq \alpha < 1$, $a, b \geq 0$. Let $x : [0, \infty) \rightarrow [0, \infty)$ be a continuous function such that for all t ,*

$$x_t \leq a + bt^\alpha \int_0^t (t-s)^{-\alpha} s^{-\alpha} x_s ds.$$

Then,

$$x_t \leq a\Gamma(1-\alpha) \sum_{n=0}^{\infty} \frac{(b\Gamma(1-\alpha)t^{1-\alpha})^n}{\Gamma((n+1)(1-\alpha))} \leq ad_\alpha \exp\{c_\alpha tb^{1/(1-\alpha)}\}.$$

Proposition 5.4. *Suppose that σ and b satisfy the hypothesis (H_1) and (H_2) respectively. Suppose also the growth condition (H_3) on σ . Then, the unique solution of (5.1) satisfies*

$$\|x\|_{\alpha,\infty} \leq C_1 \exp\{C_2 \Lambda_\alpha(g)^\kappa\},$$

where

$$\kappa = \begin{cases} \frac{1}{1-2\alpha}, & \gamma = 1, \\ > \frac{\gamma}{1-2\alpha}, & \frac{1-2\alpha}{1-\alpha} \leq \gamma < 1, \\ \frac{1}{1-\alpha}, & 0 \leq \gamma < \frac{1-2\alpha}{1-\alpha}. \end{cases}$$

Proof. Since x is solution, we can write it as $x_t = x_0 + G_t^\sigma(x) + F_t^b(x)$. From the proof of proposition 4.12 with $s = 0$ we have

$$\begin{aligned}
& |G_t^\sigma(x)| \\
& \leq \Lambda_\alpha(g) \left(\int_0^t \frac{|\sigma(r, x_r)|}{r^\alpha} + \alpha \int_0^t \int_0^r \frac{|\sigma(r, x_r) - \sigma(s, x_s)|}{(r-s)^{\alpha+1}} ds dr \right) \\
& \leq \Lambda_\alpha(g) \left(\int_0^t \frac{|\sigma(r, x_r)|}{r^\alpha} + \alpha \int_0^t \int_0^r \frac{|\sigma(r, x_r) - \sigma(r, x_s)| + |\sigma(r, x_s) - \sigma(s, x_s)|}{(r-s)^{\alpha+1}} ds dr \right) \\
& \leq \Lambda_\alpha(g) \left(K_0 \int_0^t \frac{1 + |x_r|^\gamma}{r^\alpha} + \alpha M_0 \int_0^t \int_0^r \frac{|x_r - x_s|}{(r-s)^{\alpha+1}} ds dr + \frac{\alpha M_0 t^{\beta-\alpha+1}}{(\beta-\alpha)(\beta-\alpha+1)} \right) \\
& \leq C \Lambda_\alpha(g) \left(1 + \int_0^t \left(\frac{|x_r|^\gamma}{r^\alpha} + \frac{T^\alpha}{r^\alpha} \int_0^r \frac{|x_r - x_s|}{(r-s)^{\alpha+1}} ds \right) dr \right) \\
& \leq C \Lambda_\alpha(g) \left(1 + \int_0^t \left(\frac{|x_r|^\gamma}{r^\alpha} + \frac{1}{r^\alpha} \int_0^r \frac{|x_r - x_s|}{(r-s)^{\alpha+1}} ds \right) dr \right)
\end{aligned}$$

because $T^\alpha r^{-\alpha} \geq 1$ for all $r \in (0, t]$. We also have, due to proposition 4.12

$$\begin{aligned}
& \int_0^t \frac{G_t^\sigma(x) - G_r^\sigma(x)}{(t-r)^{\alpha+1}} dr \\
& \leq C \Lambda_\alpha(g) \left(\int_0^t \frac{|\sigma(r, x_r)|}{(t-r)^{2\alpha}} dr + \int_0^t \int_0^r \frac{|\sigma(r, x_r) - \sigma(s, x_s)|}{(r-s)^{\alpha+1}} (t-s)^{-\alpha} ds dr \right),
\end{aligned}$$

which, using the hypothesis (H_1) and (H_3) yields

$$\begin{aligned}
& \int_0^t \frac{G_t^\sigma(x) - G_r^\sigma(x)}{(t-r)^{\alpha+1}} dr \\
& \leq C \Lambda_\alpha(g) \left(1 + \int_0^t \left(\frac{|x_r|^\gamma}{(t-r)^{2\alpha}} + (t-r)^{-\alpha} \int_0^r \frac{|x_r - x_s|}{(r-s)^{\alpha+1}} ds \right) dr \right).
\end{aligned}$$

Similarly, for $F^b(x)$ we have

$$\begin{aligned}
& |F_t^b(x)| + \int_0^t \frac{|F_t^b(x) - F_r^b(x)|}{(t-r)^{\alpha+1}} dr \\
& \leq C \left(\int_0^t \frac{|b(r, x_r)|}{(t-r)^\alpha} dr + c_{\alpha, T} \left(\frac{1-\alpha}{1-2\alpha} \right)^\alpha \|b_0\|_{L^{1/\alpha}(0, T)} t^{1-2\alpha} \right) \\
& \leq C \left(1 + \int_0^t \frac{|b(r, x_r)|}{(t-r)^\alpha} dr \right) \\
& \leq C \left(1 + \int_0^t \frac{|x_r|}{(t-r)^\alpha} dr \right).
\end{aligned}$$

Now, define

$$h_t = |x_t| + \int_0^t \frac{|x_t - x_r|}{(t-r)^{\alpha+1}} dr.$$

Since x is solution, $x_t = x_0 + G_t^\sigma(x) + F_t^b(x)$, so we can bound h_t as

$$h_t \leq |x_0| + |G_t^\sigma(x)| + |F_t^b(x)| + \int_0^t \frac{|G_t^\sigma(x) - G_r^\sigma(x)|}{(t-r)^{\alpha+1}} dr + \int_0^t \frac{|F_t^b(x) - F_r^b(x)|}{(t-r)^{\alpha+1}} dr$$

and, since the bounds an these quantities depend on γ , we will conclude the estimates by doing cases on γ . Actually, we will see that we can get an estimate of the form

$$h_t \leq C(1 + \Lambda_\alpha(g)) \left(1 + \int_0^t ((t-r)^{-\varepsilon(\gamma)} + r^{-\alpha}) h_r dr \right).$$

On the one hand, if $\gamma = 1$, we have

$$\begin{aligned} h_t &\lesssim (1 + \Lambda_\alpha(g)) \times \\ &\times \left[1 + \int_0^t \left(|x_r| (r^{-\alpha} + (t-r)^{-\alpha} + (t-r)^{-2\alpha}) + \right. \right. \\ &\left. \left. + (r^{-\alpha} + (t-r)^{-\alpha}) \int_0^r \frac{|x_r - x_s|}{(r-s)^{\alpha+1}} ds \right) dr \right]. \end{aligned}$$

Since $r^{-\alpha} + (t-r)^{-\alpha} + (t-r)^{-2\alpha} \leq r^{-\alpha} + (t-r)^{-2\alpha}$ and $(t-r)^{-\alpha} \leq T^\alpha (t-r)^{-2\alpha}$ we have

$$\begin{aligned} h_t &\lesssim (1 + \Lambda_\alpha(g)) \left(1 + \int_0^t (r^{-\alpha} + (t-r)^{-2\alpha}) \left(|x_r| + \int_0^r \frac{|x_r - x_s|}{(r-s)^{\alpha+1}} ds \right) dr \right) \\ &\lesssim (1 + \Lambda_\alpha(g)) \left(1 + \int_0^t (r^{-\alpha} + (t-r)^{-2\alpha}) h_r dr \right). \end{aligned}$$

For $\frac{1-2\alpha}{1-\alpha} \leq \gamma < 1$, write $\delta = \alpha\gamma$ and apply Hölder's inequality to get

$$\int_0^t \frac{|x_r|}{r^\alpha} dr = \left(\int_0^t |x_r| r^{-\alpha} dr \right)^\gamma \left(\int_0^t r^{(\alpha-\delta)/(1-\gamma)} dr \right)^{1-\gamma}.$$

Now, using that $a^\gamma < 1 + a$ if $a \geq 0$ and $\gamma < 1$ we have

$$\int_0^t \frac{|x_r|}{r^\alpha} dr \leq C \left(1 + \int_0^t \frac{|x_r|}{r^\alpha} dr \right).$$

For the other terms, take any $\varepsilon(\gamma) > 1 + \frac{2\alpha-1}{\gamma}$ and we apply Hölder's inequality with exponents $p = 1/\gamma$, $q = p/(p-1)$ and write $\delta' = \varepsilon(\gamma)\gamma$ to get

$$\int_0^t \frac{|x_r|}{(t-r)^{2\alpha}} dr \leq \left(\frac{|x_r|}{(t-r)^{\delta'/\gamma}} dr \right)^\gamma \left(\int_0^t (r-s)^{(-2\alpha+\delta')/(1-\gamma)} dr \right)^{1-\gamma},$$

and again, using that $a^\gamma < 1 + a$ if $a \geq 0$ and $\gamma < 1$ we have

$$\int_0^t \frac{|x_r|}{(t-r)^{2\alpha}} dr \leq C \left(1 + \int_0^t \frac{|x_r|}{(t-r)^{\varepsilon(\gamma)}} dr \right).$$

Joining the estimates, we are lead to

$$h_t \lesssim (1 + \Lambda_\alpha(g)) \left(1 + \int_0^t ((t-r)^{-\varepsilon(\gamma)} + r^{-\alpha}) h_r dr \right).$$

Finally, if $0 \leq \gamma < \frac{1-2\alpha}{1-\alpha}$ we can choose $\varepsilon(\gamma) = \alpha$ and, by the same procedure as the last case (using Hölder's inequality but this time with $\delta = \delta' = \alpha\gamma$) we conclude that

$$h_t \lesssim (1 + \Lambda_\alpha(g)) \left(1 + \int_0^t ((t-r)^{-\alpha} + r^{-\alpha}) h_r dr \right).$$

Observe that in this setting, we can apply the Gronwall type lemma. Indeed, if

$$h_t \leq C(1 + \Lambda_\alpha(g)) \left(1 + \int_0^t ((t-r)^{-\varepsilon(\gamma)} + r^{-\alpha}) h_r dr \right),$$

then

$$\begin{aligned} h_t &\leq C(1 + \Lambda_\alpha(g)) \left(1 + \int_0^t (t-r)^{-\varepsilon(\gamma)} r^{-\varepsilon(\gamma)} [r^{\varepsilon(\gamma)} + (t-r)^{\varepsilon(\gamma)} r^{\varepsilon(\gamma)-\alpha}] h_r dr \right) \\ &\leq C(1 + \Lambda_\alpha(g)) \left(1 + t^{\varepsilon(\gamma)} \int_0^t (t-r)^{-\varepsilon(\gamma)} r^{-\varepsilon(\gamma)} h_r dr \right). \end{aligned}$$

This implies

$$h_t \leq C_1 \exp\{C_2 t(1 + \Lambda_\alpha(g))^{1/(1-\varepsilon(\gamma))}\} \leq \tilde{C}_1 \exp\{\tilde{C}_2 t \Lambda_\alpha(g)^{1/(1-\varepsilon(\gamma))}\}.$$

Since $\sup_{t \in [0, T]} h(t) = \|x\|_{\alpha, \infty}$, we have

$$\|x\|_{\alpha, \infty} \leq \tilde{C}_1 \exp\{\tilde{C}_2 T \Lambda_\alpha(g)^{1/(1-\varepsilon(\gamma))}\}.$$

Renaming the constants, we have proved the desired result. $\square$

5.2. The Stochastic case. Consider a fractional Brownian motion $B^H = \{B_t^H; t \in [0, T]\}$ with Hurst parameter $H \in (1/2, 1)$ defined in a probability space $(\Omega, \mathcal{F}, P)$. Thanks to all the work done in the deterministic case, we can give sense to an expression of the form

$$\int_0^t u_s dB_s^H$$

for some suitable u . Once we check that this integral can be understood as a pathwise generalized Stieltjes integral we will be able to apply the theorem 5.2 of existence and uniqueness to this kind of equations. First, recall that

$$E(|B_t^H - B_s^H|^2) = |t - s|^{2H}.$$

So, for any $p \geq 1$,

$$E(|B_t^H - B_s^H|^p)^{1/p} = c_p |t - s|^H$$

because $B_t^H - B_s^H$ is Gaussian with mean zero and variance $|t - s|^{2H}$. One necessary property of B^H we need in order to define

$$\int_0^t u_s dB_s^H$$

is that $B^H \in W_T^{1-\alpha,\infty}(0,T)$ almost surely for some α . In order to check that this property holds, we have the following lemma:

Lemma 5.5. *Let B^H be a fractional Brownian motion of Hurst parameter $H \in (1/2, 1)$. Let $\alpha \in (1-H, 1/2)$. Then $B^H \in W_T^{1-\alpha,\infty}(0,T)$ almost surely.*

Proof. We know that for all $\varepsilon > 0$ and $T > 0$, there exists a random variable $\Gamma_{\varepsilon,T}$ such that

$$|B_t^H - B_s^H| \leq \Gamma_{\varepsilon,T} |t - s|^{H-\varepsilon}.$$

This in particular implies that the paths of B^H belong to $\mathcal{C}^{H-\varepsilon}([0,T])$ for all $\varepsilon > 0$. Consider then $2\varepsilon = H + \alpha - 1$. With this particular $\varepsilon > 0$ one has

$$\mathcal{C}^{H-\varepsilon}(0,T) = \mathcal{C}^{1-\alpha+\varepsilon}(0,T).$$

Since $\mathcal{C}^{1-\alpha+\varepsilon}(0,T) \subset W_T^{1-\alpha,\infty}(0,T)$, we obtain the desired result. $\square$

This means that, for a stochastic process $u = \{u_t; t \in [0,T]\}$ whose paths belong to $W^{\alpha,1}(0,T)$ for $\alpha \in (1-H, 1/2)$, then the integral

$$\int_0^t u_s dB_s^H$$

exist. Moreover, if the paths of u belong to $W_0^{\alpha,\infty}(0,T)$ then the integral has Hölder continuous paths of order $1-\alpha$. Thanks to this, we can now consider the following problem.

Problem: Given $B^H = \{B_t^H; t \in [0,T]\}$ a fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$ and X_0 a random variable, find if there exists a unique solution of the equation

$$X_t = X_0 + \int_0^t \sigma(s, X_s) dB_s^H + \int_0^t b(s, X_s) ds$$

In order to solve this problem, we will strongly use the deterministic results. Consider the set of hypothesis (H_1) and (H_2) and let

$$\alpha_0 = \frac{1}{2} \wedge \beta \wedge \frac{\delta}{1+\delta}$$

Theorem 5.6. *Suppose that X_0 is a real valued random variable and σ, b satisfy hypothesis (H_1) and (H_2) with $\beta > 1-H$, $\delta > \frac{1}{H} - 1$. Then,*

- (i) *If $\alpha \in (1-H, \alpha_0)$ and $\rho \geq 1/\alpha$, then there exists a unique stochastic process X with paths in $W_0^{\alpha,\infty}(0,T)$ solution to*

$$X_t = X_0 + \int_0^t \sigma(s, X_s) dB_s^H + \int_0^t b(s, X_s) ds$$

and, moreover, $X(\omega) \in \mathcal{C}^{1-\alpha}(0,T)$ almost surely.

- (ii) *Moreover, if the assumption (H_3) is satisfied, $\alpha \in (1-H, \alpha_0 \wedge (2-\gamma)/4)$, $\rho \geq 1/\alpha$, $X_0 \in L^\infty(\Omega)$ and the constants M_N, L_N, K_0 and the function b_0 are independent of ω , then the solution X satisfies*

$$E(\|X\|_{\alpha,\infty}^p) < \infty, \quad \forall p \geq 1.$$

Proof.

- (i) Follows directly from the deterministic theorem.
- (ii) In order to apply the second part of the deterministic theorem, we need to check that the random variable

$$G = \Lambda_\alpha(B^H) = \frac{1}{\Gamma(1-\alpha)} \sup_{0 \leq s < t \leq t} |D_{t-}^{1-\alpha}(B_{t-}^H)_s|$$

has moments of all orders. This is because then, we will have

$$E(\|X\|_{\alpha,\infty}^p) \lesssim E(\exp\{CpG^\kappa\})$$

and this last expression would be finite as long as $\kappa < 2$. Now, notice that

$$|D_{t-}^{1-\alpha}(B_{t-}^H)_s| \leq \frac{1}{\Gamma(\alpha)} \left(\frac{|B_t^H - B_s^H|}{(t-s)^{1-\alpha}} + (1-\alpha) \int_s^t \frac{|B_r^H - B_s^H|}{(r-s)^{2-\alpha}} dr \right).$$

Now, for $\varepsilon = (H + \alpha - 1)/2$ we know that B has Hölder continuous paths of order H_ε , so the last expression turns to

$$|D_{t-}^{1-\alpha}(B_{t-}^H)_s| \leq \frac{|\Gamma_{\varepsilon,T}|}{\Gamma(\alpha)} \left((t-s)^{H-\varepsilon+\alpha-1} + \int_s^t (r-s)^{H-\varepsilon-\alpha+2} dr \right).$$

Hence,

$$E \left(\frac{1}{\Gamma(1-\alpha)} \sup_{0 \leq s < t \leq t} |D_{t-}^{1-\alpha}(B_{t-}^H)_s| \right) \lesssim E(|\Gamma_{\varepsilon,T}|^p) < \infty,$$

so we have all the conditions to conclude that

$$E(\|X\|_{\alpha,\infty}^p) < \infty.$$

□

6. MALLIAVIN CALCULUS FOR STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY A FRACTIONAL BROWNIAN MOTION

The Malliavin calculus, also known as the stochastic calculus of variations, extends the classical integral and differential calculus to random variables and vectors. This allows us to give a notion of differentiability very related to the Fréchet differentiability that can be applied in order to study the regularity of solutions of stochastic differential equations. Moreover, one of the most powerful applications of the Malliavin Calculus is the study of the properties of the law of random vectors. Indeed, we will see that if a random variable F is differentiable in the Malliavin sense and it satisfies a non-degeneracy condition, then the law of F is absolutely continuous with respect to the Lebesgue measure. Hence, by adding extra hypothesis on the coefficients of the differential equations driven by a fractional Brownian motion, we will conclude that the law of the solution is absolutely continuous with respect to the Lebesgue measure.

We will follow the same scheme as in [14], using as well some tools from [13]. As it is discussed in this references, the Malliavin differentiability is intimately related with the

Fréchet differentiability. Hence, in the same way as proving existence and uniqueness of solutions, it is convenient to first study the deterministic case.

