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**OKOUNKOV BODIES AND
GIANSIRACUSA-GIANSIRACUSA
VALUATIONS**

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Abstract

Okounkov bodies are convex sets associated with divisors on algebraic varieties that allow for the study of asymptotic properties of both the divisor and the variety. This theory, developed independently by Lazarsfeld–Mustață [5], and Kaveh–Khovanskii [4], is based on tools from discrete semi-groups and valuations associated with divisors. It can be seen as a generalization of the Newton polytope in the context of projective toric varieties.

On the other hand, prevaluations, also known by their author's names as Giansiracusa–Giansiracusa valuations [16], are a more relaxed notion of valuation, replacing the value group with an idempotent semi-ring. This generalization allows for the definition of a natural product of prevaluations, which opens new paths for reinterpreting and/or extending the construction of Okounkov bodies.

We will discuss the fundamental concepts and present some preliminary developments towards developing a theory of Okounkov bodies with respect to prevaluations.

Contents

1	Introduction	4
2	Preliminaries	7
2.1	Valuations	7
2.2	Line bundles	13
2.3	Classical results in Algebraic Geometry	22
3	Okounkov bodies	25
3.1	Discrete semi-groups and graded semi-groups	25
3.2	Construction of Okounkov bodies	31
3.3	Numerical Okounkov bodies	36
4	Prevaluations	40
4.1	Prevaluations on idempotent semi-rings	40
4.2	Product of prevaluations	44
4.3	Okounkov body with respect to the product of MRR valuations	47
4.4	Conclusions and hopes	49

1 Introduction

Okounkov bodies are a wide generalization of the concept of Newton polytope in the context of projective toric varieties. They aim to generalize the relation that exists between rational polyhedra and relate convex sets associating them with divisors of an algebraic variety in such a way that allows for the study of asymptotic properties, mainly regarding the size of the space of section of large powers of the divisor.

The concept of Okounkov body is originally due to the Fields medalist Andrei Okounkov, appearing in [1], [2] as a construction allowing to apply convex geometry to study linear series of an ample line bundle in the context of the log-concavity of multiplicities. Its systematization and formalization was done in parallel by Robert Lazarsfeld and Mircea Mustața [5] on the one hand, and Kiumars Kaveh and Askold G. Khovanskii [4] on the other. They generalized the original notions expanding its utilities beyond ample line bundles and showing the potential of this construction.

The concept a priori is not hard to comprehend. It assigns a convex body to every divisor on a projective algebraic variety by introducing an appropriate valuation over the field of rational functions of said variety. This convex body encodes properties of the linear series that can be studied by methods of convex geometry. Perhaps the most direct property that Okounkov bodies represent is the following:

Theorem 1.1. *For L big line bundle of $\text{vol}(L)$ we have*

$$e(R(X, L)) = \text{vol}(L) = n! \mu_\nu(\Delta_\nu(X, L))$$

where μ_ν stands for Lebesgue measure of the \mathbb{R} -vector space spanned by the group of values after an appropriate normalization, and e is the **multiplicity** of the line bundle, which allows to generalize Hilbert-Serre's theorem for sub-algebras of the algebra of sections of an arbitrary line bundle L on a projective variety X .

The way to obtain such results lies in the characteristics of the chosen valuation. Typically the theory is developed for flag valuations, those coming from a sequence of closed subvarieties $X = X_0 \supset X_1 \supset \dots \supset X_n = \{p\}$ for which each is a Cartier divisor of the previous one at a neighborhood of the point p .

This valuations, and many others, verify the central condition:

$$\dim_k E = \#\nu(E \setminus \{0\})$$

which allows to state equidistribution theorems for the grading of the semi-group of values of the valuation centered at the variety, namely

Theorem 1.2. *Let $S \subset \mathbb{N} \times V$ a discrete graded semi-group. For any continuous function with compact support $f : V \rightarrow \mathbb{R}$ we have that*

$$\lim_{m \in \mathbb{N}(S), m \rightarrow \infty} \frac{1}{m^{\kappa(S)}} \sum_{x \in \frac{1}{m} S_m} f(x) = \int_{\Delta(S)} f d\mu_S$$

Even more generalizations of this results can be done when one consider Okounkov bodies under numerical equivalence, extending results to pseudo-effective line bundles.

Parallel to this, the concept of prevaluation, a relaxation due to the brothers Noah and Jeffrey Giansiracusa. They swap in the definition of valuation the totally ordered abelian group by an idempotent semi-ring, and this allows for a natural definition of product of prevaluations, which, together with some universality results that allow to express any prevaluation ν over a ring R as the composition of a universal prevaluation ν_R and a morphism f as in the diagram

$$\begin{array}{ccc} R & \xrightarrow{\nu} & S \\ \nu_R \searrow & & \nearrow f \\ & S_R & \end{array}$$

It may be of consequence when tackling the study of Okounkov bodies. The idea is that we may obtain further information on the linear series if instead of working with one valuation, we do so with a number of them at the same time. One can find similar efforts at [6], although the approach is different.

This Master's Final Project has the aim to first provide the necessary theoretical background to understand both Okounkov bodies and prevaluation, and then make some early steps towards relating both concepts by defining an early construction of Okounkov body for the prevaluation product of valuations of maximal rational rank.

For the first aim, the next two sections are meant to cover, respectively, the majority of preliminary concepts from valuation theory and algebraic geometry and the construction as well as the main results regarding Okounkov bodies. We have followed [3] as a main source, which itself draws from the originals [4] and [5]. In addition, for the subsections meant to cover the basics in algebraic geometry, and specifically in positivity, in order to be able to understand the theory of Okounkov bodies we have used as sources [7], [9] and [8].

The final section corresponds with the part regarding prevaluations. We expose the main definitions, borrowed not from the original concept by the Giansiracusa brothers in [16], but from [17] as a practical source, because of better commodity with the changes in the definitions. Soon however we depart from it and start working on the product of prevaluations, give some results arising from multiplying Krull valuations. We end with a first try at a coherent definition of Okounkov body from a product of valuations, and illustrate it with a simple example.