6.1. Fréchet differentiability for deterministic differential equations. In the last chapter, we proved the existence and uniqueness of the differential equation

$$(6.1) \quad x_t = x_0 + \int_0^t \sigma(s, x_s) dg_s + \int_0^t b(s, x_s) ds, \quad t \in [0, T]$$

for $g \in W_T^{1-\alpha, \infty}(0, T)$, $0 < \alpha < 1/2$, and the coefficients σ and b satisfying (H_1) , (H_2) and (H_3) . Moreover, we saw that the solution x belongs to $W_0^{\alpha, \infty}(0, T)$ (it is, in fact $(1 - \alpha)$ -Hölder continuous). Notice that, if we fix the coefficients and the initial condition x_0 , equation (6.1) defines an operator

$$\begin{aligned} x : W_T^{1-\alpha, \infty}(0, T) &\rightarrow W_0^{\alpha, \infty}(0, T) \\ g &\mapsto x(g) \end{aligned}$$

where $x(g)$ is the solution of equation (6.1) for that specific g . Our next objective is studying the differentiability in the Fréchet sense of this operator x . As it is natural, the conditions to get existence and uniqueness of solutions are not enough to ensure the differentiability of the solution. Hence, we have to add new hypothesis on σ and b :

(H_4) : $\sigma(s, x)$ and $b(s, x)$ are $\mathcal{C}_b^3(\mathbb{R})$ in the second argument and. Moreover, there exists $C > 0$ such that

$$\max_{i,j=0,1,2,3} \sup_{s \in [0, T]} (||\partial_x^i \sigma(s)||_{L^\infty(\mathbb{R})} + ||\partial_x^j b(s)||_{L^\infty(\mathbb{R})}) \leq C.$$

In other words, σ, b are $\mathcal{C}^3$, and both functions together with their partial derivatives with respect to x are uniformly bounded in $s \in [0, T]$. Under this hypothesis, we will be able to prove the differentiability. First, we need the following lemma

Lemma 6.1. *Let x be the solution of (6.1). Assume σ, b satisfy (H_4) . Then, the operator*

$$F : W_T^{1-\alpha, \infty}(0, T) \times W_0^{\alpha, \infty}(0, T) \rightarrow W_0^{\alpha, \infty}(0, T)$$

defined by

$$F(h, x)_t = x_t - x_0 - \int_0^t b(s, x_s) ds - \int_0^t \sigma(s, x_s) d(g_s + h_s)$$

is Fréchet differentiable and, for any $(h, x) \in W_T^{1-\alpha, \infty}(0, T) \times W_0^{\alpha, \infty}(0, T)$, $k \in W_T^{1-\alpha, \infty}(0, T)$ and $v \in W_0^{\alpha, \infty}(0, T)$ we have

$$\begin{aligned} D_1 F(h, x)(k)_t &= - \int_0^t \sigma(s, x_s) dk_s, \\ D_2 F(h, x)(v)_t &= v_t - \int_0^t \partial_x b(s, x_s) v_s ds - \int_0^t \partial_x \sigma(s, x_s) v_s d(g_s + h_s), \end{aligned}$$

where D_1 stands for the derivative with respect to the first argument and D_2 stands for the derivative with respect to the second argument

Proof. The proof of this lemma goes over several steps

- Step 1: Proving that F is continuous: Consider $(h, x), (\tilde{h}, \tilde{x}) \in W_T^{1-\alpha}(0, T) \times \overline{W_0^{\alpha, \infty}}(0, T)$, then notice that

(6.2)

$$\begin{aligned} F(\tilde{h}, \tilde{x}) &= \tilde{x}_t - x_0 - \int_0^t b(s, \tilde{x}_s) ds - \int_0^t \sigma(s, \tilde{x}_s) d(g_s + \tilde{h}_s) \\ &= \tilde{x}_t - x_0 - \int_0^t b(s, \tilde{x}_s) ds - \int_0^t \sigma(s, \tilde{x}_s) d(g_s + h_s) + \int_0^t \sigma(s, \tilde{x}_s) d(h_s - \tilde{h}_s). \end{aligned}$$

Hence,

$$\begin{aligned} F(h, x)_t - F(\tilde{h}, \tilde{x})_t &= x_t - \tilde{x}_t - \int_0^t (b(s, x_s) - b(s, \tilde{x}_s)) ds \\ &\quad - \int_0^t (\sigma(s, \tilde{x}_s) - \sigma(s, x_s)) d(g_s + h_s) - \int_0^t \sigma(s, \tilde{x}_s) d(h_s - \tilde{h}_s). \end{aligned}$$

Using the triangle inequality we have

$$\|F(h, x) - F(\tilde{h}, \tilde{x})\|_{\alpha, \infty} \leq \|x - \tilde{x}\|_{\alpha, \infty} + \sum_{i=1}^3 N_i,$$

where

$$\begin{aligned} N_1 &= \left\| \int_0^t (b(s, x_s) - b(s, \tilde{x}_s)) ds \right\|_{\alpha, \infty}, \\ N_2 &= \left\| \int_0^t (\sigma(s, \tilde{x}_s) - \sigma(s, x_s)) d(g_s + h_s) \right\|_{\alpha, \infty}, \\ N_3 &= \left\| \int_0^t \sigma(s, \tilde{x}_s) d(h_s - \tilde{h}_s) \right\|_{\alpha, \infty}. \end{aligned}$$

For N_1 , we use proposition 4.12 to derive the following estimate

$$\begin{aligned} N_1 &\leq C_{\alpha, T} \|b(s, x_s) - b(s, \tilde{x}_s)\|_{\alpha, \infty} \\ &\leq C_{\alpha, T} \|\partial_x b(s, \cdot)\|_{L^\infty} \|x - \tilde{x}\|_{\alpha, \infty} \\ &\leq C_{\alpha, T, \partial_x b(s)} \|x - \tilde{x}\|_{\alpha, \infty} \\ &=: C_1 \|x - \tilde{x}\|_{\alpha, \infty}. \end{aligned}$$

Again, using proposition 4.12 on N_2 we get

$$N_2 \leq C_{\alpha, T} \|g + h\|_{1-\alpha, \infty} \|\sigma(s, x_s) - \sigma(s, \tilde{x}_s)\|_{\alpha, \infty}.$$

Using lemma 4.15 with $t_1 = t_2 = s$, $x_1 = x$, $x_3 = \tilde{x}$ and $x_2 = x_4 = 0$ we get

$$\begin{aligned} &\|\sigma(s, x_s) - \sigma(s, \tilde{x}_s)\|_{\alpha, \infty} \\ &\leq \|\partial_x \sigma(s)\|_{L^\infty} \|x - \tilde{x}\|_{\alpha, \infty} + \|\partial_{xx}^2 \sigma(s)\|_{L^\infty} \|x - \tilde{x}\|_{\alpha, \infty} (\|x\|_{\alpha, \infty}^\delta + \|\tilde{x}\|_{\alpha, \infty}^\delta) \\ &\leq C(1 + \|x\|_{\alpha, \infty}^\delta + \|\tilde{x}\|_{\alpha, \infty}^\delta) \|x - \tilde{x}\|_{\alpha, \infty} \\ &=: C_2 \|x - \tilde{x}\|_{\alpha, \infty}. \end{aligned}$$

For N_3 we use proposition 4.12 to obtain

$$N_3 \leq C_{\alpha,T} \|h - \tilde{h}\|_{1-\alpha,\infty} \|\sigma(s, \tilde{x}_s)\|_{\alpha,\infty}.$$

Since σ is a Lipschitz function in space, we can write $|\sigma(s, x_s)| \leq |\sigma(s, 0)| + \|\partial_x \sigma(s)\|_{L^\infty} |x_s|$. This inequality allows us to derive the estimate

$$\begin{aligned} N_3 &\leq C(1 + \|\tilde{x}\|_{\alpha,\infty}) \|h - \tilde{h}\|_{1-\alpha,\infty} \\ &=: K_2 \|h - \tilde{h}\|_{1-\alpha,\infty}. \end{aligned}$$

If we define $K_1 := 1 + C_1 + C_2$, we have

$$\|F(h, x) - F(\tilde{h}, \tilde{x})\|_{\alpha,\infty} \leq K_1 \|x - \tilde{x}\|_{\alpha,\infty} + K_2 \|h - \tilde{h}\|_{1-\alpha,\infty},$$

which proves the continuity of F .

- Step 2: Checking that $D_2 F$ and $D_1 F$ are continuous: On the one hand, notice that for all $k \in W_T^{1-\alpha,\infty}(0, T)$ we have

$$\|D_1 F(h, x)(k)\|_{\alpha,\infty} = \left\| \int_0^t \sigma(s, x_s) dk_s \right\|_{\alpha,\infty}$$

Then, using proposition 4.12 we have

$$\left\| \int_0^t \sigma(s, x_s) dk_s \right\|_{\alpha,\infty} \leq C_{\alpha,T} \|\sigma(s, x_s)\|_{\alpha,\infty} \|k\|_{1-\alpha,\infty}$$

Now, we have that $|\sigma(s, x_s)| \leq |\sigma(s, 0)| + \|\partial_x \sigma(s)\|_{L^\infty} |x_s|$. Plugging this in the last inequality, we have

$$\|\sigma(s, x_s)\|_{\alpha,\infty} \|k\|_{1-\alpha,\infty} \leq C \|k\|_{1-\alpha,\infty}$$

Hence, $D_1 F$ is continuous. With a similar argument using proposition 4.12 and the properties of b and σ , we conclude that

$$\|D_2 F(h, x)(v)\|_{\alpha,\infty} \leq C \|v\|_{\alpha,\infty}$$

This implies that $D_2 F$ is continuous as well.

- Step 3: Checking that $D_2 F$ and $D_1 F$ are the Fréchet derivatives of F : Recall that $D_1 F(h, x)$ is the Fréchet derivative of F with respect to h if for every $k \in W_T^{1-\alpha,\infty}(0, T)$

$$\|F(h + k, x) - F(h, x) - D_1 F(h, x)(k)\|_{\alpha,\infty} \leq C \|k\|_{1-\alpha,\infty}^\delta$$

for some $\delta > 1$ But this is direct from the fact that

$$F(h + k, x) - F(h, x) = D_1 F(h, x)(k).$$

For $D_2 F$ we have to check that for all $v \in W_0^{\alpha,\infty}(0, T)$, then

$$\|F(h, x + v) - F(h, x) - D_2 F(h, x)(v)\|_{\alpha,\infty} \leq C \|v\|_{\alpha,\infty}^\delta$$

for some $\delta > 1$. By doing the computations, we have

$$\begin{aligned} & F(h, x + v)_t - F(h, x)_t - D_2F(h, x)(v)_t \\ &= \int_0^t (b(s, x_s) - b(s, x_s + v_s) + \partial_x b(s, x_s)v_s) ds \\ &+ \int_0^t (\sigma(s, x_s) - \sigma(s, x_s + v_s) + \partial_x \sigma(s, x_s)v_s) d(g_s + h_s). \end{aligned}$$

On the one hand, by the mean value theorem we have $|b(s, x_s) - b(s, x_s + v_s)| = |\partial_x b(s, \xi_s^{(1)})||v_s|$ where $\xi_s^{(1)} \in [x_s, x_s + v_s]$. Now, $|\partial_x b(s, \xi_s^{(1)}) - \partial_x b(s, x_s)||v_s| = |\partial_{xx}^2 b(s, \xi_s^{(2)})||\xi_s^{(1)} - x_s||v_s|$. Since the first 3 derivatives of b with respect to x are bounded and $|\xi_s^{(1)} - x_s| \leq |v_s|$, we have that

$$|b(s, x_s) - b(s, x_s + v_s)| \leq \|\partial_{xx}^2 b(s)\|_{L^\infty} |v_s|^2.$$

This implies that

$$\left\| \int_0^t (b(s, x_s) - b(s, x_s + v_s) + \partial_x b(s, x_s)v_s) ds \right\|_{\alpha, \infty} \leq C_{\alpha, T} \|\partial_{xx}^2 b(s)\|_{L^\infty} \|v\|_{\alpha, \infty}^2.$$

Exactly the same argument on the second term and proposition 4.12 gives us

$$\begin{aligned} & \left\| \int_0^t (\sigma(s, x_s) - \sigma(s, x_s + v_s) + \partial_x \sigma(s, x_s)v_s) d(g_s + h_s) \right\|_{\alpha, \infty} \\ & \leq C_{\alpha, T} (\|\partial_{xx}^2 \sigma(s)\|_{L^\infty} + \|\partial_{xxx}^3 \sigma(s)\|_{L^\infty} \|x\|_{\alpha, \infty}) \|g + h\|_{1-\alpha, \infty} \|v\|_{\alpha, \infty}^2. \end{aligned}$$

Adding the two estimates, we have that

$$\|F(h, x + v) - F(h, x) - D_2F(h, x)(v)\|_{\alpha, \infty} = O(\|v\|_{\alpha, \infty}^2)$$

and, in particular, this proves that $D_2F(h, x)$ is the Fréchet derivative of F with respect to x (the second argument).

□

Lemma 6.2. Fix $g \in W_T^{1-\alpha, \infty}(0, T)$ and consider the following equation

$$(6.3) \quad y_t = w_t + \int_0^t S_s y_s dg_s + \int_0^t B_s y_s ds$$

where $w, B, S \in \mathcal{C}^{1-\alpha}(0, T)$. Then, there exists a unique solution $y \in \mathcal{C}^{1-\alpha}(0, T)$ to (6.3) which satisfies

$$\|y\|_{\alpha, \infty} \leq C \|w\|_{\alpha, \infty} \exp \left(c_2 \|g\|_{1-\alpha, \infty}^{\frac{1}{1-2\alpha}} (\|B\|_{L^\infty} + \|S\|_{1-\alpha}) \right).$$

Proof. The existence, uniqueness and Hölder continuity of the solution comes from the Existence and Uniqueness theorem. In order to derive the bound of the norm of the solution, we will use a Gronwall type argument. Recall that

$$\|y\|_{\alpha, \infty} = \sup_{t \in [0, T]} \Delta_t^\alpha(y),$$

so let's try to give a bound for $\Delta_t^\alpha(y)$. Let's define $G^S(y)_t = \int_0^t S_s y_s dg_s$ and $F^B(y)_t = \int_0^t B_s y_s ds$. On the one hand, because of proposition 4.12,

$$\Delta_t^\alpha(G^S) \leq c_{\alpha,T} \|g\|_{1-\alpha,\infty} \int_0^t ((t-s)^{-2\alpha} + t^{-\alpha}) \Delta_s^\alpha(Sy) ds.$$

Now, by definition of Δ^α we have

$$\Delta_s^\alpha(Sy) = |S_s y_s| + \int_0^s \frac{|S_s y_s - S_r y_r|}{(s-r)^{\alpha+1}} dr.$$

Since $|S_s y_s - S_r y_r| \leq |S_s y_s - S_r y_s| + |S_r y_s - S_r y_r| = |y_s| |S_s - S_r| + |S_r| |y_s - y_r|$ we have

$$\Delta_s^\alpha(Sy) \leq |S_s y_s| + |y_s| \left(\int_0^s \frac{|S_s - S_r|}{(s-r)^{\alpha+1}} dr \right) + \int_0^s \frac{|S_r| |y_s - y_r|}{(s-r)^{\alpha+1}} dr.$$

By definition of the Hölder norm, $|S_s| \leq \|S\|_{1-\alpha}$ and $|S_s - S_r| \leq \|S\|_{1-\alpha} (s-r)^{1-\alpha}$. Plugging this estimates in the last inequality we have

$$\Delta_s^\alpha(Sy) \leq \|S\|_{1-\alpha} |y_s| + |y_s| \left(\int_0^s \frac{\|S\|_{1-\alpha} (s-r)^{1-\alpha}}{(s-r)^{\alpha+1}} dr \right) + \|S\|_{1-\alpha} \int_0^s \frac{|y_s - y_r|}{(s-r)^{\alpha+1}} dr.$$

Rearranging terms, we have

$$\begin{aligned} \Delta_s^\alpha(Sy) &\leq \|S\|_{1-\alpha} \left(|y_s| + \int_0^s \frac{|y_s - y_r|}{(s-r)^{\alpha+1}} dr \right) + |y_s| \left(\int_0^s \frac{\|S\|_{1-\alpha} (s-r)^{1-\alpha}}{(s-r)^{\alpha+1}} dr \right) \\ &\leq \|S\|_{1-\alpha} \Delta_s^\alpha(y) + |y_s| \left(\int_0^s \frac{\|S\|_{1-\alpha} (s-r)^{1-\alpha}}{(s-r)^{\alpha+1}} dr \right). \end{aligned}$$

Hence,

$$\begin{aligned} \Delta_t^\alpha(G^S) &\leq c_{\alpha,T} \|g\|_{1-\alpha,\infty} \left(\int_0^t ((t-s)^{-2\alpha} + t^{-\alpha}) |y_s| \left(\int_0^s \frac{\|S\|_{1-\alpha} (s-r)^{1-\alpha}}{(s-r)^{\alpha+1}} dr \right) ds \right. \\ &\quad \left. + \|S\|_{1-\alpha} \int_0^t ((t-s)^{-2\alpha} + t^{-\alpha}) \Delta_s^\alpha(y) ds \right). \end{aligned}$$

Since $|y_s| \leq \Delta_s^\alpha(y)$, we can roughly bound the last inequality by

$$\Delta_t^\alpha(G^S) \leq c_{\alpha,T} \|g\|_{1-\alpha,\infty} \|S\|_{1-\alpha} \int_0^t ((t-s)^{-2\alpha} + t^{-\alpha}) \Delta_s^\alpha(y) ds$$

For F^B the estimations are much more simpler, since again, by proposition 4.12

$$\Delta_t^\alpha(F^B) \leq c_{\alpha,T} \int_0^t \frac{|B_s y_s|}{(t-s)^\alpha} ds \leq c_{\alpha,T} \|B\|_{1-\alpha,\infty} \int_0^t \frac{|y_s|}{(t-s)^\alpha} ds.$$

Therefore, we have the estimate

$$\Delta_t^\alpha(y) \leq \|w\|_{\alpha,\infty} + c_{\alpha,T} (\|g\|_{1-\alpha,\infty} \|S\|_{1-\alpha} + \|B\|_{L^\infty}) \int_0^t ((t-s)^{-2\alpha} + t^{-\alpha}) \Delta_s^\alpha(y) ds.$$

Applying the Gronwall-type lemma that we used in theorem 5.2 we conclude

$$\|y\|_{\alpha,\infty} \leq C \|w\|_{\alpha,\infty} \exp \left(c_2 \|g\|_{1-\alpha,\infty}^{\frac{1}{1-2\alpha}} (\|B\|_{L^\infty} + \|S\|_{1-\alpha}) \right).$$

as we wanted. $\square$

Thanks to this two previous lemmas, we can prove the Fréchet differentiability of the operator given by $g \rightarrow x(g)$.

Proposition 6.3. *Let x be the solution of equation (6.1). Assume σ, b satisfy (H_4) . The mapping $g \rightarrow x(g)$ from $W_T^{1-\alpha,\infty}(0, T)$ to $W_0^{\alpha,\infty}(0, T)$ is Fréchet differentiable and, for any $h \in W_T^{1-\alpha,\infty}(0, T)$, the derivative of x in the direction of h satisfies*

$$D_h x_t = \int_0^t \phi_t(s) dh_s,$$

where $\phi_t(s)$ solves the following equation:

$$(6.4) \quad \phi_t(s) = \sigma(s, x_s) + \int_s^t \partial_x \sigma(s, x_s) \phi_u(s) dg_u + \int_s^t \partial_x b(s, x_s) \phi_u(s) du$$

and $\phi_t(s) = 0$ if $t < s$.

Proof. The strategy for proving this proposition is applying the implicit function theorem to the operator F defined in lemma 6.1. By lemma 6.1, F is continuous and Fréchet differentiable. Moreover, $F(0, x) = 0$. The last thing left to do before applying the implicit function theorem is checking that $D_2 F(0, h)$ is a linear homeomorphism, that is, $D_2 F(0, x)$ is a linear bijection, continuous, and $D_2 F(0, x)^{-1}$ is continuous. On the one hand, notice that for every $v \in W_0^{\alpha,\infty}(0, T)$ we have

$$D_2 F(0, x)(v)_t = v_t - \int_0^t \partial_x \sigma(s, x_s) v_s dg_s - \int_0^t \partial_x b(s, x_s) v_s ds.$$

Hence, thanks to the theorem of existence and uniqueness of solutions 5.2, we deduce that $D_2 F(0, x)$ is a bijection. Indeed, the existence of solutions gives us that $D_2 F(0, x)$ is onto and the uniqueness gives us that $D_2 F(0, x)$ is injective. Moreover, because of the open mapping theorem, the map $D_2 F(0, x)^{-1}$ is continuous. This proves that $D_2 F(0, x)$ is a linear homeomorphism and we can apply the implicit function theorem for operators to deduce that there exists a neighbourhood $0 \in U \subset W_T^{1-\alpha,\infty}(0, T)$ and a Fréchet differentiable function $x : U \rightarrow W_0^{\alpha,\infty}(0, T)$ satisfying $x(0) = x$ and such that for all $h \in U$,

$$F(h, x(h)) = 0.$$

Since F is Fréchet differentiable, we can differentiate with respect to h to get

$$D_1 F(h, x(h)) + D_2(h, x(h)) \circ Dx(h) = 0.$$

In particular, isolating $Dx(h)$ from the last equation and evaluating at $h = 0$ we have

$$D_2 F(0, x)(Dx) = -D_1 F(0, x) \iff Dx = -D_2 F(0, x)^{-1} \circ D_1 F(0, x).$$

From lemma 6.1, we know the exact expression for the Fréchet derivatives of F . Hence, by the relation $D_2 F(0, x)(Dx) = -D_1 F(0, x)$, we conclude that for every $k \in W_T^{1-\alpha,\infty}(0, T)$, $D_1 F(0, x)(k)$ satisfies the equation

$$D_1 F(0, x)(k)_t = -Dx(k)_t + \int_0^t \partial_x \sigma(s, x_s) Dx(k)_s dg_s + \int_0^t \partial_x b(s, x_s) Dx(k)_s ds$$

or, equivalently,

$$Dx(k)_t = \int_0^t \sigma(s, x_s) dk_s + \int_0^t \partial_x \sigma(s, x_s) Dx(k)_s dg_s + \int_0^t \partial_x b(s, x_s) Dx(k)_s ds.$$

Recall the definition of $\phi_t(s)$. Integrating the relation (6.4) with respect to $h \in W_T^{1-\alpha, \infty}(0, T)$ we have

$$\begin{aligned} \int_0^t \phi_t(s) dh_s &= \int_0^t \sigma(s, x_s) dh_s + \int_0^t \left(\int_s^t \partial_x \sigma(u, x_u) \phi_u(s) dg_u \right) dh_s \\ &\quad + \int_0^t \left(\int_s^t \partial_x b(u, x_u) \phi_u(s) du \right) dh_s. \end{aligned}$$

We would like to apply a Fubini-type theorem for Stieltjes integrals (see, for example, [3]). In order to apply it, we have to see that

$$\int_s^t |\partial_x \sigma(u, x_u) \phi_u(s)| dg_u dh_s < \infty$$

and

$$\int_0^t \int_s^t |\partial_x b(u, x_u) \phi_u(s)| du dh_s < \infty,$$

where these integrals are double Stieltjes integrals (see, for example, [4]). In order to be integrable, it is enough to check that $\partial_x \sigma(u, x_u) \phi_u(s)$ and $\partial_x b(u, x_u) \phi_u(s)$ are Hölder continuous of order $1-\alpha$ in both variables (u, s) . By hypothesis, $\partial_x \sigma(s, x_s), \partial_x b(s, x_s) \in \mathcal{C}^{1-\alpha}(0, T)$. Moreover, because of lemma 6.2, $\|\phi(s)\|_{\alpha, \infty}$ is uniformly bounded in s and this implies that $\phi_t(s)$ is Hölder continuous in t uniformly in s (again by proposition 4.12). In order to streamline the notation, we write $S_t = \partial_x \sigma(t, x_t)$ and $B_t = \partial_x b(t, x_t)$. We have, for $s' \leq s \leq t$

$$(6.5) \quad \phi_t(s) - \phi_t(s') = w(s, s') + \int_s^t S_u(\phi_u(s) - \phi_u(s')) dg_u + \int_s^t B_u(\phi_u(s) - \phi_u(s')) du,$$

where

$$(6.6) \quad w(s, s') = \sigma(s, x_s) - \sigma(s', x_{s'}) + \int_{s'}^s S_u \phi_u(s) dg_u + \int_{s'}^s B_u \phi_u(s) du.$$

Lemma 6.2 gives us that

$$(6.7) \quad \sup_{s \in [0, T]} \|\phi(s)\|_{\alpha, \infty} \leq c_1 \|\sigma(s, x_s)\|_{\alpha, \infty} \exp \left(c_2 \|g\|^{\frac{1}{1-2\alpha}} (\|B\|_{L^\infty} + \|S\|_{1-\alpha}) \right).$$

So, plugging (6.7) into (6.6) we have

$$\begin{aligned} |w(s, s')| &\leq \|\sigma(s, x_s)\|_{1-\alpha} |s - s'|^{1-\alpha} + \|B\|_{L^\infty} \left(\sup_{s \in [0, T]} \|\phi(s)\|_{\alpha, \infty} \right) |s - s'| \\ (6.8) \quad &\quad + c \|S\|_{\alpha, \infty} \|g\|_{1-\alpha, \infty} \left(\sup_{s \in [0, T]} \|\phi(s)\|_{\alpha, \infty} \right) |s - s'|^{1-\alpha} \end{aligned}$$

and we can bound roughly the last inequality by

$$|w(s, s')| \leq c |s - s'|^{1-\alpha}.$$

Applying once again lemma 6.2 to the differential equation (6.5) and using the bound on $|w(s, s')|$ we have

$$\|\phi(s) - \phi(s')\|_{\alpha, \infty} \leq c_1 |s - s'|^{1-\alpha} \exp \left(c_2 \|g\|^{\frac{1}{1-2\alpha}} (\|B\|_{L^\infty} + \|S\|_{1-\alpha}) \right),$$

which proves that $\phi_t(s)$ is $(1-\alpha)$ -Hölder continuous in s uniformly on t , as we wanted to see. This proves that $\partial_x \sigma(u, x_u) \phi_s(u)$ and $\partial_x b(u, x_u) \phi_s(u)$ belong to $\mathcal{C}^{1-\alpha}([0, T]^2)$ and we can apply Fubini's theorem for Stieltjes integrals to get

$$(6.9) \quad \begin{aligned} \int_0^t \phi_t(s) dh_s &= \int_0^t \sigma(s, x_s) dh_s + \int_0^t \partial_x \sigma(u, x_u) \left(\int_0^u \phi_u(s) dh_s \right) dg_u \\ &\quad + \int_0^t \partial_x b(u, x_u) \left(\int_0^u \phi_u(s) dh_s \right) du. \end{aligned}$$

Since $\int_0^t \phi_t(s) dh_s$ satisfies the same differential equation as $D_h x$, we can conclude by uniqueness of solutions that

$$D_h x = \int_0^t \phi_t(s) dh_s,$$

as desired. $\square$

6.2. Malliavin Calculus for the fractional Brownian motion. In this project, we have already discussed some properties of the fractional Brownian motion. This properties, however, are not enough to develop the Malliavin calculus tools we need. This section is devoted to give the proper framework as it is done for the Malliavin calculus with respect to the Wiener process (see, for example, [13]).

Instead of working with an arbitrary probability space, we will work instead with a very specific one. Consider $\Omega = \mathcal{C}_0([0, T])$ the Banach space of continuous functions $x(t)$ in $[0, T]$ such that they vanish at $t = 0$. Let P be the unique probability measure on Ω given by Kolmogorov's extension theorem such that the canonical process $\{B_t^H; t \in [0, T]\}$ defined by $B_t^H(\omega) = \omega(t)$ is a fractional Brownian motion of Hurst parameter $H > 1/2$. If we now set $\mathcal{F} = \sigma(B_t^H; t \in [0, T])$, we say that $(\Omega, \mathcal{F}, P)$ is the canonical probability space of the process B^H . We will study the Malliavin calculus for the fractional Brownian motion in this canonical probability space.

6.2.1. Stochastic Calculus with respect to the fractional Brownian motion. In order to discuss the Malliavin differentiability of a process, we need to define the *Malliavin derivative*. The definition of this derivative requires of the *isonormal process*. Defining the isonormal process is not a trivial issue, and it requires constructing an isometry between $L^2(\Omega)$ and a suitable separable Hilbert space $\mathcal{H}$. The characterization of $\mathcal{H}$ and the definition of the isometry is our immediate objective.

Definition 6.4. We denote by $\mathcal{E}$ the set of step functions on $[0, T]$, i.e. $x \in \mathcal{E}$ if and only if there exist $t_1, \dots, t_n \in [0, T]$ and $a_1, \dots, a_n \in \mathbb{R}$ with

$$(6.10) \quad x(t) = \sum_{j=1}^n a_j \mathbf{1}_{[0, t_j]}(t).$$

in $\mathcal{E}$, we can consider the scalar product given by

$$(6.11) \quad \langle \mathbb{1}_{[0,t]}, \mathbb{1}_{[0,s]} \rangle = R_H(t, s),$$

where R_H is the covariance function of B^H . In order to proceed, we shall explore some properties of R_H .

Lemma 6.5. *The covariance function of the fBm can be expressed as*

$$R_H(t, s) = \alpha_H \int_0^t \int_0^s |r - u|^{2H-2} du dr,$$

where $\alpha_H = H(2H - 1)$

Proof. Without loss of generality, assume that $s < t$. Then, for any $r \in [0, t]$ we have

$$\int_0^s |r - u|^{2H-2} du = \int_0^s (r - u)^{2H-2} \mathbb{1}_{[0,r]}(u) du + \int_0^s (u - r)^{2H-2} \mathbb{1}_{[r,s]}(u) du.$$

Then, a direct computation shows that

$$\begin{aligned} \int_0^s |r - u|^{2H-2} du &= \left(\frac{r^{2H-1}}{2H-1} - \frac{(r-s)^{2H-1}}{2H-1} \right) \mathbb{1}_{[s,t]}(r) \\ &\quad + \left(\frac{r^{2H-1}}{2H-1} + \frac{(s-r)^{2H-1}}{2H-1} \right) \mathbb{1}_{[0,s]}(r). \end{aligned}$$

Integrating with respect to r from 0 to t we get the desired result. $\square$

This lemma implies that, for any $\varphi, \psi \in \mathcal{H}$,

$$(6.12) \quad \langle \varphi, \psi \rangle_{\mathcal{H}} = \alpha_H \int_0^T \int_0^T |r - u|^{2H-2} \varphi_r \psi_u du dr$$

because of the definition of $\mathcal{H}$. The lemma also allows us to derive a representation of $R_H(t, s)$ based on a square-integrable kernel.

Proposition 6.6. *Let*

$$\begin{aligned} K_H(t, s) &= c_H s^{1/2-H} \int_s^t (u - s)^{H-3/2} u^{H-1/2} du, \\ c_H &= \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-1/2)}} \end{aligned}$$

for $t > s$ and $K_H(t, s) = 0$ in case $t \leq s$. Then, for any $t, s \in [0, T]$,

$$(6.13) \quad \int_0^{t \wedge s} K_H(t, u) K_H(s, u) du = R_H(t, s).$$

Proof. Recall the definition of the beta function:

$$\beta(z, w) = \int_0^1 t^{z-1} (1-t)^{w-1} dt.$$

Then, notice that

$$\begin{aligned}
(6.14) \quad & \int_0^{t \wedge s} K_H(t, u) K_H(s, u) du \\
&= c_H^2 \int_0^{t \wedge s} u^{1-2H} \left(\int_u^t (y-u)^{H-3/2} y^{H-1/2} dy \right) \left(\int_0^s (z-u)^{H-3/2} z^{H-1/2} dz \right) du \\
&= c_H^2 \int_0^t \int_0^s (yz)^{H-1/2} \int_0^{z \wedge y} u^{1-2H} (y-u)^{H-3/2} (z-u)^{H-3/2} du dz dy.
\end{aligned}$$

Suppose that $z \leq y$, then, by means of the change of variables $r = \frac{y-u}{z-u}$ and $x = \frac{y}{zr}$ we have

$$\begin{aligned}
& \int_0^z u^{1-2H} (y-u)^{H-3/2} (z-u)^{H-3/2} du \\
&= (y-z)^{2H-2} \int_{y/z}^\infty (zr-y)^{1-2H} r^{H-3/2} dr \\
&= (yz)^{1/2-H} (y-z)^{2H-2} \int_0^1 (1-x)^{1-2H} x^{H-3/2} dx \\
&= \beta(2-2H, H-1/2) (yz)^{1/2-H} (y-z)^{2H-2}.
\end{aligned}$$

Doing the same computations with the case $y \geq z$ we get a similar result (changing the roles of y and z), so we can summarize both cases in

$$\left(\int_0^{z \wedge y} u^{1-2H} (y-u)^{H-3/2} (z-u)^{H-3/2} du \right) \frac{(yz)^{H-1/2}}{\beta(2-2H, H-1/2)} = |y-z|^{2H-2}.$$

Then, if we plug this in the last equality of (6.14) we get that

$$\begin{aligned}
\int_0^{t \wedge s} K_H(t, u) K_H(s, u) du &= c_H^2 \beta(2-2H, H-1/2) \int_0^t \int_0^s |y-z|^{2H-2} dz dy \\
&= R_H(t, s),
\end{aligned}$$

where the last equality comes from lemma 6.5 and the definition of c_H . $\square$

We can study further properties of K_H . Notice that directly from the definition of K_H , we have

$$\partial_t K_H(t, s) = c_H \left(\frac{t}{s} \right)^{H-1/2} (t-s)^{H-3/2}.$$

Hence, if we consider the linear operator

$$K_H^* : \mathcal{E} \rightarrow L^2([0, T])$$

defined as

$$(K_H^* \varphi)(s) = \int_s^T \varphi(t) \partial_t K_H(t, s) dt,$$

then K_H^* is an isometry between $\mathcal{E}$ and $L^2([0, T])$. Notice that in particular, K_H^* satisfies

$$(K_H^* \mathbf{1}_{[0, t]})(s) = K_H(t, s) \mathbf{1}_{[0, t]}(s)$$

(this is a trivial property to check). This is an isometry because for any $t, s \in [0, T]$ then

$$\begin{aligned} \langle K_H^* \mathbb{1}_{[0,t]}, K_H^* \mathbb{1}_{[0,s]} \rangle_{L^2([0,T])} &= \langle K_H(t, \cdot) \mathbb{1}_{[0,t]}, K_H(s, \cdot) \mathbb{1}_{[0,s]} \rangle_{L^2([0,T])} \\ &= \int_0^{t \wedge s} K(t, u) K(s, u) du \\ &= R_H(t, s). \end{aligned}$$

This operator does not only establish an isometry between $L^2([0, T])$ and $\mathcal{H}$, but also establishes an isometry between $L^2(\Omega)$ and $\mathcal{H}$ in the following way: the operator $B^H(\mathbb{1}_{[0,t]}) = B_t^H$ is an isometry. Indeed,

$$E(B_t^H B_s^H) = R_H(t, s) = \langle \mathbb{1}_{[0,t]}, \mathbb{1}_{[0,s]} \rangle_{\mathcal{H}}.$$

We can extend the definition for functions in $\mathcal{E}$. Consider $\sum_{j=1}^n a_j \mathbb{1}_{[0,t_j]} \in \mathcal{E}$. Then, we define

$$B^H \left(\sum_{j=1}^n a_j \mathbb{1}_{[0,t_j]} \right) = \sum_{j=1}^n a_j B_{t_j}^H.$$

In the same way that we have defined the operators K_H and K_H^* for functions in $\mathcal{E}$, we can define operators $\mathcal{K}_H$ and $\mathcal{K}_H^*$, that are the respective extensions to $\mathcal{H}$.

Definition 6.7. We denote by $\mathcal{K}_H$ the operator

$$\mathcal{K}_H : L^2([0, T]) \rightarrow \mathcal{H}_H := \mathcal{K}_H(L^2([0, T]))$$

defined by

$$(\mathcal{K}_H h)(t) := \int_0^t K_H(t, s) h_s ds.$$

In a similar way, we define the operator

$$\mathcal{K}_H^* : \mathcal{H} \rightarrow L^2([0, T])$$

as

$$(\mathcal{K}_H^* \varphi)(s) = \int_s^T \varphi(t) \partial_t K_H(t, s) dt.$$

The operator $\mathcal{K}_H^*$, in an analogous way as K_H^* does, defines an isometry between $\mathcal{H}$ and a closed subspace of $L^2([0, T])$ and hence, we define an isometry between $\mathcal{H}$ and a closed subspace of $L^2(\Omega)$. Indeed, let $\varphi \in \mathcal{H}$. Consider a sequence $\{\varphi_n; n \geq 1\} \subset \mathcal{E}$ such that $\varphi_n \rightarrow \varphi$ in the norm of $\mathcal{H}$ then, we define

$$B^H(\varphi) = L^2(\Omega) - \lim_{n \rightarrow \infty} B^H(\varphi_n).$$

Notice that this is a good definition in the sense that $B^H(\varphi)$ does not depend in the approximating sequence. Indeed, if $\{\varphi_n; n \geq 1\}, \{\psi_n; n \geq 1\} \subset \mathcal{E}$ are sequences such that $\varphi_n, \psi_n \rightarrow \varphi$ in $\mathcal{H}$, then

$$\|B^H(\varphi_n) - B^H(\psi_n)\|_{L^2(\Omega)} = \|\varphi_n - \psi_n\|_{\mathcal{H}} \rightarrow 0$$

when $n \rightarrow \infty$.

We denote by $H_1(B^H)$ the space generated by the random variables $\{B^H(\varphi); \varphi \in \mathcal{H}, \|\varphi\|_{\mathcal{H}} = 1\}$. This isometry between $\mathcal{H}$ and $H_1(B^H)$ allows us to define the isonormal process $B^H = \{B^H(\varphi); \varphi \in \mathcal{H}\}$ associated to $\mathcal{H}$, i.e. B^H is a stochastic process in which $B^H(\varphi)$ is a centered Gaussian random variable with variance $\|\varphi\|_{\mathcal{H}}^2$. Moreover, we can identify $B^H(\varphi)$ with a Wiener-type integral of φ with respect to B^H , $\int_0^T \varphi(s) dB_s^H$. A direct computation shows the following observation

Observation 6.8. The operator $\mathcal{K}_H^*$ can be expressed in terms of fractional integrals and derivatives as follows:

$$(\mathcal{K}_H^* \varphi)(s) = c_H \Gamma(H - 1/2) s^{1/2-H} \left(I_{T-}^{H-1/2} u^{H-1/2} \varphi(u) \right) (s),$$

$$(\mathcal{K}_H^*)^{-1}(\mathbb{1}_{[0,a]}) = \frac{1}{c_H \Gamma(H - 1/2)} s^{1/2-H} \left(D_{a-}^{H-1/2} u^{H-1/2} \right) (s) \mathbb{1}_{[0,a]}(s).$$

In general, the elements of $\mathcal{H}$ are not necessarily functions, but distributions. We define $|\mathcal{H}|$ the linear subspace of functions $\varphi \in \mathcal{H}$ such that

$$\|\varphi\|_{|\mathcal{H}|}^2 = \alpha_H \int_0^T \int_0^T |\varphi_u| |\varphi_v| |u - v|^{2H-2} du dv < \infty.$$

It is routinary work to check that $|\mathcal{H}|$ equipped with $\|\cdot\|_{|\mathcal{H}|}$ is a Banach space contained in $\mathcal{H}$. However, this space is not complete when it is equipped with the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. All this discussion of $\mathcal{K}_H^*$ is very useful, because it allows us to define the fractional Brownian motion in terms of a standard Brownian motion and, moreover, it allows us to relate the generalized Stieltjes integral $\int_0^T \varphi_t dB_t^H$ with $B^H(\varphi)$. Indeed, if $B^H = \{B_t^H; t \in [0, T]\}$ is a fractional Brownian motion of Hurst index H , then the process $W = \{W_t; t \in [0, T]\}$ defined as

$$W_t = B^H((\mathcal{K}_H^*)^{-1}(\mathbb{1}_{[0,t]}))$$

is a standard Brownian motion, and moreover

$$B_t^H = \int_0^t K_H(t, s) dW_s.$$

Indeed, W_t is a centered Gaussian random variable because B^H is an isonormal process and

$$\begin{aligned} E(W_t W_s) &= \langle B^H((\mathcal{K}_H^*)^{-1}(\mathbb{1}_{[0,t]})), B^H((\mathcal{K}_H^*)^{-1}(\mathbb{1}_{[0,s]})) \rangle_{L^2(\Omega)} \\ &= \langle (\mathcal{K}_H^*)^{-1}(\mathbb{1}_{[0,t]}), (\mathcal{K}_H^*)^{-1}(\mathbb{1}_{[0,s]}) \rangle_{\mathcal{H}} \\ &= \langle \mathcal{K}_H^*(\mathcal{K}_H^*)^{-1}(\mathbb{1}_{[0,t]}), \mathcal{K}_H^*(\mathcal{K}_H^*)^{-1}(\mathbb{1}_{[0,s]}) \rangle_{L^2([0,T])} \\ &= \langle \mathbb{1}_{[0,t]}, \mathbb{1}_{[0,s]} \rangle_{L^2([0,T])} \\ &= t \wedge s. \end{aligned}$$

In [13], the author defines the Stratonovich integral with respect to the fractional Brownian motion and if the integrand φ is deterministic, the Stratonovich integral coincides with the Wiener type integral $B^H(\varphi) = W(\mathcal{K}_H^*(\varphi))$. Then, in [17] the authors showed that the Stratonovich integral defined in [13] coincides with the pathwise integral we

have defined and used in this project. Hence, for deterministic integrands, $\int_0^T \varphi_t dB_t^H$, as it coincides with $B^H(\varphi)$, it follows a Gaussian distribution of mean zero and variance $\|\varphi\|_{\mathcal{H}}^2$. Moreover, this idea of shifting the fractional Brownian motion to obtain a standard Brownian motion is the key idea to relate the Malliavin calculus with respect to the fractional Brownian motion to the Malliavin calculus with respect to the standard Brownian motion. However, we won't deal with this type of problems in this project.

In the case of the standard Brownian motion, the operators we have defined are simplified. Indeed, $K_H(t, s) = \mathbb{1}_{[0,t]}(s)$, K_H^* is the identity map and $\mathcal{H}_H$ is the space of absolutely continuous functions vanishing at zero and the first derivatives are square integrable. For $H \neq 1/2$, $\mathcal{H}_H$ is called the fractional Cameron-Martin space. Finally, we define the last operator we need as

$$\mathcal{R}_H = \mathcal{K}_H \circ \mathcal{K}_H^* : \mathcal{H} \rightarrow \mathcal{H}_H$$

with

$$(\mathcal{R}_H \varphi)_t = \int_0^T (\mathcal{K}_H^* \mathbb{1}_{[0,t]})(s) (\mathcal{K}_H^* \varphi)(s) ds = E(B_t^H B^H(\varphi)).$$

6.2.2. Wiener Chaos expansion. Even though this and the following subsections can be developed in a general context, we will use the context of the fractional Brownian motion. The Wiener chaos decomposition is, roughly speaking, a way of decomposing the space $L^2(\Omega)$ in the orthogonal sum of spaces generated by Gaussian random variables. The suitable way of approaching this decomposition is via Hermite polynomials, which we'll recall its definition and properties.