2 Preliminaries

The goal of this section is to state and run through the preliminary results that are necessary to acquire in order to begin to understand the next two sections. First, we introduce the notions regarding valuations that are involved in the construction of Okounkov bodies.

The second part is a compilation of concepts related with the study of positivity of line bundles that will become central when developing the theory of Section 3.

The last part is meant as a quick overview of basic results and theorems fundamental in the field of Algebraic Geometry. It is there in order to make the text as comprehensible as possible, so the reader is free to skim it if already familiar with the names of the theorems that appear.

2.1 Valuations

We will commence by tackling the classical theory of valuations. We will give some general notions of valuation theory, but mainly we will stay focused on those kinds of valuations and properties of them that will be of use for Okounkov bodies. For a more thorough view of valuations, we leave [19] as a fundamental source.

Let us begin then by defining what is a valuation, more precisely what is sometimes called a Krull valuation.

Definition 2.1. *Let A be an integral k -algebra, Λ a totally ordered abelian group. We say that a map $\nu : A \setminus \{0\} \rightarrow \Lambda$ is a **valuation over A** if it verifies:*

- i. $\nu(fg) = \nu(f) + \nu(g)$*
- ii. $\nu(f + g) \geq \min \{ \nu(f), \nu(g) \}$*
- iii. $\nu(a) = 0$, for all $a \in k^*$*

One may alternatively consider $\{0\}$ in the domain by setting $\nu(0) = +\infty$ together with $\nu(f) = +\infty \iff f = 0$, and establishing of course that $\lambda < +\infty \forall \lambda \in \Lambda$. We may use this later on when relating valuations with prevaluations as it will make sense in the context of semi-rings.

From this initial definition we can state some direct lemmas:

Lemma 2.2. *Let $f_1, \dots, f_m \in A \setminus \{0\}$, if $\nu(f_i) \neq \nu(f_j) \forall i \neq j$, then $\sum_i f_i \neq 0$ and $\nu(\sum_i f_i) = \min_i \nu(f_i)$.*

The preservation of this property in particular later will determine our choice of definition of prevaluation. The next lemma is a fundamental result extending the domain of valuations.

Lemma 2.3. *Any valuation over an algebra A can be extended to its field of fractions K by taking $\nu(\frac{f}{g}) = \nu(f) - \nu(g)$.*

Now we can associate some objects related to ν :

Definition 2.4. *Let ν be a valuation over K .*

1. The **group of values of ν** is defined as

$$\Lambda_\nu := \nu(K^*)$$

2. The **valuation ring of ν** is defined as

$$\mathcal{O}_\nu := \{f \in K : \nu(f) \geq 0\}$$

It is a local ring with maximal ideal

$$\mathfrak{m}_\nu = \{f \in K : \nu(f) > 0\}$$

and residue field $k(\nu) = \mathcal{O}_\nu / \mathfrak{m}_\nu \supset k$

In this context we can see that ν induces a group isomorphism such that we get $\Lambda_\nu \simeq K^* / \mathcal{O}_\nu^*$.

Of special interest is the valuation ring $\mathcal{O}_\nu \subseteq K$. The following lemma is a classical result that will give us us a way to classify valuations that we will take advantage of in Section 3.

Lemma 2.5. *Two Krull valuations ν, ν' over K such that there is an isomorphism of ordered groups $\Lambda_\nu \xrightarrow{\lambda} \Lambda_{\nu'}$ for which $\nu' = \lambda \circ \nu$ define the same valuation ring $\mathcal{O}_\nu = \mathcal{O}_{\nu'}$.*

Reciprocally, any valuation ring V of K , understood as a sub-ring of K for which either $f \in V$ or $\frac{1}{f} \in V$, defines a valuation up to the previous equivalence relation.

Proof. See [19],Chapter 2,Lemma 7.1. □

Valuations are of use to us in the context of algebraic geometry when they are related to a variety. We will call a **model** of K a k -algebraic variety whose field of rational functions is $k(X) \simeq K$.

Definition 2.6. *Let ν be a valuation over K and X a model of K , We define the **center of ν over X** as a point $\xi \in X$ such that $\mathcal{O}_{X,\xi} \subset \mathcal{O}_\nu$, $\mathfrak{m}_\xi \subset \mathfrak{m}_\nu$.*

*The valuative criterion of separatedness gives us unicity of the center if it exists. In this case, we say that ν is **centered over X** .*

Remark 2.7. *If $X = \text{Spec } A$ then ν is centered iff $A \subset \mathcal{O}_\nu$. If instead X is proper, any valuation is centered as a result of the valuative criterion of properness.*

This way, we can define the following concept to tie with the previous subsection:

Definition 2.8. *Let X be a model of K and ν a valuation centered over X . Then, the **semi-group of values of ν over X** is defined to be the sub-semi-group*

$$S_\nu(X) := \nu(\mathcal{O}_{X,\xi} \setminus \{0\}) \subset \Lambda_\nu$$

where ξ is the center of ν over X .

It is important to note here that in this case $S_\nu(X)$ does depend on the choice of model, thus the notation.

As a side note, one can prove that, given any sub-semi-group of finite type of a totally ordered abelian group $S \subset \Lambda$ it is possible to build an affine algebraic variety (in fact, a toric variety if S is saturated in Λ and finitely generated) which is a model of K and has a valuation ν centered over it with semi-group of values $S_\nu = S$. Details of this construction can be found in the corresponding section of [3].

Remark 2.9. *Any positive semi-group S of finite type of a totally ordered abelian group Λ , then S is **well-ordered**, i.e, any nonempty subset has a minimal element.*

We will now introduce one of the crucial hypothesis needed on a valuation for the construction of meaningful Okounkov bodies, together with two types

of valuations related to it. One will be useful for its comprehensibility, and the other because it will become central in the next section.