Definition 6.9. Let $H_n(x)$ denote the n -th Hermite polynomial, which is defined by

$$\begin{cases} H_0(x) = 1, \\ H_n(x) = \frac{(-1)^n}{n!} \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n} \exp\left(-\frac{x^2}{2}\right), \quad n \geq 1. \end{cases}$$

Notice that these polynomials are the coefficients of the Taylor expansion in powers of t of the function $F(t, x) = \exp(tx - \frac{t^2}{2})$, that is,

$$F(t, x) = \sum_{n \geq 0} H_n(x) t^n.$$

From here, one can deduce the following properties about these polynomials

- $H'_n(x) = H_{n-1}(x)$ for all $n \geq 1$.
- $(n+1)H_{n+1}(x) = xH_n(x) - H_{n-1}(x)$ for all $n \geq 1$.
- $H_n(-x) = (-1)^n H_n(x)$ for all $n \geq 1$.

The reason why we are choosing Hermite polynomials and not any other family of orthogonal polynomials for our decomposition of $L^2(\Omega)$ is the following interaction between these polynomials and Gaussian random variables:

Lemma 6.10. *Let X, Y be two random variables with joint Gaussian distribution and $E(X) = E(Y) = 0$, $\text{Var}(X) = \text{Var}(Y) = 1$. Then, for any $n, m \geq 0$ we have*

$$E(H_n(X)H_m(Y)) = \begin{cases} 0, & n \neq m, \\ \frac{1}{n!}(E(XY))^n, & n = m. \end{cases}$$

Proof. See, for example, [13]. □

Recall the definition of the isonormal process associated to the Hilbert space $\mathcal{H}$, that is a process B^H indexed by $\mathcal{H}$ with the property that $B^H(\varphi) \sim N(0, \|\varphi\|_{\mathcal{H}}^2)$. The isonormal process plays a huge role in the Malliavin calculus in general. The first big appearance of this process is in this preliminary stage, in which we want to prepare the framework needed in order to work with 3 important operators. The following lemma is key in order to decompose $L^2(\Omega)$ in orthogonal subspaces.

Lemma 6.11. *The random variables $\{\exp(B^H(h)); h \in \mathcal{H}\}$ form a total subset of $L^2(\Omega, \mathcal{G}, P)$, where Ω and P are the same as in the implicit probability space $(\Omega, \mathcal{F}, P)$ we are working with and $\mathcal{G}$ is the σ -field generated by the random variables $\{B^H(h); h \in \mathcal{H}\}$.*

Proof. Can be found in [13]. □

For any $n \geq 1$, we define $H_n(B^H)$ (or H_n if there is no risk of confusion with another isonormal process) the linear subspace of $L^2(\Omega, \mathcal{F}, P)$ generated by the random variables $\{H_n(B^H(\varphi)); \|\varphi\|_{\mathcal{H}} = 1\}$, and for $n = 0$ H_0 is just the space of all constants. Notice that thanks to lemma 6.10, these spaces are orthogonal. We say that H_n is the Wiener chaos of order n . The following result ensures that we can decompose the space $L^2(\Omega, \mathcal{G}, P)$ in the sum of the Wiener chaos'

Theorem 6.12.

$$L^2(\Omega, \mathcal{G}, P) = \bigoplus_{n \geq 0} H_n.$$

Proof. Let $X \in L^2(\Omega, \mathcal{G}, P)$ such that X is orthogonal to H_n for all $n \geq 1$. We shall show that $X = 0$. Since $E(XH_n(B^H(\varphi))) = 0$ for all $n \geq 1$, $\varphi \in \mathcal{H}$ unitary and the Hermite polynomials span all the polynomials, we deduce by linearity of the expectation that $E(XB^H(\varphi)^n) = 0$ for all $n \geq 0$ and φ unitary. Hence, by the dominated convergence theorem, we have that

$$E(X \exp(tB^H(\varphi))) = 0, \quad \forall \varphi \in \mathcal{H}, \|\varphi\|_{\mathcal{H}} = 1, t \in \mathbb{R}.$$

Then, thanks to lemma 6.11 we conclude that $X = 0$ which is exactly what we wanted to show. □

From now on, J_n will denote the projection operator to the n -th Wiener chaos. This is all the framework we need in order to fully study 3 of the most important operators in the Malliavin calculus context, which are the Malliavin derivative, the divergence operator and the Ornstein-Uhlenbeck operator.

6.2.3. *The Malliavin derivative.* This operator allows us to, roughly speaking, differentiate a random variable $F(\omega)$ with respect to ω . Since we are in the canonical probability space of the fractional Brownian motion, ω is a function and therefore F is a functional. Hence, the reader with a background in functional analysis will see that this concept resembles the Fréchet differentiability of a functional. In this thesis we will discuss the relation between being Fréchet differentiable and Malliavin differentiable. A non-degeneracy condition on this Malliavin derivative operator will give us the absolute continuity of the law, as we will see in the next section. The definition of the operator is not straightforward. Indeed, we will first define the operator for a class of random variables and we will extend the definition to a broader space using an approximation argument.

Fix $H \in (1/2, 1)$. Consider $B^H = \{B^H(\varphi); \varphi \in \mathcal{H}\}$ the isonormal process associated to $\mathcal{H}$.

Definition 6.13. *A random variable F is cylindrical if F has the form*

$$(6.15) \quad F = f(B^H(\varphi_1), \dots, B^H(\varphi_n))$$

for some $\varphi_1, \dots, \varphi_n \in \mathcal{H}$ and $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$. We denote by $\mathcal{S}$ the set of all cylindrical random variables.

Notice that $\mathcal{S}$ is a dense subspace of $L^2(\Omega)$. First, we will define the Malliavin derivative for cylindrical random variables.

Definition 6.14. *Let $F \in \mathcal{S}$. We define the Malliavin derivative of F as the $\mathcal{H}$ -valued random variable defined by*

$$DF = \sum_{i=1}^n \partial_i f(B^H(\varphi_1), \dots, B^H(\varphi_n)) \varphi_i,$$

where ∂_i stands for the partial derivative with respect to the i -th argument. Given $h \in \mathcal{H}$, we can define the directional derivative of F in the h -th direction as

$$D_h F = \sum_{i=1}^n \partial_i f(B^H(\varphi_1), \dots, B^H(\varphi_n)) \langle \varphi_i, h \rangle_{\mathcal{H}} = \langle DF, h \rangle_{\mathcal{H}}.$$

In the same way as we have defined one derivative, we can define iterated derivatives. Indeed, let $(s_1, \dots, s_k) \in [0, T]^k$. Then, for $F \in \mathcal{S}$, $D_{s_1, \dots, s_k}^k F$ is the random variable with values in $\mathcal{H}^{\otimes k}$ of the form

$$D_{s_1, \dots, s_k}^k F = \sum_{j_1, \dots, j_k=1}^n \frac{\partial f}{\partial_{j_1} \dots \partial_{j_k}} f(B^H(\varphi_1), \dots, B^H(\varphi_n)) \varphi_{j_1}(s_1) \dots \varphi_{j_k}(s_k).$$

We won't be dealing with high order derivatives until section 8. That's why we will mainly focus on random variable with one Malliavin derivative well-defined.

Let $\{e_i\}_{i \in I}$ be an orthonormal system in $\mathcal{H}$. Since the isonormal process is linear, there exists a linear map A such that

$$(\varphi_1, \dots, \varphi_n)^T = A(e_1, \dots, e_n)^T.$$

Now, if $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$ then $g = f \circ A \in \mathcal{C}_b^\infty(\mathbb{R}^n)$ and

$$g(B^H(e_1), \dots, B^H(e_n)) = f(B^H(\varphi_1), \dots, B^H(\varphi_n)).$$

So, if $F \in \mathcal{S}$, we can assume that F has the form (6.15) with $\varphi_1, \dots, \varphi_n$ orthonormal elements of $\mathcal{H}$. The idea now is to explore some properties of the Malliavin derivative that coincide with the classical derivative of a function.

Proposition 6.15 (Product Rule). *Let $F, G \in \mathcal{S}$, then*

$$D(FG) = DF \cdot G + F \cdot DG.$$

Proof. Suppose that $F = f(B^H(\varphi_1), \dots, B^H(\varphi_n))$ and $G = g(B^H(\psi_1), \dots, B^H(\psi_m))$. We can write FG as

$$\begin{aligned} FG &= h((B^H(\varphi_1), \dots, B^H(\varphi_n), B^H(\psi_1), \dots, B^H(\psi_m))) \\ &= f(B^H(\varphi_1), \dots, B^H(\varphi_n))g(B^H(\psi_1), \dots, B^H(\psi_m)). \end{aligned}$$

Then

$$\begin{aligned} DF &= \sum_{i=1}^n \partial_i f(B^H(\varphi_1), \dots, B^H(\varphi_n)) \varphi_i, \\ DG &= \sum_{i=1}^m \partial_i g(B^H(\psi_1), \dots, B^H(\psi_m)) \psi_i, \end{aligned}$$

and

$$\begin{aligned} &\partial_i h((B^H(\varphi_1), \dots, B^H(\varphi_n), B^H(\psi_1), \dots, B^H(\psi_m))) \\ &= \partial_i f(B^H(\varphi_1), \dots, B^H(\varphi_n))g(B^H(\psi_1), \dots, B^H(\psi_m))\varphi_i \\ &+ f(B^H(\varphi_1), \dots, B^H(\varphi_n))\partial_i g(B^H(\psi_1), \dots, B^H(\psi_m))\psi_i \end{aligned}$$

for $i \in \{1, \dots, n\}$ and

$$\begin{aligned} &\partial_i h((B^H(\varphi_1), \dots, B^H(\varphi_n), B^H(\psi_1), \dots, B^H(\psi_m))) \\ &= f(B^H(\varphi_1), \dots, B^H(\varphi_n))\partial_i g(B^H(\psi_1), \dots, B^H(\psi_m))\psi_i \end{aligned}$$

for $i \in \{n+1, \dots, n+m\}$. Adding all derivatives we get the result. $\square$

Proposition 6.16. *Let $F \in \mathcal{S}$ and let $h \in \mathcal{H}$. The following relation holds:*

$$E(\langle DF, h \rangle_{\mathcal{H}}) = E(FB^H(h)).$$

Proof. Using Gram-Schmidt algorithm, we can construct an orthonormal system $\{e_i\}_{i \in I}$ such that $e_1 = h$, so we can express F as

$$F = f(B^H(e_1), \dots, B^H(e_n)).$$

Since e_i have norm 1, $B^H(e_i)$ are standard Gaussian random variables. Hence,

$$\langle DF, h \rangle_{\mathcal{H}} = \langle DF, e_1 \rangle_{\mathcal{H}} = \partial_1 f(B^H(e_1), \dots, B^H(e_n)).$$

Thus,

$$\begin{aligned} E(\langle DF, h \rangle_{\mathcal{H}}) &= \int_{\mathbb{R}^n} \partial_1 f(x_1, \dots, x_n) \phi_n(x_1, \dots, x_n) dx_1 \dots dx_n \\ &= - \int_{\mathbb{R}^n} f(x_1, \dots, x_n) \partial_1 \phi_n(x_1, \dots, x_n) dx_1 \dots dx_n, \end{aligned}$$

where, if we define $x = (x_1, \dots, x_n)$, then $\phi_n(x) = \phi_n(x_1, \dots, x_n) = (2\pi)^{-n/2} \exp\left(-\frac{\|x\|_2^2}{2}\right)$ is the density of a standard Gaussian random variable. This density satisfies

$$\partial_1 \phi_n(x) = -x_1 \phi_n(x),$$

so

$$E(\langle DF, h \rangle_{\mathcal{H}}) = \int_{\mathbb{R}^n} x_1 f(x_1, \dots, x_n) \phi_n(x_1, \dots, x_n) dx_1 \dots dx_n = E(B^H(e_1)F).$$

Finally, the result follows from the fact that, by definition, $e_1 = h$. $\square$

Proposition 6.17. *Let $F, G \in \mathcal{S}$ and $h \in \mathcal{H}$. Then,*

$$E(G \langle DF, h \rangle_{\mathcal{H}}) = E(FGB^H(h) - F \langle DG, h \rangle_{\mathcal{H}}).$$

Proof. Apply the proposition 6.16 to FG and use the product rule. $\square$

As mentioned before, we would like to extend the Malliavin derivative operator to a broader class of random variables. In order to do an approximation argument, we need that the operator is closable. Otherwise, if we take a sequence $\{F_n; n \geq 1\} \subset \mathcal{S}$, the limit of the sequence $\{DF_n; n \geq 1\}$ may not be well-defined. Luckily, the Malliavin derivative is indeed closable:

Proposition 6.18. *The Malliavin derivative is a closable operator from $L^2(\Omega)$ to $L^2(\Omega; \mathcal{H})$.*

Proof. Let $\{F_N; N \geq 1\}$ be a sequence of cylindrical random variables such that $F_N \rightarrow 0$ in $L^2(\Omega)$. We have to see that if the sequence of derivatives DF_N converges to η in $L^2(\Omega)$, then $\eta = 0$. For any $h \in \mathcal{H}$, Consider a cylindrical random variable F such that $FB^H(h)$ is bounded. Then

$$\begin{aligned} E(\langle \eta, h \rangle_{\mathcal{H}} F) &= \lim_{N \rightarrow \infty} E(\langle DF_N, h \rangle_{\mathcal{H}} F) \\ &= \lim_{N \rightarrow \infty} E(-F_N \langle DF, h \rangle_{\mathcal{H}} + F_N FB^H(h)) = 0, \end{aligned}$$

because $\langle DF, h \rangle_{\mathcal{H}}$ and $FB^H(h)$ are bounded and $F_N \rightarrow 0$ in $L^2(\Omega)$. Notice that the set of F 's such that $FB^H(h)$ is bounded is dense in $\mathcal{S}$. Indeed, given $G \in \mathcal{S}$ we can consider the approximating sequence $G_\varepsilon = G \exp(-\varepsilon B^H(h)^2)$. The fact that this set of F 's such that $FB^H(h)$ is bounded is dense in $\mathcal{S}$, and $\mathcal{S}$ is dense in $L^2(\Omega)$, implies that

$$\langle \eta, h \rangle_{\mathcal{H}} = 0.$$

Since this holds for every $h \in \mathcal{H}$, we deduce that $\eta = 0$ as we wanted to see. $\square$

We will denote by $\mathbb{D}^{1,2}$ the domain of D in $L^2(\Omega)$, that is, the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,2} = [E(|F|^2) + E(\|DF\|_{\mathcal{H}}^2)]^{1/2},$$

so the Malliavin derivative is well defined for $F \in \mathbb{D}^{1,2}$. An immediate observation is that $\mathbb{D}^{1,2}$ is a Hilbert space with the inner product given by

$$\langle F, G \rangle_{1,2} := E(FG) + E(\langle DF, DG \rangle_{\mathcal{H}}).$$

In the same way as we defined the space $\mathbb{D}^{1,2}$, we can define the spaces $\mathbb{D}^{k,p}$ in a similar way. Indeed, consider the following seminorm on $\mathcal{S}$:

$$\|F\|_{k,p} = \left[E(|F|^p) + \sum_{j=1}^k E(\|D^j F\|_{\mathcal{H}^{\otimes j}}^p) \right]^{1/p}.$$

Then we define $\mathbb{D}^{k,p}$ as the completion of $\mathcal{S}$ with respect to $\|\cdot\|_{k,p}$. Moreover, we define $\mathbb{D}^\infty$ as the space

$$\mathbb{D}^\infty = \bigcap_{k \geq 1} \bigcap_{p \geq 1} \mathbb{D}^{k,p}.$$

We can also define the localized version of this Malliavin-Sobolev spaces. This can be done thanks to the local property of the Malliavin derivative

Lemma 6.19. *Let $F \in \mathbb{D}^{1,2}$ such that $F = 0$ a.s. in $A \in \mathcal{F}$. Then, $DF = 0$ a.s. in A .*

Proof. See [13]. □

This property allows us to define the space $\mathbb{D}_{\text{loc}}^{1,2}$ as the set of random variables F such that there exist a sequence $\{(\Omega_n, F_n); n \geq 1\}$ with $\Omega_n \uparrow \Omega$ a.s. (that is, $P(\Omega \setminus \Omega_n) \rightarrow 0$ as $n \rightarrow \infty$), $F_n \in \mathbb{D}^{1,2}$ and $F_n = F$ almost surely in Ω_n . For any, $F \in \mathbb{D}_{\text{loc}}^{1,2}$, DF is well defined. Indeed, we can define $DF := DF_n$ in Ω_n for all $n \geq 1$.

An important property of the Malliavin derivative is that, even though we will see that some aspects of the classical differential calculus are not preserved, this operator has some properties in common with the classical derivatives. One of them, is the famous chain rule.

Lemma 6.20 (Chain Rule). *Let F be a random variable belonging to $\mathbb{D}^{1,2}$. Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be a continuously differentiable function with bounded derivative. Then, $\varphi(F) \in \mathbb{D}^{1,2}$ and*

$$D\varphi(F) = \varphi'(F) \cdot DF.$$

Proof. Notice that the statement holds true if $F \in \mathcal{S}$ and $\varphi \in \mathcal{C}_b^\infty(\mathbb{R})$. Now, let's check it for $\varphi \in \mathcal{C}^1(\mathbb{R})$ with bounded derivative. Consider $\{\alpha_n(x); n \geq 1\}$ the standard mollifier and consider, as well, the sequence $\{\varphi_n(x); n \geq 1\}$ defined as $\varphi_n(x) = (\varphi * \alpha_n)(x)$. Notice that all $\varphi_n(x)$'s belong to $\mathcal{C}_b^\infty(\mathbb{R})$. Then, for a general $F \in \mathbb{D}^{1,2}$, if we consider a sequence $\{F_n; n \geq 1\} \rightarrow F$ in $\mathbb{D}^{1,2}$ we have that

$$D\varphi_n(F_n) = \varphi'_n(F_n) \cdot DF_n.$$

On the one hand,

$$\begin{aligned} E(|\varphi_n(F_n) - \varphi(F)|^2) &\lesssim E(|\varphi_n(F_n) - \varphi(F_n)|^2) + E(|\varphi(F_n) - \varphi(F)|^2) \\ &\leq E(|\varphi_n(F_n) - \varphi(F_n)|^2) + \|\varphi'\|_{L^\infty(\mathbb{R})}^2 E(|F_n - F|^2). \end{aligned}$$

Now, when $n \rightarrow \infty$ the first term goes to zero because since φ is continuous, $\varphi_n(x) \rightarrow \varphi(x)$ uniformly in x and the second term goes to zero due to the boundedness of φ' and that $F_n \rightarrow F$ in $L^2(\Omega)$. For the convergence of the derivatives:

$$\begin{aligned} E(\|\varphi'_n(F_n)DF_n - \varphi'(F)DF\|_{\mathcal{H}}^2) &\lesssim E(\|\varphi'_n(F_n)DF_n - \varphi'_n(F_n)DF\|_{\mathcal{H}}^2) \\ &+ E(\|\varphi'_n(F_n)DF - \varphi'(F_n)DF\|_{\mathcal{H}}^2) + E(\|\varphi'(F_n)DF - \varphi'(F)DF\|_{\mathcal{H}}^2). \end{aligned}$$

Concerning the first term, notice that for all $x \in \mathbb{R}$

$$|\varphi'_n(x)| \leq \int_{\mathbb{R}} |\varphi'(x-t)|\alpha_n(t)dt \leq \|\varphi'\|_{L^\infty(\mathbb{R})}.$$

so, by choosing any $C > 0$ with $\|\varphi'\|_{L^\infty} \leq C$ we have

$$E(\|\varphi'_n(F_n)DF_n - \varphi'_n(F_n)DF\|_{\mathcal{H}}^2) \leq CE(\|DF_n - DF\|_{\mathcal{H}}^2).$$

Now, this last expression tends to zero because $F_n \rightarrow F$ in $\mathbb{D}^{1,2}$. For the second term, using the uniform convergence of φ_n to φ and the dominated convergence theorem we see that it tends to zero when $n \rightarrow \infty$. The third term also tends to zero by the dominated convergence theorem (since the statement of the dominated convergence theorem also holds when the convergence is in measure). We have then a sequence $\{F_n; n \geq 1\}$ such that $\varphi_n(F_n) \rightarrow \varphi(F)$ in $L^2(\Omega)$ and $D\varphi_n(F_n) \rightarrow \varphi'(F)DF$ in $L^2(\Omega; \mathcal{H})$ as $n \rightarrow \infty$. By closedness of the derivative operator, $\varphi(F) \in \mathbb{D}^{1,2}$ and $D\varphi(F) = \varphi'(F)DF$. $\square$

6.2.4. The divergence operator (Skorohod integral). The next operator we will introduce is the divergence operator. It is closely related to the Malliavin derivative, and it is fundamental in the computations of probability laws as we'll see later on. We first give a definition.

Definition 6.21. *The divergence operator δ is the adjoint of the operator D . That is, $\delta : L^2(\Omega; \mathcal{H}) \rightarrow L^2(\Omega)$ is an unbounded operator such that*

- i. *The domain of δ , $\text{Dom}(\delta)$, is the set of square integrable random variables $u \in L^2(\Omega; \mathcal{H})$ such that*

$$|E(\langle DF, u \rangle_{\mathcal{H}})| \leq c\|F\|_2$$

for all $F \in \mathbb{D}^{1,2}$.

- ii. *If $u \in \text{Dom}(\delta)$, then $\delta(u)$ is the unique element of $L^2(\Omega)$ characterized by*

$$(6.16) \quad E(F\delta(u)) = E(\langle DF, u \rangle_{\mathcal{H}}).$$

Remarks 6.22.

- i. In the same way D is a closed unbounded operator, δ is a closed and unbounded operator.
- ii. The relation (6.16) is sometimes called the *duality relation between D and δ* .

Let's see how δ acts on a simple class of functionals. We denote by $\mathcal{S}_{\mathcal{H}}$ be the class of smooth elementary elements of the form

$$(6.17) \quad u = \sum_{j=1}^n F_j h_j$$

where $F_j \in \mathcal{S}$ and $h_j \in \mathcal{H}$. Notice that $\mathcal{S}_{\mathcal{H}} \subset \text{Dom}(\delta)$. Indeed, if we consider u of the form (6.17), then

$$\begin{aligned} |E(\langle DF, u \rangle_{\mathcal{H}})| &= \left| \sum_{j=1}^n E(F_j \langle DF, h_j \rangle_{\mathcal{H}}) \right| \\ &\leq \sum_{j=1}^n (|E(F \langle DF_j, h_j \rangle_{\mathcal{H}})| + |E(F F_j B^H(h_j))|) \\ &\leq \|F\|_{L^2(\Omega)} \left(\sum_{j=1}^n \|\langle DF_j, h_j \rangle_H\|_{L^2(\Omega)} + \|F_j B^H(h_j)\|_{L^2(\Omega)} \right) \\ &\leq c \|F\|_{L^2(\Omega)}. \end{aligned}$$

The Malliavin derivative and the divergence operator, even though one can be thought as a derivative and the other as an integral, they are not inverse operators in the sense that $\delta(DF) \neq F$. In the next section we will talk about the Ornstein-Uhlenbeck operator, which gives an answer to what is the relation between $\delta(DF)$ and F (among other properties).

6.2.5. The Ornstein-Uhlenbeck operator. This is the last operator we need in order to have all the machinery ready to prove the main results of this project. The definition of this operator is not as straight-forward as in the Divergence and Derivative operators. We consider again the isonormal process B^H defined in the canonical probability space $(\Omega, \mathcal{F}, P)$. Recall that J_n is the projection to the n -th Wiener chaos.

Definition 6.23. *The Ornstein-Uhlenbeck semigroup is the one-parameter semigroup $\{P_t; t \geq 0\}$ of operators on $L^2(\Omega)$ defined by*

$$P_t F = \sum_{n=0}^{\infty} e^{-nt} J_n F.$$

There exists an alternative and equivalent definition of this semigroup which is, in fact, more suitable for our purposes. Consider $B^{H'}$ an independent copy of the isonormal process B^H such that both processes are defined in the probability space $(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', P \times P')$. For any $t > 0$, we consider the process

$$Z(\varphi) = e^{-t} B^H(\varphi) + \sqrt{1 - e^{-2t}} B^{H'}(\varphi)$$

Notice that this process is Gaussian (because B^H and $B^{H'}$ are Gaussian). Moreover, it is trivial that it has zero mean. The variance function of Z is, by independence of the isonormal processes,

$$E(Z(\varphi)Z(\psi)) = \langle \varphi, \psi \rangle_{\mathcal{H}}$$

Let $B^H : \Omega \rightarrow \mathbb{R}^{\mathcal{H}}$, $B^{H'} : \Omega' \rightarrow \mathbb{R}^{\mathcal{H}}$ be the canonical mappings associated to B^H and $B^{H'}$ respectively. This mappings are defined as follows:

- For $\mathbb{1}_{[0,t]}$, $B^H(\mathbb{1}_{[0,t]})(\omega) = B_t^H(\omega) = \omega(t)$.
- $B^H(\cdot)(\omega) : \mathcal{H} \rightarrow \mathbb{R}$ is the map defined such as, for any $\varphi \in \mathcal{H}$, consider a sequence of simple functions $\{s_n; n \geq 1\} \rightarrow \varphi$ in the norm of $\mathcal{H}$. Then, $B^H(\varphi)(\omega) = \lim_{n \rightarrow \infty} B^H(s_n)(\omega)$

Hence, for any ω , $B^H(\cdot)(\omega)$ can be seen as a map from $\mathcal{H}$ to $\mathbb{R}$. The canonical mapping is hence the map $\omega \rightarrow B^H(\cdot)(\omega)$. We will usually denote the image of ω by the canonical mapping as $B^H(\omega)$.

Now, given $F \in L^2(\Omega)$, let ψ_F be the measurable mapping defined $P \circ (B^H)^{-1}$ -a.s. by $\psi_F = F \circ B^H$. Then,

$$\psi_F(Z(\omega, \omega')) = \psi_F(e^{-t}B^H(\omega) + \sqrt{1 - e^{-2t}}B^{H'}(\omega'))$$

is well defined $P \times P'$ -a.s. The alternative definition of the Ornstein-Uhlenbeck semigroup relies on the following proposition

Proposition 6.24 (Mehler's formula).

$$P_t F = E'[\psi_F(e^{-t}B^H + \sqrt{1 - e^{-2t}}B^{H'})],$$

where E' denotes the expectation with respect to P' .

Proof. Can be found in [13]. □

The Ornstein-Uhlenbeck operator introduced in the next definition will be the infinitesimal generator of the one parameter family of operators P_t .

Definition 6.25. The Ornstein-Uhlenbeck operator L is the operator defined in $L^2(\Omega)$ defined by

$$LF = \sum_{n=0}^{\infty} -n J_n F$$

provided the sum is convergent in $L^2(\Omega)$.

From this definition we readily check that

$$\text{Dom}(L) = \{F \in L^2(\Omega); \sum_{n=0}^{\infty} n^2 \|J_n F\|_2^2 < \infty\}.$$

It can be checked, using multiple Wiener integrals that $\text{Dom}(L) \subset \mathbb{D}^{1,2}$. Moreover, it can also be shown that L is a symmetric unbounded operator, i.e. for every $F, G \in \text{Dom}(L)$, $E[F(LG)] = E[(LF)G]$. The next proposition shows that L is the infinitesimal generator of the semigroup $\{P_t; t \geq 0\}$.

Proposition 6.26. The Ornstein-Uhlenbeck operator L coincides with the infinitesimal generator of the Ornstein-Uhlenbeck semigroup $\{P_t; t \geq 0\}$

Proof. We shall prove that $F \in \text{Dom}(L)$ if and only if

$$\lim_{t \downarrow 0} \frac{P_t F - F}{t} = G,$$

where the limit is taken in the $L^2(\Omega)$, and in such case $G = LF$. Assume first $F \in \text{Dom}(L)$. Applying the very definitions, we have

$$\begin{aligned} F &= \sum_{n=0}^{\infty} J_n F \\ P_t F &= \sum_{n=0}^{\infty} e^{-nt} J_n F \\ LF &= \sum_{n=0}^{\infty} -n J_n F. \end{aligned}$$

Hence,

$$E \left[\left(\frac{P_t F - F}{t} - LF \right)^2 \right] = E \left[\left(\sum_{n=0}^{\infty} \left(\frac{e^{-nt} - 1}{t} + n \right) J_n F \right)^2 \right].$$

Now, using that the Wiener chaos spaces are orthogonal, we readily check that

$$E \left[\left(\sum_{n=0}^{\infty} \left(\frac{e^{-nt} - 1}{t} + n \right) J_n F \right)^2 \right] = \sum_{n=0}^{\infty} \left(\frac{e^{-nt} - 1}{t} + n \right)^2 E(J_n F^2).$$

Since $(e^{-nt} - 1)/t \leq 0$, the last expression is less or equal than $\sum_{n=0}^{\infty} n^2 \|J_n F\|_2^2$, which is a finite quantity by hypothesis. Hence, we can switch the limit as $t \downarrow 0$ with the sum, and using the elementary equality

$$\lim_{t \downarrow 0} \frac{e^{-nt} - 1}{t} = -n,$$

we deduce that $G = LF$ (by the uniqueness of the L^2 limit). Conversely, assume that $\lim_{t \downarrow 0} \frac{P_t F - F}{t} = G$ in $L^2(\Omega)$. Notice that $J_n P_t F = e^{-nt} J_n F$. Hence,

$$J_n G = J_n F \lim_{t \downarrow 0} \frac{e^{-nt} - 1}{t} = -n J_n F.$$

From the fact that $G \in L^2(\Omega)$ and consequently $\sum_{n=0}^{\infty} \|J_n G\|^2 < \infty$ we deduce that $F \in \text{Dom}(L)$ and, moreover, $G = LF$. $\square$

The following relation between the Malliavin derivative, the divergence operator and the Ornstein-Uhlenbeck operator is a necessary tool in order to bound the density functions of the solutions to the SDEs:

Proposition 6.27. *Let $F \in L^2(\Omega)$. Then, $F \in \text{Dom}(L)$ if and only if $F \in \text{Dom}(\delta D)$ (i.e., $F \in \mathbb{D}^{1,2}$ and $DF \in \text{Dom}(\delta)$). In this case, $\delta DF = -LF$.*

Proof. The proof uses techniques that haven't been developed in this project. Nevertheless, these techniques are not very difficult and one can check that the proof in [13] is short and direct. $\square$

6.3. Study of probability laws. This subsection is devoted to show how, depending on the actions δ and D have on a random variable F tell us information about the law of F . We first give a general result that ensures the absolute continuity of F and then we will give a criterion to check if the law of F is absolutely continuous with respect to the Lebesgue measure. The first result, even though is very important in order to compute density functions, is not suitable for the differential equations setting. Indeed, the hypothesis are hard to check. On the other hand, the absolute continuity criterion that is presented is easier to check in the differential equations setting, but it doesn't give us information about the density of the solution. This is the reason why we want to go further, and in section 7 we study the properties of the density function of the solution to stochastic differential equations that is ensured to exist thanks to this criterion.

6.4. Computation of probability laws. One of the most powerful applications of the duality relation between D and δ is that it allows us to compute in a (somehow) explicit way the density function of a random variable. Indeed, we have the following result:

Proposition 6.28. *Let F be a random variable in $\mathbb{D}^{1,2}$. Suppose that $DF/||DF||_{\mathcal{H}}^2$ belongs to the domain of δ . Then, the law of F has a continuous and bounded density $p_F(x)$ given by*

$$p_F(x) = E \left[\mathbb{1}_{\{F > x\}} \delta \left(\frac{DF}{||DF||_{\mathcal{H}}^2} \right) \right].$$

Proof. Let ψ be a non-negative smooth function with compact support. Let φ be the antiderivative of ψ given by

$$\varphi(x) = \int_{-\infty}^x \psi(t) dt.$$

By the chain rule, $\varphi(F) \in \mathbb{D}^{1,2}$ and $D\varphi(F) = \varphi'(F)DF = \psi(F)DF$. Hence

$$\langle D\varphi(F), DF \rangle_{\mathcal{H}} = \psi(F) ||DF||_{\mathcal{H}}^2.$$

In other words,

$$\psi(F) = \left\langle D\varphi(F), \frac{DF}{||DF||_{\mathcal{H}}^2} \right\rangle_{\mathcal{H}}.$$

Applying expectations to both sides, and using the duality relation between D and δ we have

$$(6.18) \quad E[\psi(F)] = E \left[\left\langle D\varphi(F), \frac{DF}{||DF||_{\mathcal{H}}^2} \right\rangle_{\mathcal{H}} \right] = E \left[\varphi(F) \delta \left(\frac{DF}{||DF||_{\mathcal{H}}^2} \right) \right].$$

Given any interval $[a, b] \subset \mathbb{R}$, we can consider a sequence of smooth positive compact-supported functions ψ_n such that $\psi_n(x) \rightarrow \mathbb{1}_{[a,b]}(x)$ and (6.18) holds for every $n \geq 1$. By the dominated convergence theorem,

$$P(a \leq F \leq b) = E[\mathbb{1}_{[a,b]}(F)] = E \left[\left(\int_{-\infty}^F \mathbb{1}_{[a,b]}(x) dx \right) \delta \left(\frac{DF}{\|DF\|_{\mathcal{H}}^2} \right) \right].$$

Now, notice that

$$\int_{-\infty}^F \mathbb{1}_{[a,b]}(x) dx = \int_{a \wedge F}^{b \wedge F} dx = \int_a^b \mathbb{1}_{\{F > x\}} dx.$$

Applying now Fubini's theorem, we deduce

$$P(a \leq F \leq b) = \int_a^b E \left[\mathbb{1}_{\{F > x\}} \delta \left(\frac{DF}{\|DF\|_{\mathcal{H}}^2} \right) \right] dx.$$

□

Notice that, in particular, if F belongs to $\mathbb{D}^{1,2}$ and $DF/\|DF\|_{\mathcal{H}}^2$ belongs to the domain of δ , the law of F is absolutely continuous with respect to the Lebesgue measure. Our objective is relaxing a bit this condition and trying to make an absolute continuity criterion not very restrictive and easy to check.

6.4.1. An absolute continuity criterion. As mentioned before, the Malliavin calculus is sometimes used to prove that the law of a certain random vector is absolutely continuous with respect to the Lebesgue measure. The way both aspects are related lies in the following proposition.

Proposition 6.29. *Let $F \in \mathbb{D}_{loc}^{1,2}$ be a random variable such that $\|DF\|_{\mathcal{H}} > 0$ a.s.. Then, the law of F is absolutely continuous with respect to the Lebesgue measure.*

Proof. We must show that every subset $E \in \mathcal{B}(\mathbb{R})$ of measure zero satisfies $\mathbb{P}(F \in E) = 0$. Equivalently, we must show that $\mathbb{1}_E(F) = 0$ almost surely. Let E be any borel set. Then since $F \in \mathbb{D}^{1,2}$ there exists $\Omega_n \subset \Omega$ and $F_n \in \mathbb{D}^{1,2}$ such that $F_n = F$ in Ω_n . Moreover, since $\Omega_n \uparrow \Omega$, there exists an index $n \geq 0$ such that

$$P(F \in E) = P((F \in E) \cap \Omega_n) = P(F_n \in E).$$

Hence, we can assume by this localization argument that $F \in \mathbb{D}^{1,2}$ without loss of generality. Note that if F has an absolutely continuous law with respect to the Lebesgue measure and ϕ is a diffeomorphism, then the law of $\phi(F)$ is also absolutely continuous with respect to the Lebesgue measure. Hence, we can assume that F takes values in $(-1, 1)$, and thus we can restrict ourselves to Borel sets $E \subset (-1, 1)$ of measure zero.

Let P_F denote the law of F . There exists a sequence of continuous functions f_n converging pointwise to $\mathbb{1}_E$ almost everywhere with respect to the measure $P_F + dx$ (i.e., $f_n(x) \rightarrow \mathbb{1}_E(x)$ almost everywhere in $(-1, 1)$ and $f_n(F) \rightarrow \mathbb{1}_E(F)$ almost surely). Define $\varphi_n(t) = \int_{-1}^t f_n(x) dx$. Then φ_n is of class $\mathcal{C}^1$ and each φ_n has bounded derivative. Applying the chain rule, we find that $\varphi_n(F) \in \mathbb{D}^{1,2}$ and $D\varphi_n(F) = \varphi_n'(F)DF = f_n(F)DF$.

Since $f_n(x) \rightarrow \mathbf{1}_E$ pointwise almost everywhere, $\varphi_n(x) \rightarrow 0$ almost everywhere. By the dominated convergence theorem, we have:

$$\begin{aligned}\varphi_n(F) &\rightarrow 0 \quad \text{in } L^2(\Omega), \\ f_n(F)DF &\rightarrow \mathbf{1}_E(F)DF \quad \text{in } L^2(\Omega; \mathcal{H})\end{aligned}$$

Since the Malliavin derivative is a closed operator, we have $\mathbf{1}_E(F)DF = 0$ almost surely, and thus $\mathbf{1}_E(F)\|DF\|_{\mathcal{H}}^2 = 0$ almost surely. Since $\|DF\|_{\mathcal{H}} > 0$ almost surely, we deduce that $\mathbf{1}_E(F) = 0$ almost surely. $\square$

Hence, the study of the law of a random variable is reduced to the study of the non-degeneracy of its Malliavin derivative.

Recall that, unless otherwise indicated, the probability space in which we are working on is the canonical probability space associated to the fractional Brownian motion, so we have to keep in mind that each $\omega \in \Omega$ is a function and random variables are now functionals. In the next subsection, we will argue that if a random variable is Fréchet differentiable in the directions of the Cameron-Martin space, then it is locally Malliavin differentiable. Once we apply this results to the deterministic equations, we will be able to conclude that the law of the solution to an SDE is absolutely continuous.

6.5. Continuously differentiable random variables in the Cameron-Martin space directions. Let $\mathcal{H}$ be a Hilbert space and let $(\Omega, \mathcal{H}, P)$ be an abstract Wiener space in the sense of Gross, that is, the underlying probability space is $(\Omega, \mathcal{F}, P)$ where Ω is a separable Banach space, P is a Gaussian probability measure on Ω and $\mathcal{F}$ is the completion of the Borel σ -field of Ω (with respect to P). Moreover, $\mathcal{H}$ is a separable Hilbert space that is continuously embedded in Ω via the injection $i : \mathcal{H} \hookrightarrow \Omega$ and the following property is satisfied:

$$\int_{\Omega} e^{i\langle y, \omega \rangle} P(d\omega) = \exp\left(\frac{-1}{2}\|y\|_{\mathcal{H}}^2\right)$$

for any $y \in \Omega^* \subset H^* \cong H$. We will denote by $\mathcal{H}^1$ the image of $\mathcal{H}$ under the injection i . This space $\mathcal{H}^1$ is usually denoted as the Cameron-Martin space. In the case of the fractional Brownian motion, we will use the notation $\mathcal{H}_H$ instead of $\mathcal{H}^1$, but in order to study a general setting, we'll keep $\mathcal{H}^1$ for the moment. This is all the information we need about abstract Wiener spaces for this project. If the reader is interested in this topic, they can refer to the paper of L. Gross [9].

Definition 6.30. *We say that a random variable F is $\mathcal{H}$ -continuously differentiable if for almost every $\omega \in \Omega$, the mapping $h \rightarrow F(\omega + i(h))$ is continuously differentiable in $\mathcal{H}$. The set of all $\mathcal{H}$ -continuously differentiable random variables is denoted by $\mathcal{C}_{\mathcal{H}}^1$.*

The objective of this section is to prove that $\mathcal{C}_{\mathcal{H}}^1 \subset \mathcal{D}_{\text{loc}}^{1,2}$ because we want to prove that if a random variable is Fréchet differentiable in the directions of the Cameron-Martin space, then it is also locally Malliavin differentiable. Moreover,

Consider, for any $A \in \mathcal{F}$, the function

$$\rho_A(\omega) = \inf\{\|h\|_{\mathcal{H}}; \omega + i(h) \in A\}.$$

From this definition, we can clearly notice that $\rho_A(\omega) = 0$ if $\omega \in A$ and $\rho_A(\omega) = \infty$ if $\omega \notin A + \mathcal{H}^1$. Moreover, given any $A \in \mathcal{F}$, we can construct an ε -neighbourhood of A with the distance of $\mathcal{H}^1$. Indeed, by density we can extend the distance of $\mathcal{H}^1$ to a distance $d_1 : \Omega \times \Omega \rightarrow \mathbb{R}$ and construct a neighbourhood of A with respect to this distance.

Proposition 6.31. *The function ρ_A satisfies the following properties:*

- (1) *If $A \subset B$ then $\rho_A(\omega) \geq \rho_B(\omega)$.*
- (2) *Fixed $A \in \mathcal{F}$, ρ satisfies the following Lipschitz condition:*

$$|\rho_A(\omega + i(h)) - \rho_A(\omega)| \leq \|h\|_{\mathcal{H}}$$
- (3) *If $\{A_n; n \geq 1\}$ is such that $A_n \uparrow A$, then $\rho_{A_n}(\omega) \downarrow \rho_A(\omega)$.*
- (4) *If G is σ -compact and $\phi \in \mathcal{C}_0^\infty(\mathbb{R})$ then $\phi(\rho_G) \in \mathbb{D}^{1,2}$ (with the convention $\phi(\infty) = 0$) and $\|D(\phi(\rho_G))\|_{\mathcal{H}} \leq \|\phi'\|_{L^\infty}$.*

In order to prove this proposition, we will use the following lemma:

Lemma 6.32. *Let F be a random variable satisfying*

$$|F(\omega + i(h)) - F(\omega)| \leq c\|h\|_{\mathcal{H}} \quad \forall h \in \mathcal{H}.$$

Then $F \in \mathbb{D}^{1,2}$ and $\|DF\|_{\mathcal{H}} \leq c$ a.s.