Every valuation ν over K defines a decreasing filtration of sub- \mathcal{O}_ν -modules of the form:

$$F_\nu^\lambda K := \{f \in K : \nu(f) \geq \lambda\}$$

If we denote by $F_\nu^\lambda E = E \cap F_\nu^\lambda K$ for any E finite dimensional k -vector subspace, we obtain a graded algebra

$$Gr_\nu^\lambda E = \frac{\{f \in E : \nu(f) \geq \lambda\}}{\{f \in E : \nu(f) > \lambda\}}$$

The next lemma will be essential in the next section, as will be necessary when stating and proving properties of Okounkov bodies:

Lemma 2.10. *The residue field of a valuation ν over K verifies*

$$k(\nu) = k \iff \dim_k E = \#\nu(E \setminus \{0\})$$

$\forall E \subset K$ finite-dimensional k -vector subspaces. These conditions are also equivalent to:

- i. $\dim Gr_\nu^\lambda K = 1 \forall \lambda \in \Lambda_\nu$*
- ii. $\exists A \subset K$ sub- k -algebra such that $\dim_k Gr_\nu^\lambda A \leq 1 \forall \lambda \in \Lambda_\nu$*

Proof. Note first that for $E \subset K$ finite-dimensional vector space we have that

$$\dim_k E = \sum_{\lambda \in \Lambda_\nu} \dim_k Gr_\nu^\lambda E$$

Since $Gr_\nu^\lambda E \neq 0$ if and only if $\lambda \in \nu(E \setminus \{0\})$ we have that our property is equivalent to seeing that $\dim_k Gr_\nu^\lambda E \leq 1$ for every $\lambda \in \Lambda_\nu$.

Going first from left to right, suppose $k(\nu) = k$ and fix $E \subset K$ a finite-dimensional sub- k -vector space. We want to see that for any $f, g \in E \setminus \{0\}$ with $\nu(f) = \lambda = \nu(g)$ we have k -linear dependency on $Gr_\nu^\lambda E$, i.e., that there exists $a \in k$ so that $\nu(f - ag) > \lambda$.

This is not hard because, by definition of the valuation extended to the field of fractions, we have that $\nu\left(\frac{f}{g}\right) = \nu(f) - \nu(g) = 0 \Rightarrow \frac{f}{g} \in U(\mathcal{O}_\nu)$.

Therefore $\mathcal{O}_\nu/\mathfrak{m}_\nu = k(\nu) = k$ implies $\exists a \in k = k(\nu)$ such that $\frac{f}{g} - a \in \mathfrak{m}_\nu$, that is, $\nu(\frac{f}{g} - a) > 0$. Therefore we get $\nu(f - ag) = \nu((\frac{f}{g} - a) \cdot g) = \nu(\frac{f}{g} - a) + \nu(g) > \nu(g) = \lambda$.

On the other hand, suppose that $\dim_k E = \#\nu(E \setminus \{0\})$ holds. Let us see that necessarily $k(\nu) = k$. The inclusion $k \subset k(\nu)$ is trivial by definition, so we need only to check the other one. Elements of $k(\nu)$ can be identified with a representative $f \in \mathcal{O}_\nu \setminus \mathfrak{m}_\nu$ such that $\nu(f) = 0$. We have to see that f is in k , i.e., $\exists a \in k$ such that $\nu(f - a) > 0$. However, this is a consequence of the equivalent property $\dim_k Gr_\nu^\lambda E \leq 1$ for $\lambda = 0$ and $E = \langle f, 1 \rangle$ the k -vector space generated by f and 1.

As to the equivalence with properties *i.* and *ii.*, the first one comes from taking $E = K$ and reasoning as we did at the beginning of the proof, while the second one should come as a consequence of the containment of any finite-dimensional vector space $E \subset A \subset K$. \square

Definition 2.11. A *monomial valuation* ν over $k(T_1, \dots, T_n)$ is a valuation uniquely determined by the values $\nu(T_i)$ via the formula

$$\nu\left(\sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha\right) = \min \left\{ \sum_i \alpha_i \nu(T_i) : \alpha \in \mathbb{N}^n, a_\alpha \neq 0 \right\}$$

for any polynomial $\sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in k[T_1, \dots, T_n]$.

These valuations are nice since $\Lambda_\nu = \sum_i \mathbb{Z}\nu(T_i)$ and, if centered over \mathbb{A}_k^n , then $S_\nu(\mathbb{A}_k^n) = \sum_i \mathbb{N}\nu(T_i)$. The following lemma allows us to easily recognize when we have a monomial valuation.

Lemma 2.12. Any valuation ν over $k(T_1, \dots, T_n)$ such that the elements $\nu(T_i)$ are \mathbb{Z} -linearly independent is automatically a monomial valuation. Moreover, the valuation has trivial residue field $k(\nu) = k$.

Proof. The monomial condition is verified as a result of $\nu(T_i)$ being pairwise different when applying Lemma 2.2 to the expression $\nu(T^\alpha) = \sum_i \alpha_i \nu(T_i)$.

For the triviality of the residue field, we will use the equivalent condition given in Lemma 2.10.*ii.* with the sub-algebra $A := k[T_1, \dots, T_n]$. Since $Gr_\nu^\lambda A \neq 0$ if and only if $\lambda \in \nu(A \setminus \{0\})$, then $\exists! \alpha \in \mathbb{N}^n$ such that $\lambda = \sum_i \alpha_i \lambda_i$, and so we can see that $Gr_\nu^\lambda A \neq 0$ is generated over k by $\nu(T^\alpha)$. \square

We will now introduce the second and central type of valuation, that which will be the one we use to base our theory from now on. In order to do so, we need to define a preliminary concept.

Definition 2.13. Let ν be a valuation over K and let $V_\nu := \Lambda_\nu \otimes_{\mathbb{Z}} \mathbb{R}$. The **rational rank of ν** is the maximal number of \mathbb{Z} -linearly independent elements of Λ_ν in V_ν .

It can be seen and should come as no surprise that the rational rank of a variety over K cannot be larger than the transcendence degree of the field extension K/k .

It comes as a result of the following lemma, which we will leave without proof. (For a more complete proof one can look to [20] p.50 – 52, and for a sketch of it we refer to in [3])

Lemma 2.14. For ν a valuation over K and K' an arbitrary sub-field of K containing k we have that:

- i. $\#(\nu(K^*)/\nu(K'^*)) \leq [K : K']$.
- ii. $\text{rk}(\nu(K^*)/\nu(K'^*)) \leq \text{tr. deg}(K/K')$.

We have now motivation for the next definition.

Definition 2.15. We say ν is **of maximal rational rank** if

$$\text{rat. rk } \nu = \text{tr. deg}(K/k) = n$$

Maximal rational rank valuations (from now on *MRR* valuations for shortness) verify some nice properties that will be key in the construction of Okounkov bodies. The following proposition states those of interest to us:

Proposition 2.16. Let ν be a *MRR* valuation. Then

- i. Λ_ν is a free \mathbb{Z} -module of rank n .
- ii. $\forall E \subset K$ finite-dimensional vector space $\dim_k E = \#\nu(E \setminus \{0\})$.
- iii. $\forall K'/k$ subfield extension, $\nu|_{K'}$ is also *MRR*.

The proof of this proposition relies on a very useful result that takes us back to monomial valuations:

Lemma 2.17. ν is *MRR* iff there exists a finite field extension $K/k(T_1, \dots, T_n)$ such that $\nu|_{k(T_1, \dots, T_n)}$ is monomial with all $\nu(T_i)$ \mathbb{Z} -linearly independent.

Proof. Let ν be a *MMR* valuation over K . By definition, there are elements $f_1, \dots, f_n \in K^*$ for which $\nu(f_i)$ are \mathbb{Z} -linearly independent. This condition implies that the f_i themselves are algebraically independent, therefore inducing an embedding $k(T_1, \dots, T_n) \hookrightarrow K$. This embedding is finite since K/k is a finite extension of $\text{tr. deg}(K/k) = n$. We have then that $\nu|_{k(T_1, \dots, T_n)}$ has $\nu(T_i) = \nu(f_i)$ \mathbb{Z} -linearly independent, and thus $\nu|_{k(T_1, \dots, T_n)}$ is monomial by Lemma 2.12.

On the other hand, for ν any monomial valuation over $k(T_1, \dots, T_n)$ has as group of values $\mathbb{Z}\nu(T_1) + \mathbb{Z}\nu(T_2) + \dots + \mathbb{Z}\nu(T_n)$, which makes it of rational rank n . Any other valuation obtained as an extension of ν over a finite extension of $k(T_1, \dots, T_n)$ is of again rank n , as a result of the second inequality in Lemma 2.14. \square

Proof. (of proposition) By Lemma 2.10, property *ii.* is equivalent to check triviality of the residue field. Applying Lemma 2.17 we can restrict ν to ν' a monomial valuation over $k(T_1, \dots, T_n)$ with group of values $\Lambda_{\nu'} = \sum_i^n \mathbb{Z}\nu(T_i)$ and $k(\nu') = k$. Let us see that we can lift this properties up to ν .

Since $K/k(T_1, \dots, T_n)$ is of finite type, then Lemma 2.14 tells us that $\Lambda_{\nu'}$ of finite index over Λ_{ν} , giving us property *i.*

On the other hand, $K/k(T_1, \dots, T_n)$ being finite entails that $k(\nu)/k(\nu')$ is of finite type, but $k(\nu') = k$ with k algebraically closed, therefore $k(\nu) = k$, getting *ii.*

Finally, property *iii.* comes from the inequalities in Lemma 2.14 and the equalities

$$\begin{aligned} \text{rk } \nu(K^*) &= \text{rk}(\nu(K^*)/\nu(K'^*)) + \nu(K'^*) \\ \text{tr. deg}(K/k) &= \text{tr. deg}(K/K') + \text{tr. deg}(K'/k) \end{aligned}$$

\square

We have run through the essential properties of valuations that are used in the classical theory of Okounkov bodies. Afterwards, we will return to them to expand notions when working with valuations.

2.2 Line bundles

Here we will cover some definitions and results which are basic to our understanding of Okounkov bodies. We will discuss what are ample, very ample,

big and nef bundles, as well as how these properties relate to one another in regards to the context where we will apply them.

Throughout this section, unless stated otherwise, we will work with schemes or varieties over \mathbb{C} . We will commence by clarifying some details of the definitions of divisors and their relation with line bundles.

First of all, we will consider X an irreducible complex variety, and denote by \mathfrak{K}_X the constant sheaf of rational functions.

Definition 2.18. A **Cartier divisor** is a global section of the sheaf of abelian multiplicative groups $\mathfrak{K}_X^*/\mathcal{O}_X^*$. We denote by $\text{Div}(X)$ the group of Cartier divisors, i.e.,

$$\text{Div}(X) = \Gamma(X, \mathfrak{K}_X^*/\mathcal{O}_X^*)$$

This will be our main concept of divisor. An element $D \in \text{Div}(X)$ is represented by a pair $\{\{U_i\}_i, \{f_i\}_i\}$ consistent of an open affine cover $X = \bigcup_i U_i$ and $f_i \in \Gamma(U_i, \mathfrak{K}_X^*)$ local equations such that on $U_{ij} = U_i \cap U_j$ we have that $f_i = g_{ij}f_j$ for some $g_{ij} \in \Gamma(U_{ij}, \mathcal{O}_X^*)$.

The sum of two Cartier divisors D, D' , represented by $\{\{U_i\}_i, \{f_i\}_i\}$ and $\{\{U_i\}_i, \{f'_i\}_i\}$ respectively, corresponds to the Cartier divisor represented by:

$$D + D' = \{\{U_i\}_i, \{f_i f'_i\}_i\}$$

This representation comes evident when the divisor is defined by a global section $f \in \Gamma(X, \mathfrak{K}_X)$. We will call such divisors **principal divisors**, and form a subgroup $\text{Princ}(X) \subseteq \text{Div}(X)$.

We define an equivalence relation from the principal divisors as follows.

Definition 2.19. Two divisors D_1, D_2 are said to be **linearly equivalent** if $D_1 - D_2 \in \text{Princ}(X)$.

We define the **class group of X** as the quotient

$$\text{Cl}(X) = \text{Div}(X)/\text{Princ}(X)$$

Cartier divisors should be thought of as cohomological objects. Its homological analog is what we call Weil divisors.