The proof of this lemma is omitted because it is out of the scope of this project. We are ready now to prove the proposition.

Proof.

- (1) Let $A \subset B$. Notice that if $h \in \mathcal{H}$ is such that $\omega + i(h) \in A$, then $\omega + i(h) \in B$. Hence, $A'_\omega = \{h \in \mathcal{H}; \omega + i(h) \in A\} \subset \{h \in \mathcal{H}; \omega + i(h) \in B\} = B'_\omega$. Since $\|\cdot\|_{\mathcal{H}}$ is a continuous map and $A'_\omega \subset B'_\omega$, then

$$\rho_A(\omega) = \inf_{h \in A'_\omega} \|h\|_{\mathcal{H}} \geq \inf_{h \in B'_\omega} \|h\|_{\mathcal{H}} = \rho_B(\omega).$$

- (2) Define the sets

$$\begin{aligned} A_\omega &= \{g \in \mathcal{H}; \omega + i(g) \in A\} \\ A_{\omega,h} &= \{g \in \mathcal{H} : \omega + i(h) + i(g) \in A\}. \end{aligned}$$

Notice that $g \in A_{h,\omega} \iff h + g \in A_\omega$. Now,

$$\begin{aligned} \rho_A(\omega + i(h)) &= \inf\{\|g\|_{\mathcal{H}}; g \in A_{\omega,h}\} \\ &= \inf\{\|g\|_{\mathcal{H}}; g + h \in A_\omega\} \\ &= \inf\{\|\varphi - h\|_{\mathcal{H}}; \varphi \in A_\omega\}. \end{aligned}$$

Hence

$$|\rho_A(\omega + i(h)) - \rho_A(\omega)| \leq |\inf\{\|\varphi - h\|_{\mathcal{H}} - \|\varphi\|_{\mathcal{H}}; \varphi \in A_\omega\}|.$$

Since $\|\varphi - h\|_{\mathcal{H}} - \|\varphi\|_{\mathcal{H}} \leq \|h\|_{\mathcal{H}}$ for all $\varphi \in A_\omega$, we get the desired result.

- (3) The sequence $\rho_{A_n}(\omega)$ is decreasing on n , so in order to ensure the convergence, we will see that $\inf_n \rho_{A_n} \leq \rho_A(\omega)$ for all $\omega \in \Omega$. First, if $\omega \notin A + \mathcal{H}^1$ then $\rho_{A_n}(\omega) = \rho_A(\omega) = \infty$ and hence the result is clear. If $\omega \in A + \mathcal{H}^1$ then for every $\varepsilon > 0$ there exists $h \in \mathcal{H}$ such that $\omega + i(h) \in A$ and $\|h\|_{\mathcal{H}} \leq \rho_A(\omega) + \varepsilon$. Since $A_n \uparrow A$, there exists n_0 such that for all $n \geq n_0$, $h \in A_n$. Then, it is clear that

$$\rho_{A_n}(\omega) \leq \|h\|_{\mathcal{H}} \leq \rho_A(\omega) + \varepsilon.$$

Taking the infimum on n and then the infimum on $\varepsilon > 0$ we prove the result.

- (4) Consider $G = \bigcup_{n \geq 1} K_n$ where K_n is an increasing sequence of compact sets. By the last item, we have seen that $\rho_{K_n} \downarrow \rho_G$, hence $\phi(\rho_{K_n}) \rightarrow \phi(\rho_G)$. Since ρ_{K_n} is measurable for all n $\phi(\rho_G)$ is bounded and measurable. Moreover, by the second item and the mean value theorem we have

$$|\phi(\rho_G(\omega + i(h))) - \phi(\rho_G(\omega))| \leq \|\phi'\|_{L^\infty} \|h\|_{\mathcal{H}}.$$

Now, using lemma 6.32 we derive the result.

Now we are ready to prove the result that relates the $C_{\mathcal{H}}^1$ class of random variables with $\mathbb{D}_{\text{loc}}^{1,2}$.

Proposition 6.33. *We have $C_{\mathcal{H}}^1 \subset \mathbb{D}_{\text{loc}}^{1,2}$.*

Proof. Let $F \in C_{\mathcal{H}}^1$. We define the following sets

$$A_n = \left\{ \omega \in \Omega; \sup_{\|h\|_{\mathcal{H}} \leq 1/n} |F(\omega + i(h))| \leq n, \right. \\ \left. \sup_{\|h\|_{\mathcal{H}} \leq 1/n} \|DF(\omega + i(h))\|_{\mathcal{H}} \leq n \right\}.$$

Since $F \in C_{\mathcal{H}}^1$, then $\Omega = \bigcup_{n \geq 1} A_n$. Now, for any A_n let G_n be any σ -compact set such that $G_n \subset A_n$ and $P(G_n) = P(A_n)$. Consider $\phi \in \mathcal{C}_0^\infty(\mathbb{R})$ be a Urysohn function associated to $[-1/3, 1/3]$ and $[-2/3, 2/3]$ such that $|\phi'(t)| \leq 4$. Define the random variables

$$F_n = \phi(n\rho_{G_n})F.$$

Then, F_n satisfies the following properties:

- (1) $F_n = F$ on G_n .
- (2) F_n is bounded. To see it, notice that

$$|F_n| \leq \mathbb{1}_{\{\rho_{G_n} < \frac{2}{3n}\}} |F| \leq n.$$

- (3) For any h with $\|h\|_{\mathcal{H}} \leq \frac{1}{3n}$, F_n satisfies the following Lipschitz condition:

$$|F_n(\omega + i(h)) - F_n(\omega)| \leq (8n^2 + n)\|h\|_{\mathcal{H}}.$$

By lemma 6.32, $F_n \in \mathbb{D}^{1,2}$ and properties 1 and 2 ensures us that $F_n = F$ along an increasing subsequence of sets. Hence, by definition, $F \in \mathbb{D}^{1,2}$. $\square$

$\square$

6.6. Application to SDEs driven by a fractional Brownian motion. Let's consider the following SDE

$$(6.19) \quad X_t = x_0 + \int_0^t \sigma(s, X_s) dB_s^H + \int_0^t b(s, X_s) ds,$$

where $x_0 \in \mathbb{R}$ and σ, b satisfy (H_i) for $i = 1, 2, 3$. Under this hypothesis, we have the existence and uniqueness of solutions. We want to establish a connection between the Malliavin derivative and the Fréchet derivative. To do so, assume that σ and b satisfy (H_4) (so that we have Fréchet differentiability of the solutions). We will make use of the following lemma:

Lemma 6.34. *Let $\varphi \in \mathcal{H}$. Then, for all $\varepsilon > 0$ and $h \in \mathcal{H}$ we have*

$$B^H(\varphi)(\omega + \varepsilon \mathcal{R}_H(h)) = B^H(\varphi)(\omega) + \varepsilon \langle \varphi, h \rangle_{\mathcal{H}}.$$

Proof. First let's prove the result for $\varphi = \mathbb{1}_{[0,t]}$. In this case

$$B_t^H(\omega + \varepsilon \mathcal{R}_H(h)) = (\omega + \varepsilon \mathcal{R}_H(h))_t = \omega_t + \varepsilon \mathcal{R}_H(h)_t.$$

Now, by definition,

$$\mathcal{R}_H(h)_t = \langle K_H^* \mathbb{1}_{[0,t]}, h \rangle_{L^2[0,T]}.$$

Hence,

$$B_t^H(\omega + \varepsilon \mathcal{R}_H(h)) = \omega_t + \varepsilon \langle \mathbb{1}_{[0,t]}, h \rangle_{\mathcal{H}} = B_t^H(\omega) + \varepsilon \langle \mathbb{1}_{[0,t]}, h \rangle_{\mathcal{H}},$$

so the result is true for indicators. Since the map $\varphi \rightarrow B^H(\varphi)$ is linear, we know that by linearity the result is true for functions in $\mathcal{E}$. To prove the result for $\varphi \in \mathcal{H}$ we will use an approximation argument. It is clear that if we have a sequence $\{\varphi_n; n \geq 1\} \subset \mathcal{E}$ such that $\varphi_n \rightarrow \varphi$ in $\mathcal{H}$ then

$$B^H(\varphi) + \varepsilon \langle \varphi, h \rangle_{\mathcal{H}} = L^2(\Omega) - \lim_{n \rightarrow \infty} B^H(\varphi_n) + \varepsilon \langle \varphi_n, h \rangle_{\mathcal{H}}.$$

In particular, the convergence holds in $L^1(\Omega)$. Consider the shift operator $T_h^\varepsilon(\omega) = \omega + \varepsilon \mathcal{R}_H(h)$. We have to see that

$$B^H(\varphi) \circ T_h^\varepsilon = L^2(\Omega) - \lim_{n \rightarrow \infty} B^H(\varphi_n) \circ T_h^\varepsilon.$$

Using Girsanov's theorem,

$$\begin{aligned} E \left(|B^H(\varphi_n) \circ T_h^\varepsilon - B^H(\varphi) \circ T_h^\varepsilon| \right) \\ = E \left(|B^H(\varphi_n) - B^H(\varphi)| \exp \left(\varepsilon B^H(h) - \frac{\varepsilon^2}{2} \|h\|_{\mathcal{H}}^2 \right) \right). \end{aligned}$$

By Cauchy-Schwarz's inequality, the convergence in $L^1(\Omega)$ of $B^H(\varphi_n) \circ T_h^\varepsilon$ to $B^H(\varphi) \circ T_h^\varepsilon$ holds. Now, by uniqueness of the $L^1(\Omega)$ limit, we have

$$B^H(\varphi)(\omega + \varepsilon \mathcal{R}_H(h)) = B^H(\varphi)(\omega) + \varepsilon \langle \varphi, h \rangle_{\mathcal{H}}.$$

as wanted □

Proposition 6.35. *Let X be the solution to (6.19) with σ and b satisfying (H_4) . Then, the derivative of X in the sense of the Malliavin calculus satisfies the equation*

$$(6.20) \quad D_s X_t = \sigma(s, X_s) + \int_s^t \partial_x \sigma(u, X_u) D_s X_u dB_u^H + \int_s^t \partial_x b(u, X_u) D_s X_u du$$

if $s \leq t$ and 0 if $s > t$.

Proof. The first observation is that the Fréchet derivative of X in the direction $\mathcal{R}_H h$ satisfies

$$D_{\mathcal{R}_H h} X_t = \langle DX_t, h \rangle_{\mathcal{H}},$$

where DX_t in the right hand side of the equality is the Malliavin derivative of X at time t . To see it, consider a cylindrical random variable of the form

$$F = f(B^H(\varphi_1), \dots, B^H(\varphi_n)).$$

Then,

$$\begin{aligned} \langle DF, h \rangle_{\mathcal{H}}(\omega) &= \sum_{i=1}^n \partial_i f(B^H(\varphi_1)(\omega), \dots, B^H(\varphi_n)(\omega)) \langle \varphi_i, h \rangle_{\mathcal{H}} \\ &= \frac{d}{d\varepsilon} f(B^H(\varphi_1)(\omega) + \varepsilon \langle \varphi_1, h \rangle_{\mathcal{H}}, \dots, B^H(\varphi_n)(\omega) + \varepsilon \langle \varphi_n, h \rangle_{\mathcal{H}}). \end{aligned}$$

by lemma 6.34, the last equality coincides with $\frac{d}{d\varepsilon} F(\omega + \varepsilon \mathcal{R}_H(h))$ which is precisely the Fréchet derivative of F in the $\mathcal{R}_H h$ direction. The general case follows from an approximation argument. This is precisely the relation between the Fréchet derivative and the Malliavin derivative that we have been talking about in the whole project. Now, using the representation for the deterministic case, we have

$$D_{\mathcal{R}_H h} X_t = \int_0^t \phi_t(r) d(\mathcal{R}_H h)(r).$$

Notice that, using Fubini's theorem, we have the following equality

$$(\mathcal{R}_H h)(r) = \int_0^r \int_0^u (\partial_u K_H(u, s) (\mathcal{K}_H^* h)(s)) ds du.$$

Hence,

$$d(\mathcal{R}_H(h))(r) = \int_0^r \partial_r K_H(r, u) (\mathcal{K}_H^* h)(u) du.$$

This implies that

$$\begin{aligned}
D_{\mathcal{R}_H(h)}X_t &= \int_0^t \phi_t(r) d(\mathcal{R}_H h)(r) \\
&= \int_0^t \int_0^r \phi_t(r) \partial_r K_H(r, u) (\mathcal{K}_H^* h)(u) du dr \\
&= \int_0^T \int_0^r \phi_t(r) \partial_r K_H(r, u) (\mathcal{K}_H^* h)(u) du dr \\
&= \int_0^T (\mathcal{K}_H^* h)(u) \left(\int_u^T \phi_t(r) \partial_r K_H(r, u) dr \right) du \\
&= \int_0^T (\mathcal{K}_H^* h)(u) (\mathcal{K}_H^* \phi_t)(u) du \\
&= \langle \phi_t, h \rangle_{\mathcal{H}}.
\end{aligned}$$

This implies that $\phi_t(s) = D_s X_t$ and, because of the differential equation that satisfies ϕ_t , we conclude that the malliavin derivative satisfies equation (6.20). $\square$

Theorem 6.36. *Assume, in addition, that $\sigma(0, x_0) \neq 0$. Then, the solution to (6.19) has an absolutely continuous law with respect to the Lebesgue measure.*

Proof. We have to see that $\|DX_t\|_{\mathcal{H}} > 0$ almost surely. To do so, let's assume that $P(\|DX_t\|_{\mathcal{H}} = 0) > 0$. We have seen that the operator $\mathcal{K}_H^*$ is an isometry between $\mathcal{H}$ and a closed subspace $E \subset L^2([0, T])$. Let $\{e_n; n \geq 1\}$ be a complete orthonormal system of E and we define $f_n := (\mathcal{K}_H^*)^{-1}(e_n)$. Hence, we can write

$$DX_t = \sum_{n=1}^{\infty} \langle DX_t, f_n \rangle_{\mathcal{H}} f_n,$$

so

$$\|DX_t\|_{\mathcal{H}}^2 = \sum_{n=1}^{\infty} |\langle DX_t, f_n \rangle_{\mathcal{H}}|^2.$$

If we assume that $P(\|DX_t\|_{\mathcal{H}} = 0) > 0$ then, with probability different from zero, we have $\langle DX_t, f_n \rangle = 0$ for all $n \geq 1$. We will use now the deterministic representation. From the relation between the Fréchet derivative and the Malliavin derivative, we have that

$$0 = \langle DX_t, f_n \rangle_{\mathcal{H}} = D_{\mathcal{R}_H f_n} X_t.$$

By definition of the Fréchet derivative and lemma 5.1 we have that

$$0 = D_{\mathcal{R}_H f_n} X_t = D_2 F(0, X)^{-1} \circ D_1 F(0, X)(\mathcal{R}_H f_n).$$

Since $D_2 F(0, X)$ is a linear homeomorphism, then

$$D_2 F(0, X)^{-1} \circ D_1 F(0, X)(\mathcal{R}_H f_n) = 0 \iff D_1 F(0, X)(\mathcal{R}_H f_n) = 0.$$

By definition of $D_1 F$ we have that

$$D_1 F(0, X)(\mathcal{R}_H f_n)(t) = - \int_0^t \sigma(s, X_s) d(\mathcal{R}_H f_n)(s).$$

We have done a similar computation in the previous lemma. Indeed,

$$\begin{aligned} \int_0^t \sigma(s, X_s) d(\mathcal{R}_H f_n)(s) &= \int_0^T \sigma(s, X_s) \mathbb{1}_{[0,t]}(s) d(\mathcal{R}_H f_n)(s) \\ &= \langle \sigma(\cdot, X) \mathbb{1}_{[0,t]}, f_n \rangle_{\mathcal{H}}, \end{aligned}$$

so $\langle \sigma(\cdot, X) \mathbb{1}_{[0,t]}, f_n \rangle_{\mathcal{H}} = 0$ for all $n \geq 1$. Adding all this terms, we have

$$0 = \sum_{n=1}^{\infty} |\langle \sigma(\cdot, X) \mathbb{1}_{[0,t]}, f_n \rangle_{\mathcal{H}}|^2 = \|\sigma(\cdot, X) \mathbb{1}_{[0,t]}\|_{\mathcal{H}}^2.$$

Hence, $\sigma(s, X_s) = 0$ for all $s \in [0, t]$. In particular, for $s = 0$ we have $\sigma(0, x_0) = 0$ which contradicts the hypothesis of $\sigma(0, x_0) \neq 0$. The contradiction arises from assuming that $P(\|DX_t\|_{\mathcal{H}}^2 = 0) > 0$ so $\|DX_t\|_{\mathcal{H}} > 0$ almost surely. By the criterion of absolute continuity 6.29, we conclude that the law of X is absolutely continuous with respect to the Lebesgue measure. $\square$

7. GAUSSIAN BOUNDS FOR THE DENSITY OF SDES DRIVEN BY A FRACTIONAL BROWNIAN MOTION

In the last section we proved one big result, that is that whenever σ, b satisfy (H_4) , the solution X_t to

$$X_t = x_0 + \int_0^t \sigma(s, X_s) dB_s^H + \int_0^t b(s, X_s) ds$$

has a density $p_t(x)$. Moreover, thanks to the Malliavin calculus, we have an explicit expression of the density $p_t(x)$. The drawback of this expression is that computing $p_t(x)$ in terms of elementary functions is not an easy problem; it is sometimes impossible to give such expression. Our objective now is, in order to understand a bit better the behaviour of $p_t(x)$, giving lower and upper bounds for $p_t(x)$ in terms of Gaussian density functions i.e. we are looking for Gaussian upper and lower bounds for $p_t(x)$. This is, again, not an easy problem. However, there exists a general method for bounding density functions of random variables that can be applied to very simple equations. We first give a discussion about how this method works:

7.1. A general bounding technique. This technique was introduced by Nourdin and Viens in [12]. In general, the tools that the authors use are more accessible than other popular techniques (see, for example, [10]). The price we pay for having a more accessible technique is that the family of equations for which this method is suitable is not very broad. The method relies on the following theorem:

Theorem 7.1. *Let $F \in \mathbb{D}^{1,2}$ with zero mean, let $g_F(x)$ be the function defined as*

$$g_F(x) = E \left(\langle DF, DL^{-1}F \rangle_{\mathcal{H}} | F = x \right).$$

The law of F has a density $p_F(x)$ if and only if the random variable $g_F(F)$ is strictly positive almost surely. In this case, the support of p_F , $\text{supp } p_F$, is a closed interval of

$\mathbb{R}$ containing zero and, for almost $x \in \text{supp } p_F$,

$$p_F(x) = \frac{E(|F|)}{2g_F(x)} \exp\left(-\int_0^x \frac{z}{g_F(z)} dz\right).$$

Before proving this theorem, we shall give an explanation about what does L^{-1} mean. Let $F \in L^2(\Omega)$, then we define $L^{-1}F$ as

$$L^{-1}F = \sum_{n \geq 1} -\frac{1}{n} J_n F.$$

Notice that $L^{-1}F \in \text{Dom}(L)$, indeed, using that J_n are projections to jointly orthogonal spaces

$$\sum_{n \geq 1} n^2 \|J_n L^{-1}F\|_2^2 = \sum_{n \geq 1} \|J_n F\|_2^2 \leq E(F^2) < \infty.$$

Moreover, since the sum that defines L^{-1} does not contain the term $n = 0$ that corresponds to $E(F)$, we have

$$LL^{-1}F = F - E(F).$$

Hence, L^{-1} behaves as the inverse operator of L in the case of centered random variables. Now we can give the proof:

Proof. First, let's prove that for every $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ in $\mathcal{C}^1(\mathbb{R})$ we have

$$E(F\varphi(F)) = E(\varphi'(F)g_F(F)).$$

Indeed,

$$\begin{aligned} E(F\varphi(F)) &= E(L(L^{-1}F) \cdot \varphi(F)) \\ &= E(\delta(D(-L^{-1}F)) \cdot \varphi(F)) \\ &= E(\langle D\varphi(F), -DL^{-1}F \rangle_{\mathcal{H}}) \\ &= E(\varphi'(F) \langle DF, DL^{-1}F \rangle_{\mathcal{H}}) \\ &= E(\varphi'(F)g_F(F)), \end{aligned}$$

where in the second equality we have used that $\delta(DX) = -LX$ for a random variable $X \in \mathbb{D}^{1,2}$, in the third equality we have used the duality relation between δ and D , in the fourth equality we have used the chain rule and in the last equality we applied the definition of $g_F(F)$. Now, given $B \in \mathcal{B}(\mathbb{R})$, we can approximate $\varphi(x) = \int_{-\infty}^x \mathbb{1}_B(z) dz$ by $\mathcal{C}^1$ functions and by the dominated convergence theorem and previous equality we have

$$E\left(F \int_{-\infty}^F \mathbb{1}_B(z) dz\right) = E(\mathbb{1}_B g_F(F)).$$

Now, if B has Lebesgue measure zero, then from the fact that $g_F(F) > 0$ a.s. we have that $E(\mathbb{1}_B g_F(F)) = 0$ implies $P(F \in B) = 0$ a.s. By Radon-Nikodym's theorem, F has a density.

Let's prove the converse implication. Assume F has a density $p_F(x)$. Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function and let Φ be a primitive function of φ . Then,

$$\begin{aligned} E(\varphi(F)g_F(F)) &= E(\Phi(F)F) \\ &= \int_{\mathbb{R}} \Phi(x)xp_F(x)dx \\ &= \int_{\mathbb{R}} \varphi(x) \left(\int_x^{\infty} zp_F(z)dz \right) dx \\ &= E \left(\varphi(F) \frac{\int_F^{\infty} xP_F(x)dx}{p_F(F)} \right). \end{aligned}$$

Hence,

$$g_F(F) = \int_F^{\infty} zp_F(z)dz.$$

Since $F \in \mathbb{D}^{1,2}$, we know by [13] that $\text{supp } p_F = [\alpha, \beta]$ for $-\infty \leq \alpha < \beta \leq \infty$. Since F has zero mean, it has to be $\alpha < 0$ and $\beta > 0$. For every $x \in (\alpha, \beta)$, we define

$$G(x) = \int_x^{\infty} zp_F(z)dz.$$

This functions is differentiable almost everywhere in (α, β) with derivative $-xp_F(x)$. On the other hand,

$$(7.1) \quad G(x) = p_F(x)g_F(x).$$

We are lead therefore to the following ode:

$$\frac{G'(x)}{G(x)} = -\frac{x}{g_F(x)}.$$

Hence,

$$\log G(x) = \log G(0) - \int_0^x \frac{z}{g_F(z)}dz.$$

Taking exponentials in both sides and taking into account that $2G(0) = E(|F|)$ we have

$$G(x) = \frac{1}{2}E(|F|) \exp \left(- \int_0^x \frac{z}{g_F(z)}dz \right).$$

Now, from equation (7.1), we deduce that

$$p_F(x) = \frac{E(|F|)}{2g_F(x)} \exp \left(- \int_0^x \frac{z}{g_F(z)}dz \right).$$

□

As a corollary, we deduce a method to find bounds on density functions:

Corollary 7.2. *If there exist $\sigma_{min}, \sigma_{max} > 0$ such that*

$$\sigma_{min}^2 \leq g_F(F) \leq \sigma_{max}^2,$$

then F has density function satisfying

$$\frac{E(|F|)}{2\sigma_{\max}^2} \exp\left(-\frac{x^2}{2\sigma_{\min}^2}\right) \leq p_F(x) \leq \frac{E(|F|)}{2\sigma_{\min}^2} \exp\left(-\frac{x^2}{2\sigma_{\max}^2}\right).$$

Computing $\langle DF, DL^{-1}F \rangle_{\mathcal{H}}$ is still a non-trivial issue. However, if F is a solution to an SDE driven by a Gaussian process that can be identified with an isonormal process we can find methods to compute $g_F(x)$.

Concerning our case, let B^H be a fractional Brownian motion with Hurst parameter $H > 1/2$. As an abuse of notation, we will denote by B^H the isonormal process asociated to $\mathcal{H}$. A way of computing $g_F(x)$ is the following:

Proposition 7.3. *Assume $DF = \Phi_F(B^H)$ for a measurable function $\Phi_F : \mathbb{R}^{\mathcal{H}} \rightarrow \mathcal{H}$. Then, we have*

$$\langle DF, DL^{-1}F \rangle_{\mathcal{H}} = \int_0^\infty e^{-\theta} \langle \Phi_F(B^H), E' \left(\Phi_F(e^{-\theta} B^H + \sqrt{1 - e^{-2\theta}} B^{H'}) \right) \rangle_{\mathcal{H}} d\theta$$

and, therefore,

$$g_F(F) = \int_0^\infty e^{-\theta} \mathbb{E} \left(\langle \Phi_Z(B^H), \Phi_Z(e^{-\theta} B^H + \sqrt{1 - e^{-2\theta}} B^{H'}) \rangle_{\mathcal{H}} | F \right) d\theta,$$

where $B^{H'}$ stands for an independent copy of B^H such that B^H and $B^{H'}$ are defined in the probability space $(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', P \times P')$, E' denotes the expectation with respect to P' and $\mathbb{E}$ denotes the expectation with respect to $P \times P'$.

In general, finding bounds for the density of stochastic differential equations driven by a fractional Brownian motion is a hard problem. In this section, we give an example on how to apply this method for SDEs where the diffusion term is deterministic.

Theorem 7.4. *Consider the following SDE*

$$X_t = x_0 + \int_0^t \sigma_s dB_s^H + \int_0^t b(s, X_s) ds,$$

where b satisfies (H_4) , $\sigma : [0, T] \rightarrow \mathbb{R}$ is a deterministic function satisfying (H_4) and moreover is (λ, Λ) -elliptic, that is, there exists two constants $0 < \lambda < \Lambda$ such that $\lambda \leq |\sigma_t| \leq \Lambda$ for all $t \geq 0$. Then, the density function $p_t(x)$ of X_t satisfies

$$\frac{E(|X_t - m_t|)}{2\Lambda^2 t^{2H}} \exp\left(-\frac{(x - m_t)^2}{2\lambda^2 t^{2H}}\right) \leq p_t(x) \leq \frac{E(|X_t - m_t|)}{2\lambda^2 t^{2H}} \exp\left(-\frac{(x - m_t)^2}{2\Lambda^2 t^{2H}}\right),$$

where $m_t = E(X_t)$.

Proof. A first comment is that, from the hypothesis of the theorem, the existence of $p_t(x)$ is ensured. A second comment is that we can assume without loss of generality that $\lambda \leq \sigma_t \leq \Lambda$. Indeed, if $\sigma_t < 0$ then we can consider $-B^H$ as a driving noise, which is still a fractional Brownian motion of Hurst parameter $H > 1/2$. The last comment is that if we want to apply theorem 7.1 and corollary 7.2, we need that the random variable we study is centered. This is not a problem, since we can consider the random variable

$F = X_t - E(X_t)$ (which is centered) and, since $E(X_t)$ is deterministic, $DF = DX_t$. Let's first study the Malliavin derivative of X_t . Since σ is deterministic, we have for every $s < t$,

$$\begin{aligned} D_s X_t &= \sigma_s + \int_s^t \sigma'_u D_s \sigma_u dB_u^H + \int_s^t \partial_x b(u, X_u) D_s X_u du \\ &= \sigma_s + \int_s^t \partial_x b(u, X_u) D_s X_u du. \end{aligned}$$

This equation has an explicit solution given by

$$D_s X_t = \sigma_s \exp \left(\int_s^t \partial_x b(u, X_u) du \right)$$

From hypothesis (H_4) , we know that there exists a constant C such that

$$e^{-C(t-s)} \leq \exp \left(\int_s^t \partial_x b(u, X_u) du \right) \leq e^{C(t-s)}.$$

Moreover, from the fact that $t, s \in [0, T]$ and $s < t$, we have

$$e^{-CT} \leq \exp \left(\int_s^t \partial_x b(u, X_u) du \right) \leq e^{CT}.$$

Using now the ellipticity condition on σ_t , we find that

$$\lambda e^{-CT} \leq D_s X_t \leq \Lambda e^{CT},$$

which is a lower and upper bound for $D_s X_t$ that is uniform in s and ω . Hence, for every $s < t$ it also holds that

$$\lambda e^{-CT} \leq D_s X_t^\theta \leq \Lambda e^{CT}$$

for every $\theta \geq 0$ where

$$F^\theta(\omega, \omega') = F(e^{-\theta}\omega + \sqrt{1 - e^{-2\theta}}\omega').$$

Now, using that for every $\varphi, \psi \in \mathcal{H}$,

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \int_0^T \int_0^T \varphi_u \psi_v |u - v|^{2H-2} du dv,$$

we have that

$$\begin{aligned} g_F(F) &= \int_0^\infty e^{-\theta} \mathbb{E} \left(E' \left[\langle DF, DF^\theta \rangle_{\mathcal{H}} | F \right] \right) d\theta \\ &= \int_0^\infty e^{-\theta} \mathbb{E} \left(E' \left[\langle DX_t, DX_t^\theta \rangle_{\mathcal{H}} | F \right] \right) d\theta \\ &= \int_0^\infty e^{-\theta} \mathbb{E} \left(E' \left[\int_0^t \int_0^t D_u X_t D_v X_t^\theta |u - v|^{2H-2} du dv | F \right] \right) d\theta. \end{aligned}$$

We now recall the following facts:

- (i) $\lambda^2 e^{-2CT} \leq D_u X_t D_v X_t^\theta \leq \Lambda^2 e^{2CT}$,
- (ii) $\int_0^t \int_0^t |u - v|^{2H-2} du dv = \alpha_H t^{2H}$,

$$(iii) \int_0^\infty e^{-\theta} d\theta = 1.$$

With this information, one can readily check that naming $c_1 = \alpha_H e^{-2CT}$ and $c_2 = \alpha_H e^{2CT}$,

$$c_1 \lambda^2 t^{2H} \leq g_F(F) \leq c_2 \Lambda^2 t^{2H}.$$

By corollary 7.2, we then deduce that $F = X_t - E(X_t)$ has a density function $p_F(x)$ and, moreover, for every x in the support of $p_F(x)$ we have

$$(7.2) \quad \frac{E(|F|)}{2\Lambda^2 t^{2H}} \exp\left(-\frac{x^2}{2\lambda^2 t^{2H}}\right) \leq p_F(x) \leq \frac{E(|F|)}{2\lambda^2 t^{2H}} \exp\left(-\frac{x^2}{2\Lambda^2 t^{2H}}\right).$$

Now, in order to deduce the bound for $p_t(x)$ we use the formula for the density function of a change of variables. Notice that $X_t = h(F)$ where $h(x) = x + m_t$ and $m_t = E(X_t)$. Hence, $h^{-1}(x) = x - m_t$ and $|(h^{-1})'(x)| = 1$. Then, the density of X_t is

$$p_t(x) = p_F(h^{-1}(x)) = p_F(x - m_t).$$

Thus, going back to (7.2) with $F = X_t - m_t$ we find

$$\frac{E(|X_t - m_t|)}{2\Lambda^2 t^{2H}} \exp\left(-\frac{(x - m_t)^2}{2\lambda^2 t^{2H}}\right) \leq p_t(x) \leq \frac{E(|X_t - m_t|)}{2\lambda^2 t^{2H}} \exp\left(-\frac{(x - m_t)^2}{2\Lambda^2 t^{2H}}\right)$$

as desired. $\square$

8. FURTHER WORK AND OPEN PROBLEMS: GAUSSIAN BOUNDS FOR STOCHASTIC DELAY DIFFERENTIAL EQUATIONS

As I mentioned in the introduction, my advisor Carles and me tried to follow the lines of [1], [15] and [10] in order to get lower bounds for the density of stochastic delay differential equations. We made a good progress, but we weren't able to show the desired final result due to some mistakes in the paper [1] that one of the authors pointed out when we contacted him. This section is a compilation of results we managed to prove and some results that we conjecture to be true but we haven't done a proof yet. My advisor and me are currently working on this problem.

8.1. Stochastic Delay Differential Equations. This section is devoted to the study of a particular family of SDEs. Consider a fractional Brownian motion B^H of Hurst parameter $H > 1/2$. Consider also a real constant $r > 0$. A Stochastic Delay Differential Equation (SDDE) is an equation of the form

$$(8.1) \quad \begin{cases} X_t = \eta_0 + \int_0^t \sigma(X_{s-r}) dB_s^H + \int_0^t b(X_s) ds, & t \in [0, T], \\ X_t = \eta_t, & t \in [-r, 0]. \end{cases}$$

Where $\eta \in \mathcal{C}([-r, 0])$. In general, SDDEs appear biological evolution models and it is also a good model for rough volatilities in the financial context (see [6]). In this project, we will cover briefly the problem of existence and uniqueness of solutions, as well as the problem of existence of a density function of the solution. However, we will put our focus on giving bounds on this density function (when it exists).

8.1.1. *Existence and uniqueness of solutions. Absolute continuity of the solutions.* The fact that we have a delayed diffusion term allows us to do an induction argument. Indeed, we will prove that there exists a solution in each interval $[kr, (k+1)r]$, $k \geq 0$ and, in each of these intervals, we will use that in the diffusion term we deal with the solution found for the previous interval. This strategy of inductively finding a solution allows us to reduce the regularity. The existence and uniqueness of solutions relies on the following result:

Theorem 8.1. *Assume σ and b satisfy (H_1) and (H_2) respectively with $\rho = 1/\alpha$ and $0 < \alpha < 1/2 \wedge \beta$. Let $g \in W_T^{1-\alpha, \infty}(-r, T)$. Then, there exists a unique solution to equation*

$$(8.2) \quad \begin{cases} x_t = \eta_0 + \int_0^t \sigma(x_{s-r}) dg_s + \int_0^t b(x_s) ds & t \in [0, T], \\ x_t = \eta_t & t \in [-r, 0). \end{cases}$$

Moreover, the solution x belongs to $W_0^{\alpha, \infty}(0, T)$.

Proof. The proof goes through similar lines as the proof of theorem 5.2. The full proof is a simplified version of the one in [7]. $\square$

As it was discussed, if $u = \{u_t; t \in [0, T]\}$ is a stochastic process whose trajectories belong to $W_0^{\alpha, \infty}(0, T)$ then the integral

$$\int_0^T u_s dB_s^H$$

exists pathwise in the generalized Stieltjes sense, and the paths of the integral process are Hölder continuous of order $1 - \alpha$. Hence, we can apply theorem 8.1 to conclude that there exists a unique Hölder continuous solution to equation (8.1).

The next objective is studying the Malliavin regularity of the solution to equation (8.1). Notice that the hypothesis needed to prove that the delayed equation has a unique solution are no different than the hypothesis required to prove theorem 5.2. However, under less conditions we can prove that the solution has a density.

Theorem 8.2. *Assume $\sigma, b \in \mathcal{C}_b^2(\mathbb{R})$ and $\eta \in \mathcal{C}^{1-\alpha}(-r, 0)$. Set X the solution of (8.1) and assume $\sigma > 0$. Then, for any $t \in (0, T]$, the law of X_t is absolutely continuous with respect to the Lebesgue measure.*

Proof. The proof is an adaptation of the more complex result in [2]. $\square$

Our main objective will be using conditional Malliavin calculus tools in order to give bounds on the density of the solution to equation (8.1) under a bit stronger conditions than in theorem 8.2.

8.2. Gaussian Bounds for the Density of SDDE. The main result that we wanted to prove is the following:

Conjecture 8.3. Assume $\sigma, b \in \mathcal{C}_b^\infty(\mathbb{R})$ and $\eta \in \mathcal{C}^{1-\alpha}(-r, 0)$. Assume that σ is (λ, Λ) -elliptic, that is, there exists two real constants $0 < \lambda < \Lambda$ such that $\lambda \leq |\sigma(x)| \leq \Lambda$. Then, there exists constants $c_1, c_2 > 0$ such that for all $t \in (0, T]$, the solution of the equation (8.1) has a density function $p_t(x)$ satisfying

$$(8.3) \quad \frac{c_1}{t^H} \exp\left(-\frac{c_2(x - m_t)^2}{t^{2H}}\right) \leq p_t(x),$$

where $m_t = E(X_t)$.

8.2.1. *The case $t \in [0, r]$.* The strategy we use to prove this result depends on the time interval we are studying. First, in order to treat the case $t \in [0, r]$, we will use the same approach as in theorem 7.4. The result for this time interval is the following:

Proposition 8.4. *Let X be the solution (8.1) with σ, b satisfying the hypothesis of the theorem 8.2. If σ is (λ, Λ) -elliptic, then for any $t \in (0, r]$, the density function $p_t(x)$ of X_t satisfies*

$$\begin{aligned} \frac{E(|X_t - E(X_t)|)}{2\Lambda^2 t^{2H}} \exp\left(-\frac{(x - E(X_t))^2}{2\lambda^2 t^{2H}}\right) &\leq p_t(x) \\ &\leq \frac{E(|X_t - E(X_t)|)}{2\lambda^2 t^{2H}} \exp\left(-\frac{(x - E(X_t))^2}{2\Lambda^2 t^{2H}}\right). \end{aligned}$$

Proof. We will apply the method recently introduced. Recall that for $t \in (0, r]$, X_t solves the equation

$$X_t = \eta_0 + \int_0^t \sigma(\eta(s - r)) dB_s^H + \int_0^t b(X_s) ds.$$

Hence, the Malliavin derivative of X_t in the direction $s < t$ satisfies

$$D_s X_t = \sigma(\eta(s - r)) + \int_0^t \sigma'(\eta(u - r)) D_s \eta(u - r) dB_u^H + \int_0^t b'(X_u) D_s X_u du.$$

Since η is deterministic, $D_s \eta(u - r) = 0$, so the equation satisfied by $D_s X_t$ reduces to

$$D_s X_t = \sigma(\eta(s - r)) + \int_0^t b'(X_u) D_s X_u du.$$

This is an ode with initial condition $\sigma(\eta(s - r))$, so the explicit solution to this equation is

$$D_s X_t = \sigma(\eta(s - r)) \exp\left(\int_0^t b'(X_u) du\right).$$

Moreover, from the fact that $|b'(X_u)| \leq \|b'\|_{L^\infty}$ and the ellipticity condition on σ we deduce that

$$(8.4) \quad \lambda e^{-Mr} \leq D_s X_t \leq \Lambda e^{Mr}.$$

The important conclusion of this bound is that (8.4) holds uniformly with respect to ω . From theorem 8.2, we have $X_t \in \mathbb{D}^{1,2}$, so we will apply the method to the random

variable $F = X_t - E(X_t)$. Notice that since $E(X_t)$ is a real number, $DF = DX_t$. Moreover, recall that if $\varphi, \psi \in \mathcal{H}$, then

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \int_0^T \int_0^T |\varphi_u| |\psi_v| |u - v|^{2H-2} du dv$$

If we define $F^\theta := F(e^{-\theta}\omega + \sqrt{1 - e^{-2\theta}}\omega')$ we can write g_F as

$$g_F(F) = \int_0^\infty e^{-\theta} \mathbb{E} \left[E' \left(\int_0^t \int_0^t D_u X_t D_v X_t^\theta |u - v|^{2H-2} du dv | F \right) \right] d\theta$$

using the bounds (8.4) we can easily find that

$$\lambda^2 t^{2H} \lesssim g_F(F) \lesssim \Lambda^2 t^{2H}$$

One can also check that $O(\lambda t^H) \leq E(|X_t - E(X_t)|) \leq O(\Lambda t^H)$, so the lower bound is consistent with equation (8.3). $\square$

Repeating the same strategy for the intervals $[kr, (k+1)r]$ turns out to be inefficient. Let's check for example the interval $[r, 2r]$ to illustrate the problem. In this interval of times, the Malliavin derivative satisfies the equation

$$D_s X_t = \sigma(X_{s-r}) + \int_{r+s}^t \sigma'(X_{u-r}) D_s X_{u-r} dB_u^H + \int_s^t b'(X_u) D_s X_u du,$$

which has an explicit solution

$$D_s X_t = \left(\sigma(X_{s-r}) + \int_{r+s}^t \sigma'(X_{u-r}) D_s X_{u-r} dB_u^H \right) \exp \left(\int_s^t b'(X_u) du \right).$$

The exponential term can be easily bounded by

$$e^{-\|b'\|_{L^\infty} r} \leq \exp \left(\int_s^t b'(X_u) du \right) \leq e^{\|b'\|_{L^\infty} r}.$$

However, the lack of information on the sign of σ' and the impossibility to give a uniform lower bound for B^H makes it difficult to use the method that we have been using until now. We need therefore to change the strategy.

8.2.2. *The case $t \geq r$.* Inspired by [1], [10], and [15] we will make use of the representation of the density of X_t given by the following proposition:

Proposition 8.5. *If $\sigma, b \in \mathcal{C}_b^\infty(\mathbb{R})$ and σ satisfies is (λ, Λ) -elliptic, then the solution X_t is a non-degenerate random variable for all $t \in (0, T]$ in the sense that*

- (i) $X_t \in \mathbb{D}^\infty$.
- (ii) $\|DX_t\|_{\mathcal{H}} > 0$ almost surely and $\|DX_t\|_{\mathcal{H}}^{-1} \in \bigcap_{p \geq 1} L^p(\Omega)$.