Definition 2.20. Let X be a scheme of pure dimension n . A **k -cycle of X** is a \mathbb{Z} -linear combination of irreducible subvarieties of dimension k . We denote by $Z_k(X)$ the group of k -cycles.

For $k = n - 1$ we have special notation: $Z_{n-1}(X) \equiv W \operatorname{Div}(X)$ and its elements, $(n - 1)$ -cycles, are called **Weil divisors of X** .

These two definitions as one may assume by now, are related. We do so via what is called the cycle map:

$$\operatorname{Div}(X) \rightarrow W \operatorname{Div}(X), \quad D \mapsto [D] = \sum_{V \text{ prime}} \operatorname{ord}_V(D)[V]$$

Here, ord_V is a valuation associated to the order of D along V prime Weil divisor, i.e., $(n - 1)$ -dimensional subvariety. Later we will see how to define a valuation over sections of a divisor, but for now we will say that if the divisor is principal $D = \operatorname{div}(f)$, then the cycle map identifies D with the linear combination of prime Weil divisors with coefficients $\operatorname{ord}_V(f)$ the order of zeros and poles of f along the subvariety.

This homomorphism is neither injective nor surjective in general. The following proposition tells us when the cycle map has good properties:

Proposition 2.21. *Let X be a scheme. If X is normal, then the cycle map above is an injection. Moreover, if X is smooth, then it is an isomorphism.*

We will for the most part not assume smoothness and focus on the relation between Cartier divisors and line bundles, which still holds.

In any case, two divisors that are linearly equivalent define the same line bundle, uniquely determined by an open affine cover, and a set of local equations with regular transition functions.

$$\operatorname{Cl}(X) \xrightarrow{\sim} \operatorname{Pic}(X) \quad , \quad [D] \mapsto \mathcal{O}_X(D)$$

Now, one way to study a variety is through the positivity of line bundles defined over it. The theory of Okounkov bodies works essentially with positivity characterized under numerical equivalence, namely with the concepts of ample, nef, big and pseudoeffective line bundles.

Numerical equivalence relies on the definition of an intersection number between divisors and irreducible curves $C \subset X$. In general, we can define an intersection number of the form

$$(D_1 \cdot \dots \cdot D_k \cdot V) \in \mathbb{Z}$$

for $D_1, \dots, D_k \in \operatorname{Div}(X)$ and $V \subseteq X$ a k -dimensional irreducible subvariety. To do so, we will follow the approach at [7] and work topologically, but

there are other ways to see this construction. In particular, we will refer to [9] whenever in doubt of any property.

In such a situation, we have that each divisor has associated a line bundle $\mathcal{O}_X(D_i)$ with a Chern class $c_1(\mathcal{O}_X(D_i)) \in H^2(X; \mathbb{Z})$, the singular cohomology of X with the classical topology.

Thinking like this, one can take the cup product of the D_i s we get an element:

$$c_1(D_1) \smile c_1(D_2) \smile \dots \smile c_1(D_k) \in H^{2k}(X; \mathbb{Z})$$

If we denote the fundamental class¹ of V by $[V] \in H_{2k}(X; \mathbb{Z})$ we can use the cap product to get an integer:

$$(c_1(D_1) \smile c_1(D_2) \smile \dots \smile c_1(D_k)) \frown [V] \in \mathbb{Z}$$

From now on we will not burden the notation and refer to the intersection product using either of the following notations:

$$(D_1 \cdot \dots \cdot D_k \cdot V) \quad , \quad \int_V D_1 \cdot \dots \cdot D_k$$

By linearity, V can be replaced by an arbitrary k -cycle and this product is invariant under linear equivalence of the D_i 's. If $D_1 = D_2 = \dots = D_k$, we write $(D^k \cdot V)$ and if $V = X$ of dimension n , then simply $(D_1 \cdot \dots \cdot D_n) \in \mathbb{Z}$. For the non-reduced case we can do the same safe minor adaptations.

More generally, for V a d -dimensional subvariety, $d \geq k$, we get an intersection number in the sense $(D_1 \cdot \dots \cdot D_k \cdot [V]) \in H_{2d-2k}(X; \mathbb{Z})$.

Let us close the preliminaries by defining numerical equivalence and the Neron-Severi group in order to finally be able to define positivity in the contexts proposed in the latter sections. We assume now X to be a complete algebraic variety over \mathbb{C} .

Definition 2.22. *Let $D_1, D_2 \in \text{Div}(X)$. We say they are **numerically equivalent**, $D_1 \sim_{num.} D_2$ if $(D_1 \cdot C) = (D_2 \cdot C)$ for every irreducible curve $C \subseteq X$.*

Equivalently, two divisors are numerically equivalent if $(D_1 \cdot \gamma) = (D_2 \cdot \gamma)$ for every 1-cycles of X . We say D is **numerically trivial** if $D \sim_{num.} 0$.

¹This is, the homology class that generates $H_{\dim V}(X; \mathbb{Z}) \simeq \mathbb{Z}$.

Remark 2.23. *Since the line bundle associated to a divisor is invariant under linear equivalence, and the Chern class of a line bundle is invariant under rational equivalence, we have the implications*

$$D_1 \sim_{lin.} D_2 \implies D_1 \sim_{rat.} D_2 \implies D_1 \sim_{num.} D_2$$

From this new, weaker equivalence relation we can define the Neron-Severi group as follows.

Definition 2.24. *The **Neron-Severi group of X** is the quotient group*

$$N^1(X) = \text{Div}(X) / \text{Num}(X)$$

where $\text{Num}(X) \subseteq \text{Div}(X)$ is the subgroup of numerically trivial divisors.

The first thing to know about the Neron-Severi group is that it is finitely generated.

Proposition 2.25. *The Neron-Severi group $N^1(X)$ is a free abelian group of finite rank.*

Proof. A divisor D of X defines a cohomology class by way of its first Chern class $[D]_{hom} = c_1(\mathcal{O}_X(D)) \in H^2(X; \mathbb{Z})$. Evidently, $[D]_{hom} = 0$ implies that D is numerically trivial, as the Chern class conditions the intersection number. This means that the group of cohomologically trivial divisors is contained in $\text{Num}(X)$.

Therefore, one can identify $N^1(X)$ with a subgroup of $H^2(X; \mathbb{Z})$, and in particular finitely generated. It will be torsion-free by construction. \square

Contra-intuitively to what its name would suggest, we define the **Picard number of X** to be $\rho(X) = \text{rk } N^1(X)$.

Importantly, we have that numerically equivalent divisors give the same intersection number:

Lemma 2.26. *Let X be a complete \mathbb{C} -scheme and let $D_1, \dots, D_k, D'_1, \dots, D'_k$ be divisors such that $D_i \sim_{num.} D'_i$ for each i . Then*

$$(D_1 \cdot \dots \cdot D_k \cdot [V]) = (D'_1 \cdot \dots \cdot D'_k \cdot [V]) \quad \forall V \subseteq X \text{ } k\text{-dim. subscheme.}$$

Proof. Since $(c_2(D_2) \cdot \dots \cdot c_1(D_k)) \sim [V]$ defines a 1-cycle γ on X , we have that $D_1 \sim_{num.} 0$ implies that

$$(D_1 \cdot \dots \cdot D_k \cdot [V]) = (D_1 \cdot \gamma) = 0$$

by definition of numerical equivalence. It follows that $(D_1 \cdot \gamma) = (D'_1 \cdot \gamma)$ if they are numerically equivalent, and by iterating over i we are done. \square

Remark 2.27. *Regarding line bundles, L being numerically trivial is equivalent to stating that $\exists m \in \mathbb{Z}^*$ such that $mL \in \text{Pic}^0(X)$, that is, is a deformation of the trivial line bundle.*

Having now established numerical equivalence, we will now study what positivity means in different contexts. Perhaps the most easily accesible concept of positivity is that of effectiveness.

Definition 2.28. $D \in \text{Div}(X)$ is an **effective divisor** when the image under the cycle map takes it to a linear combination of positive coefficients.

q This condition ensures the existence of global sections of the line bundle associated to it.

While this condition is very nice, it is also very much strict. If we expand our horizons a bit further, we find ampleness.

Definition 2.29. *Let X be a complete scheme. A divisor D is said to be **very ample** if there exists a closed immersion φ of X into some projective space \mathbb{P}^N such that $\mathcal{O}_X(D) = \varphi^* \mathcal{O}_{\mathbb{P}^N}(1)$.*

*A divisor D is **ample** if mD is very ample for some $m \in \mathbb{Z}_{>0}$.*

Ample line bundles can be characterized a number of different ways. The previous definition corresponds with the geometrical approach, but there is also a cohomological way, for which we leave [8] as a reference for further details, and we will only state this result arising from it.

Corollary 2.30. *Suppose L is globally generated, i.e, generated by its global sections, and let $\phi = \phi_{|L|}$ be the map to the projective space defined by the complete linear series. Then L is ample if and only if ϕ is a finite mapping, or equivalently if and only if*

$$\int_C c_1(L) > 0$$

for every irreducible curve $C \subseteq X$.

We are however interested in the numerical approach. The following theorem, usually called the **Nakai-Moishezon criterion**, gives us how to characterize ampleness numerically.

Theorem 2.31 (Nakai's criterion). *Let L be a line bundle on a projective scheme X . Then L is ample if and only if*

$$\int_V c_1(L)^{\dim V} > 0$$

$\forall V \subseteq X$ irreducible subvariety of positive dimension (including irreducible components).

Proof. (Sketch) One implication is somewhat trivial, as taking L to be ample implies that the intersection number

$$\int_V c_1(mL)^{\dim V} = m^{\dim V} \int_V c_1(L)^{\dim V}$$

corresponds with the degree of V in the projective embedding of X , therefore this number is positive and as a consequence so is the intersection number of L .

For the converse reasoning, it can be seen that positivity of the intersection number leads to the conditions in **Corollary 2.30** holding, implying L is ample. \square

This theorem motivates the following corollary, which itself leads to the next definition.

Corollary 2.32. *If $D_1, D_2 \in \text{Div}(X)$ are numerically equivalent divisors on a projective scheme, then D_1 is ample if and only if D_2 is ample.*

Definition 2.33. *A numerical class $\delta \in N^1(X)$ is **ample** if it is the class of an ample divisor.*

Now, this criterion also allows for a more relaxed notion of divisor, captured in the coming definition.

Definition 2.34. *Let L be a line bundle on a projective scheme X . Then L is **nef** (numerically effective) if and only if*

$$\int_V c_1(L)^{\dim V} \geq 0 \quad , \quad \forall V \subseteq X \text{ irred. subvar. of } \dim V > 0$$

We define a class $\delta \in N^1(X)$ to be nef in an analogous manner to ample classes.

Nef and ample line bundles are related in a sense visible when thinking of them in terms of the classes of the Neron-Severi group, specially when tensored with \mathbb{Q} or \mathbb{R} (where its equivalence classes are that of \mathbb{Q} -divisors and \mathbb{R} -divisors, linear combinations of Cartier divisors with coefficients in those respective fields)

Ample classes of \mathbb{R} -divisors have many properties in the projective case, namely:

Proposition 2.35. *If X is a projective variety and H an ample \mathbb{R} -divisor on X , then for finitely many \mathbb{R} -divisors E_1, \dots, E_r the divisor*

$$H + \varepsilon_1 E_1 + \varepsilon_2 E_2 + \dots + \varepsilon_r E_r$$

is ample for $0 \leq |\varepsilon_i| \ll 1$ sufficiently small.

which itself leads to the result:

Corollary 2.36. *If X is a projective variety, then the finite-dimensional vector space $N^1(X)_{\mathbb{R}} = N^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$ is spanned by the classes of ample divisors on X .*

Taking together that ampleness is an **open condition**, i.e, stable under small perturbations, with the definition of nef bundles being a relaxation of the inequality in Nakai's criterion should have given away the fact that nef line bundles correspond to limits of ample ones.