In particular, the density of X_t admits the representation $p_t(x) = E(\delta_x(X_t))$ where δ_x stands for the Dirac delta measure at the point x .

Notice that if X_t is non-degenerate random variable, then X_t satisfies the hypothesis of proposition 8.4. Hence, the bound still holds in this new setting. For $t \in [r, T]$, the analysis of the density $p_t(x)$ relies on the analysis of the quantity $E(\delta_x(X_t))$, which will be done by evaluating the conditional densities of an approximating sequence $\{F_n; 0 \leq n \leq N\}$ such that $F_N = X_t$. To do so, we will construct a sequence of partitions $\pi_N = \{r = t_0 < \dots < t_N = t\}$ and associated random variables $F_n \in \mathcal{F}_{t_n}$. We define the random variables F_n in an iterative way as follows: Since we want $F_N = X_t$, we have to define

$$F_N = \eta_0 + \int_0^{t_N} \sigma(X_{s-r}) dB_s^H + \int_0^{t_N} b(X_s) ds.$$

Moreover, we would also like that, for all $n \geq 0$,

$$F_n = F_{n-1} + I_n + R_n,$$

where I_n is a random variable that is Gaussian when conditioned to $\mathcal{F}_{t_{n-1}}$ and R_n is the remainder term. Hence, we define

$$I_n = \int_0^{t_n} \sigma(X_{s-r}) dB_s^H, \quad I_n = \int_0^{t_n} b(X_s) ds.$$

In order to bound from below quantities related to the Hölder norm of the fractional Brownian motion and some Malliavin derivatives that will appear, we will make use of some bump functions. For any $p, \gamma > 0$, we define the functionals

$$N_{\gamma,p}^n(B^H) := \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} \frac{|B_u^H - B_v^H|^{2p}}{|v - u|^{2\gamma p + 2}} du dv.$$

This functionals control the Hölder norms of the fractional Brownian motion in the following way:

Proposition 8.6. *Let $H > 1/2$ and p such that $0 < \gamma < H - \frac{1}{2p}$. Then,*

$$\|B^H\|_{C^\gamma([t_{n-1}, t_n])} \leq C_{\gamma,p} [N_{\gamma,p}^n(B^H)]^{1/2p}.$$

Proof. See for instance [8]. □

Now, for any $M, \varepsilon > 0$ we introduce the family of bump functions $\Phi_{M,\varepsilon} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ which are regularizations of $\mathbf{1}_{\{x \leq M\}}(x)$. They are any smooth function such that $\Phi_{M,\varepsilon}(x) = 0$ if $x > M + \varepsilon$ and $\Phi_{M,\varepsilon}(x) = 1$ if $x \in [0, M - \varepsilon]$. For a specific construction, see for example [1]. If $\varepsilon = 1$, we will simply write Φ_M .

The first localization we will take into account is $\Phi_M := \Phi_M(N_{\gamma,p}^n(B^H))$ and M is an arbitrarily big positive constant. The other important localization is going to be

$$\Phi_{c_n, \varepsilon_n} := \Phi_{c_n, \varepsilon_n} (\|DR_n\|_{\mathcal{H}[t_{n-1}, t_n]}^2),$$

for some constants c_n, ε_n of the form

$$(8.5) \quad c_n = \frac{\lambda}{4} \int_{t_{n-1}}^{t_n} K(t, s)^2 ds, \quad \varepsilon_n = \frac{c_n}{2},$$

where

$$||\varphi||_{\mathcal{H}[t_{n-1}, t_n]}^2 = \alpha_H \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} |\varphi_u| |\varphi_v| |u - v|^{2H-2} du dv.$$

This constants are chosen in a convenient way and the importance of this choice will be visible later on. In order to streamline the notation, we set

$$E_s(\cdot) := E(\cdot | \mathcal{F}_s).$$

Since, for every $n \geq 0$, $E(\delta_x(F_n)) = E[E_{t_{n-1}}(\delta_x(F_n))]$, we will give bounds for $E_{t_{n-1}}(\delta_x(F_n))$ and we will then conclude (8.3) by applying expectations. Notice that

$$E_{t_{n-1}}(\delta_x(F_n)) = E_{t_{n-1}}[\delta_x(F_n)(1 - \Phi_M \Phi_{c_n, \varepsilon_n})] + E_{t_{n-1}}[\delta_x(F_n) \Phi_M \Phi_{c_n, \varepsilon_n}],$$

and since $0 \leq \Phi_M \Phi_{c_n, \varepsilon_n} \leq 1$, we conclude

$$E_{t_{n-1}}(\delta_x(F_n)) \geq E_{t_{n-1}}[\delta_x(F_n) \Phi_M \Phi_{c_n, \varepsilon_n}].$$

Now, we have that $F_n = F_{n-1} + I_n + R_n$, so using Taylor we have

$$E_{t_{n-1}}[\delta_x(F_n) \Phi_M \Phi_{c_n, \varepsilon_n}] = J_{1,n} + J_{2,n} + J_{3,n}$$

where

$$(8.6) \quad J_{1,n} = E_{t_{n-1}}[\delta_x(F_{n-1} + I_n)],$$

$$(8.7) \quad J_{2,n} = E_{t_{n-1}}[\delta_x(F_{n-1} + I_n)(\Phi_M \Phi_{c_n, \varepsilon_n} - 1)],$$

$$(8.8) \quad J_{3,n} = E_{t_{n-1}}[\Phi_M \Phi_{c_n, \varepsilon_n} \int_0^1 \frac{d}{dx} \delta_x(F_{n-1} + I_n + \rho R_n) R_n d\rho].$$

The idea is showing that the main contribution comes from $J_{1,n}$, while $|J_{2,n}|$ and $|J_{3,n}|$ will tend to 0 as $N \rightarrow \infty$. If we are able to prove that, then because

$$E_{t_{n-1}}[\delta_x(F_n)] \geq J_{1,n} - |J_{2,n}| - |J_{3,n}|,$$

we will be able to proof the lower bound. In order to get bound for this quantities, we need some notions of conditional Malliavin calculus.

Definition 8.7. For any F random variable and $t \in [0, T]$, let $||F||_{k,p,t}$ and $\Gamma_{F,t}$ be the quantities defined by

$$(8.9) \quad ||F||_{k,p,t} = \left[E_t(|F|^p) + \sum_{j=1}^k E_t(||DF||_{\mathcal{H}[t,T]^{\otimes j}}^p) \right]^{1/p},$$

and

$$(8.10) \quad \Gamma_{F,t} = ||DF||_{\mathcal{H}[t,T]}^2.$$

Another key property is the conditional integration by parts formula that we find in [10] and the authors in [1] have adapted to a setting suitable for our purposes.

Proposition 8.8. *Suppose F is a non-degenerate random variable, let $Z \in \mathbb{D}^\infty$ be an $\mathcal{F}_s$ -measurable random variable such that $F + Z$ is a non-degenerate random variable and let $G \in \mathbb{D}^\infty$ such that $F + Z$. Then, for any function $g \in \mathcal{C}_p^\infty(\mathbb{R})$ (i.e. g is smooth and g , together with all its partial derivatives, have at most polynomial growth) then there exists a random variable $H_{(1)}(F, G)$ such that*

$$E_s[g'(F + Z)G] = E_s[g(F + Z)H_{(1)}^s(F, G)],$$

and, for q_1, q_2, q_3 such that $\frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3}$ then the following estimate holds:

$$\|H_{(1)}^s(F, G)\|_{p,s} \leq c \|\Gamma_{F,s}^{-1}\|_{2^{n-1}q_1,s}^n \|F\|_{n+2,2^n q_2,s}^{2(n+1)} \|G\|_{n,q_3,s}.$$

Let's start with the bound for $J_{1,n}$.

Proposition 8.9. *Let $J_{1,n}$ be defined as (8.6). Then, if the partition π_N is such that $t_{j+1} - t_j < r$ and $\int_{t_{n-1}}^{t_n} K(t, s)^2 ds = \alpha_H \frac{(t-r)^{2H}}{N} =: \sigma_N^2$, then*

$$J_{1,n} \geq \frac{1}{\sqrt{2\pi\Lambda^2\sigma_N^2}} \exp\left(-\frac{(x - F_{n-1})^2}{2\lambda^2\sigma_N^2}\right).$$

Proof. We need to find the conditional density of $F_{n-1} + I_n$ with respect to $\mathcal{F}_{t_{n-1}}$. On the one hand, F_{n-1} is $\mathcal{F}_{t_{n-1}}$ -measurable, so it behaves like a constant. On the other hand, recall that

$$I_n = \int_{t_{n-1}}^{t_n} \sigma(X_{s-r}) dB_s^H.$$

from the fact that $t_{n-1} - t_n = \sigma_N^2 < r$, we have that $\sigma(X_{s-r})$ is $\mathcal{F}_{t_{n-1}}$ measurable for all $s \in [t_{n-1}, t_n]$. If we name $f(s-r) = E_{t_{n-1}}[\sigma(X_{s-r})]$ then we have that $F_{n-1} + I_n$, when conditioned to $\mathcal{F}_{t_{n-1}}$ follows a Normal distribution with mean F_{n-1} and variance $\|f(\cdot - r)\|_{\mathcal{H}[t_{n-1}, t_n]}^2$. Moreover, from the fact that

$$\lambda^2\sigma_N^2 \leq \|f(\cdot - r)\|_{\mathcal{H}[t_{n-1}, t_n]}^2 \leq \Lambda^2\sigma_N^2,$$

we conclude that

$$J_{1,n} \geq \frac{1}{\sqrt{2\pi\Lambda^2\sigma_N^2}} \exp\left(-\frac{(x - F_{n-1})^2}{2\lambda^2\sigma_N^2}\right),$$

as wanted. $\square$

The proof of this proposition requires some conditions on the partition. The next lemma ensures that this partition exists:

Lemma 8.10. *Let $t_0, \dots, t_N$ be the partition of $[r, t]$ satisfying the conditions of proposition 8.9. Then:*

1. *The partition is constructed in a unique way.*
2. *We have $0 \leq t_{n+1} - t_n \leq c_H N^{-1/2H}$.*
3. *The constants c_n are $\alpha_H \frac{\lambda(t-r)^{2H}}{4N}$.*

The proof of this lemma is omitted, but it's a matter of computations and estimations based on $K(t, s)$. Our next objective is bounding $|J_{2,n}|$.

Proposition 8.11. *Let $J_{2,n}$ be defined as in (8.7). Then, there exists positive constants $c_{\lambda,\Lambda}$, k_1 , p_1 and a real constant k , all of them independent of N , such that*

$$|J_{2,n}| \leq c_{\lambda,\Lambda}(\sigma_N^2)^k L_{n,N}^{\gamma,p}(k_1, p_1),$$

where

$$L_{n,N}^{\gamma,p}(k_1, p_1) = \|1 - \Phi_M \Phi_{c_n, \varepsilon_n}\|_{k_1, p_1, t_{n-1}}.$$

Proof. By the conditional integration by parts formula of the Malliavin calculus, we have

$$|J_{2,n}| = \left| E_{t_{n-1}} \left[\mathbf{1}_{F_{n-1}+I_n > x} H_{(1)}^{t_{n-1}}(I_n, 1 - \Phi_M \Phi_{c_n, \varepsilon_n}) \right] \right|.$$

Moreover, by [13] we have that there exists constants $k_i, p_i = 1, \dots, 4$ such that

$$\begin{aligned} & \left| E_{t_{n-1}} \left[\mathbf{1}_{F_{n-1}+I_n > x} H_{(1)}^{t_{n-1}}(I_n, 1 - \Phi_M \Phi_{c_n, \varepsilon_n}) \right] \right| \\ & \leq c \|\Gamma_{I_n, t_{n-1}}^{-1}\|_{p_3, t_{n-1}}^{k_3} \|I_n\|_{k_2, p_2, t_{n-1}}^{k_4} \|1 - \Phi_M \Phi_{c_n, \varepsilon_n}\|_{k_1, p_1, t_{n-1}}. \end{aligned}$$

Since, by definition, $L_N^n(k_1, p_1) = \|1 - \Phi_M \Phi_{c_n, \varepsilon_n}\|_{k_1, p_1, t_{n-1}}$, we only need to bound the first two terms. Recall that $\Gamma_{I_n, t_{n-1}} := \|DI_n\|_{\mathcal{H}[t_{n-1}, t_n]}^2$. Since we know the explicit expression of I_n , it is rather direct to check that, for all $s \in [t_{n-1}, t_n]$,

$$D_s I_n = \sigma(X_{s-r}) + \int_{t_{n-1}}^{t_n} \sigma'(X_{u-r}) D_s X_{u-r} dB_u^H = \sigma(X_{s-r}),$$

because $D_s X_{u-r} = 0$ due to the fact that $s > u - r$. Moreover, this implies that all the derivatives of order greater than 2 vanish. From here we deduce that

$$\lambda^2 \sigma_N^2 \leq \Gamma_{I_n, t_{n-1}} \leq \Lambda^2 \sigma_N^2,$$

and therefore $\Gamma_{I_n, t_{n-1}}^{-1} \leq \frac{1}{\lambda^2 \sigma_N^2}$. Concerning the Malliavin norm of I_n , we can conclude, using the conditional Gaussianity of I_n and the information we extracted from the derivatives that

$$\begin{aligned} \|I\|_{k_2, p_2, t_{n-1}}^{k_4} & \leq C(\sigma_N^2 \Lambda^2)^{k_4/2}, \\ \|\Gamma_{I_n, t_{n-1}}^{-1}\|_{p_3, t_{n-1}}^{k_3} & \leq (\lambda^2 \sigma_N^2)^{-k_3}. \end{aligned}$$

These two estimates prove the statement. $\square$

This last proposition suggests that we should give a bound on $L_N^n(k_1, p_1)$. The bound on this functional relies on the following statement:

Proposition 8.12. *Consider $\gamma \in (1/2, H)$ and $k_1, p_1 \geq 1$. Let $L_{n,N}^{\gamma,p}(k_1, p_1)$ be the random variable defined as in proposition 8.11. Then, for any $p \geq \frac{k_1}{2}$, $\gamma > 0$ such that $2p(H - \gamma) - 2 > k_1 H$, the following holds true: For any $\alpha > 0$ there exists a positive constant c independent of N such that*

$$E(L_{n,N}^{\gamma,p}(k_1, p_1)) \leq cN^{-\alpha}.$$

The constant c depends on $p, k_1, p_1, \gamma, H, M$ and α .

Proof. See [1] $\square$

Hence, with this two propositions we have that for a $\beta > 0$ big enough, $|J_{2,n}| \leq cN^{-\beta}$. Our aim now is showing a similar estimate for $|J_{3,n}|$. Recall that

$$J_{3,n} = E_{t_{n-1}}[\Phi_M \Phi_{c_n, \varepsilon_n} \int_0^1 \frac{d}{dx} \delta_x(F_{n-1} + I_n + \rho R_n) R_n d\rho].$$

We introduce a new random variable defined by $\sigma_N U_n = I_n + \rho R_n$. Hence, $J_{3,n}$ can now be written as

$$J_{3,n} = E_{t_{n-1}} \left[\Phi_M \Phi_{c_n, \varepsilon_n} \int_0^1 \frac{d}{dx} \delta_x(F_{n-1} + \sigma_N U_n) R_n d\rho \right].$$

Integration by parts formula now yields

$$J_{3,n} = \sigma_N^{-2} \int_0^1 E_{t_{n-1}} \left[\mathbb{1}_{\{I_n + \rho R_n > x - F_{n-1}\}} H_{(1)}^{t_{n-1}}(U_n, R_n \Phi_M \Phi_{c_n, \varepsilon_n}) \right] d\rho.$$

This last expression can be bounded as follows

$$|J_{3,n}| \leq c\sigma_N^{-2} A_1 \int_0^1 A_2(\rho) A_3(\rho) d\rho,$$

where

$$A_1 = \|R_n \Phi_M\|_{k_1, p_1, t_{n-1}},$$

$$A_2(\rho) = \|\Gamma_{U_n, t_{n-1}}^{-1} \Phi_M \Phi_{c_n, \varepsilon_n}\|_{p_3, t_{n-1}}^{k_3},$$

and

$$A_3(\rho) = \|U_n \Phi_{M'}\|_{k_2, p_2, t_{n-1}}^{k_4}.$$

The idea is to first bound A_1 . We tried to follow a similar strategy as in [1], but we contacted one of the authors and he said that there were mistakes in the proof of the bound for A_1 . The result we wanted to get is something of the form:

Conjecture 8.13. In this setting,

$$\|R_n \Phi_{M'}\|_{k_1, p_1, t_{n-1}} \leq O(N^{-\kappa})$$

for some $\kappa > 0$.

Even though we weren't able to prove this result, we can still give some bounds for the other terms:

Lemma 8.14. $A_2(\rho)$ is uniformly bounded in N, ρ and $\omega \in \Omega$.

Proof. Recall that $A_2(\rho) = \|\Gamma_{U_n, t_{n-1}}^{-1} \Phi_M \Phi_{c_n, \varepsilon_n}\|_{p_3, t_{n-1}}^{k_3}$. From the definition of U_n , we have

$$DU_n = \sigma_N^{-1} DI_n + \sigma_N^{-1} DR_n.$$

Hence, using that $(x + y)^2 \geq x^2/2 - y^2$ we have

$$\sigma_N^2 \|DU_n\|_{\mathcal{H}[t_{n-1}, t_n]}^2 \geq \frac{1}{2} \|DI_n\|_{\mathcal{H}[t_{n-1}, t_n]}^2 - \rho \|DR_n\|_{\mathcal{H}[t_{n-1}, t_n]}^2.$$

From the fact that $0 \leq \rho \leq 1$ we can make a rougher estimate

$$\sigma_N^2 \|DU_n\|_{\mathcal{H}[t_{n-1}, t_n]}^2 \geq \frac{1}{2} \|DI_n\|_{\mathcal{H}[t_{n-1}, t_n]}^2 - \|DR_n\|_{\mathcal{H}[t_{n-1}, t_n]}^2.$$

Now, using that

$$\|DI_n\|_{\mathcal{H}[t_{n-1}, t_n]}^2 \geq \lambda^2 \sigma_N^2,$$

we find that

$$\Gamma_{U_n, t_{n-1}} \Phi_{c_n, \varepsilon_n} \leq \sigma_N^2 \left(\frac{\lambda^2 \sigma_N^2}{2} - \|DI_n\|_{\mathcal{H}[t_{n-1}, t_n]}^2 \right)^{-1} \Phi_{c_n, \varepsilon_n}.$$

Since we have localized $\|DI_n\|_{\mathcal{H}[t_{n-1}, t_n]}^2$ by $\Phi_{c_n, \varepsilon_n}$ with $c_n = \frac{\lambda^2 \sigma_N^2}{4}$ and $\varepsilon_n = \frac{c_n}{2}$ we conclude that

$$\Gamma_{U_n, t_{n-1}}^{-1} \Phi_{c_n, \varepsilon_n} \leq C \sigma_N^2 \sigma_N^{-2}$$

for some constant $C > 0$. Since this quantity is uniformly bounded, we can apply Hölder's inequality to see that

$$\|\Gamma_{U_n, t_{n-1}}^{-1} \Phi_M \Phi_{c_n, \varepsilon_n}\|_{p_3, t_{n-1}}^{k_3} \leq \|\Gamma_{U_n, t_{n-1}}^{-1} \Phi_{c_n, \varepsilon_n}\|_{p_4, t_{n-1}}^{k_3} \|\Phi_M\|_{p_5, t_{n-1}}^{k_3}.$$

And, by the previous computations, this quantity is uniformly bounded in N , ρ and ω . $\square$

The following result relies on the conjecture 8.13.

Lemma 8.15. $A_3(\rho)$ is uniformly bounded in N , ρ and $\omega \in \Omega$.

Proof. Recall that $A_3(\rho) = \|U_n \Phi_{M'}\|_{k_2, p_2, t_{n-1}}^{k_4}$. From the definition of U_n we have, by triangular inequality,

$$\|U_n \Phi_{M'}\|_{k_2, p_2, t_{n-1}} \leq \|I_n \Phi_{M'}\|_{k_2, p_2, t_{n-1}} + \|R_n \Phi_{M'}\|_{k_2, p_2, t_{n-1}}.$$

For the first term, apply Hölder's inequality to get

$$\|I_n \Phi_{M'}\|_{k_2, p_2, t_{n-1}} \leq \|I_n\|_{k_2, p_3, t_{n-1}} \|\Phi_{M'}\|_{k_2, p_4, t_{n-1}}.$$

With a proof of conjecture 8.13, one can finish the proof of this result. $\square$

The final objective is ending up with something of the form

$$|J_{3,n}| \leq O(N^{-\alpha})$$

for some $\alpha > 0$. Once this result is proved, we can give a proof of conjecture 8.3 in the same way as in [1].

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