Proposition 2.37. *Ample and nef classes of divisors span cones in the \mathbb{R} -vector space spanned by $N^1(X)$, denoted by $\text{Amp}(X) \subset \text{Nef}(X) \subset N^1(X)_{\mathbb{R}}$ in such a way that*

$$\overline{\text{Amp}(X)} = \text{Nef}(X) \quad , \quad \text{Nef}^{\circ}(X) = \text{Amp}(X)$$

with respect to the Euclidean topology associated to the \mathbb{R} -vector space.

The previous characterization is a result of **Kleiman's theorem**. It allows also to reduce Nakai's criterion to be checked only on irreducible curves.

Corollary 2.38 (Kleiman’s theorem). *Let X be a projective scheme, H an ample \mathbb{R} -divisor on X . For D an \mathbb{R} -divisor on X , D is ample if and only if there exists $\varepsilon > 0$ such that*

$$\frac{(D \cdot C)}{(H \cdot C)} \geq \varepsilon$$

for every irreducible curve $C \subset X$.

Another important result we will use is **Fujita’s vanishing theorem**, which somewhat mimics Serre’s when dealing with twists by nef divisors.

Theorem 2.39 (Fujita’s vanishing theorem). *Let X be a complex projective scheme and let H be an ample (integral) divisor on X . Given any \mathcal{F} coherent sheaf on X , there exist an integer m_0 dependent on H, \mathcal{F} such that*

$$H^i(X, \mathcal{F} - mH + P) = 0 \quad \text{for } i > 0, m \geq m_0$$

and any P nef divisor on X .

The topic is still rich and one could delve further into it. Since constraints on time and scope keep us here, [7] is recommended to read further about either of the last two theorems.

We will move on instead to an even more relaxed positivity condition than ampleness, that of **big bundles**. This notion has to do with the asymptotic growth of sections for large powers of the line bundle.

Big bundles are traditionally defined as those line bundles of maximal **Iitaka dimension** $\kappa(L) = \dim R(X, L) - 1$ for $R(X, L)$ the ring of sections of L . We will however take as a definition its characterization by the volume function instead.

Definition 2.40. *The **volume function** is a continuous function from the Neron-Severi group of an irreducible variety of dimension n defined by*

$$\text{vol} : N^1(X) \longrightarrow \mathbb{R}_{\geq 0} \quad , \quad \text{vol}(\delta) = \limsup_{m \rightarrow \infty} \frac{h^0(X, mL)}{m^n \cdot n!}$$

for L any line bundle representative of the class δ .

Definition 2.41. *A line bundle L is **big** if its numerical class $[L] \in N^1(X)$ has $\text{vol}([L]) > 0$.*

Big bundles span a cone $\text{Big}(X) \subset N^1(X)$, and subsequently there is a notion equivalent to nefness for big divisors, that of **pseudoeffectiveness**, related by their cones as

$$\text{Eff}^\circ(X) = \text{Big}(X) \quad , \quad \text{Eff}(X) = \overline{\text{Big}(X)}$$

Now we will state some properties that relate big, ample, nef and pseudoeffective bundles which we will use throughout the next section.

Proposition 2.42. *Let L be a line bundle on X . Then, L is big iff*

$$H^0(X, mL - H) \neq 0$$

for all ample line bundles H on X , $m \gg 1$.

Another important characterization is that of big classes inside $N^1(X)_{\mathbb{R}}$. The following proposition tells us how to express them in terms of ampleness.

Proposition 2.43. *A numerical class $\delta \in N^1(X)$ is big if there is an effective \mathbb{R} -divisor E such that $\delta - E$ is ample.*

2.3 Classical results in Algebraic Geometry

To finalize the preliminaries, we will give the statement to some of the theorems invoked in the proofs of the next section, and will give reference as to where their proofs can be found in context. Some of them may have already been used in through the preceding section but linearity was sacrificed to put them all here in a more orderly fashion. These theorems and results can all be found in Hartshorne's book [8].

Firstly, the valuative criterions, which were used when determining the existence and uniqueness of the center of a valuation in a variety.

Theorem 2.44 (Valuative Criterion of Separatedness). *Let $f : X \rightarrow Y$ be a morphism of schemes, and assume that X is Noetherian. Then f is separated if and only if the following condition holds: $\forall K$ field, R valuation ring with $\text{Frac}(R) = K$, let $T = \text{Spec } R$, let $U = \text{Spec } K$, and let $i : U \rightarrow T$ be the morphism induced by the inclusion $R \hookrightarrow K$. Given morphisms from T to Y , and from U to X such that they make a commutative diagram:*

there is at most one morphism $T \rightarrow X$ making the whole diagram commutative.

$$\begin{array}{ccc}
 U & \longrightarrow & X \\
 i \downarrow & \nearrow & \downarrow f \\
 T & \longrightarrow & Y
 \end{array}$$

Proof. See [8].p 97. Theorem 4.3. □

Theorem 2.45 (Valuative Criterion of Properness). *Let $f : X \rightarrow Y$ be a morphism of finite type, with X Noetherian. Then f is proper if and only if for every valuation ring R and for every morphism from $U = \text{Spec}(R)$ to X , and $T = \text{Spec}(K)$ to Y forming a commutative diagram (as in the previous theorem):*

$$\begin{array}{ccc}
 U & \longrightarrow & X \\
 i \downarrow & \nearrow & \downarrow f \\
 T & \longrightarrow & Y
 \end{array}$$

there exists a unique morphism $T \rightarrow X$ making the whole diagram commutative.

Proof. See [8].p 101.Theorem 4.7. □

Next in our list is Chow's lemma. It will allow us to work with proper varieties by being able to pull back to a projective case, for which more concrete tools will be available.

Theorem 2.46 (Chow's Lemma). *Let X be a proper scheme over S a Noetherian scheme. Then there is a scheme X' and a morphism $\pi : X' \rightarrow X$ such that X' is projective over S , and there is an open dense subset $U \subseteq X$ such that π induces an isomorphism of $\pi^{-1}(U)$ to U .*

Hilbert-Serre's theorem is fundamental in our study of Okounkov bodies, and one of the results will amount to a generalization of it for a non-finitely generated situation.

Theorem 2.47 (Hilbert-Serre). *Let k be an algebraically closed field, M be a finitely generated graded $k[x_0, \dots, x_n]$ -module. Then, there is a unique rational polynomial $P_M(z) \in \mathbb{Q}[z]$ such that $\dim_k M_m = P_M(m)$ for all $m \gg 0$.*

Furthermore, $\deg P_M(z) = \dim Z(\text{Ann}M)$, where Z denotes the zero set in \mathbb{P}_k^n of a homogeneous ideal.

As a result of this, the Hilbert function of a finitely generated algebra is of the form

$$H_A(m) = e \frac{m^\kappa}{\kappa!} + o(m^\kappa)$$

with $e \in \mathbb{Q}_+$.

Finally, Riemann-Roch theorem gives information about the dimension of the sections of a vector bundle. For our purposes, we only used the case of algebraic curves.

Theorem 2.48. *Let D be a divisor on a curve C of genus g . Then,*

$$\chi(\mathcal{O}_X(D)) = \deg D + 1 - g$$

where $\chi(\mathcal{O}_X(D)) = \sum_i (-1)^i h^i(C, \mathcal{O}_X(D))$.

Proof. See [8].p 295. Theorem 1.3. □

With this last result, we give as concluded the preliminaries. Let us now enter the main sections of the text.

3 Okounkov bodies

Our goal here will be to give a compressed notion of the construction and properties of Okounkov bodies. The first part of this section aims to cover the main results from discrete semi-groups used in the theory behind Okounkov bodies, in order to give an idea of what should still hold true later when translating to the language of prevaluations. to motivate its translation to the language of prevaluations to expand its reach and utility as a tool for study of line bundles.

3.1 Discrete semi-groups and graded semi-groups

Let V be an n -dimensional \mathbb{R} -vector space. We define a **semi-group** $S \subset V$ to be a subset closed under addition. For S semi-group we will denote by $\mathbb{Z}S$ and $\mathbb{R}S$ the group and \mathbb{R} -vector space generated by S respectively.

Parallel to this, we also consider $C(S) = \{\lambda r + \mu s \mid \lambda, \mu \in \mathbb{R}_+, r, s \in S\}$ the convex cone generated by S , and will denote by $\overline{C}(S)$ and $\overset{\circ}{C}(S)$ its closure and interior respectively.

For these definitions one can immediately find useful characterizations of these objects. One can express $\mathbb{Z}S$ as

$$\mathbb{Z}S = \{x - y \mid x, y \in S\}$$

and $\overline{C}(S)$ as

$$\overline{C}(S) = \overline{\{s/m : s \in S, m \in \mathbb{N}^*\}}$$

Additionally, we will define the **regularization** of S to be

$$S^{reg} = \overline{C}(S) \cap \mathbb{Z}S$$

This S^{reg} is also a semi-group, and will allow us to state certain properties for it instead than for the original semi-group itself.

We will give now the first central definition:

Definition 3.1. *Let S be a semi-group. We will say that S is a **discrete semi-group** if $\mathbb{Z}S$ is discrete, i.e., the subspace topology of $\mathbb{Z}S \subset \mathbb{R}S$ is the discrete topology.*

Remark 3.2. *In the given definition we look for discreteness of the group and not directly for discreteness of S to avoid cases where S may be discrete but not $\mathbb{Z}S$*

Now, consider instead $S \subset \mathbb{N} \times V$ a discrete semi-group and $\pi_V : \mathbb{N} \times V \rightarrow V$ the projection onto V . This projection induces a grading of the form

$$S_m = \pi_V(S \cap \{m\} \times V)$$

We call this a **graded discrete semi-group**. One can reciprocally build a graded discrete semi-group by taking any additive sequence $(S_m)_{m \in \mathbb{N}}$ of subsets of V , i.e., a sequence of subsets that respect $S_m + S_l \subset S_{m+l}$.

For graded semi-groups we have the following notion:

Definition 3.3. *Let S be a discrete semi-group. We define the **base of S** as*

$$\Delta(S) = \pi_V(\overline{C(S)} \cap (\{1\} \times V))$$

It is a closed convex set of dimension $\kappa(S) = \dim_{\mathbb{R}} S - 1$.

We have that π_V induces an affine isomorphism between $\mathbb{R}S \cap \{1\} \times V$ and the affine subspace generated by the base $\Delta(S)$, call it $\text{Aff}(S)$. The underlying vector space is identified via this isomorphism with $\mathbb{R}S \cap \{0\} \times V$.

Note that by construction $\frac{1}{m}S_m \subset \Delta(S)$, which means that we can use the previous characterization of $C(S)$ to derive a characterization of the base, namely

$$\Delta(S) = \overline{\bigcup_{m \geq 1} \frac{1}{m}S_m}$$

As well as the base, one can induce from S another related semi-group

$$\mathbb{N}(S) = \{m \in \mathbb{N} : S_m \neq \emptyset\}$$

Since $\mathbb{N}(S) \subset \mathbb{N}$, we have that $\mathbb{N}(S) \subseteq Nm(S)$ for a certain $m(S) \in \mathbb{N}^*$, and equality holds up to a finite set.

With all the pieces present, we aim now to give an equidistribution theorem related to the $\frac{1}{m}S_m$ inside the base of a graded semi-group S .

Definition 3.4. Let $S \subset \mathbb{N} \times V$ be a discrete graded semi-group. The affine subspace $\text{Aff}(S) \subset V$ generated by $\Delta(S)$ has a natural integral structure defined by the lattice:

$$\vec{\text{Aff}}(S)_{\mathbb{Z}} := \pi_V(\mathbb{Z}S \cap (\{0\} \times V))$$

We will denote by μ_S is the Lebesgue measure normalized by the lattice $\vec{\text{Aff}}(S)_{\mathbb{Z}} = \pi_V(\mathbb{Z}S \cap \{0\} \times V)$.

Lemma 3.5. There exists an affine isomorphism $\phi : \text{Aff}(S) \rightarrow \mathbb{R}^{\kappa(S)}$ of linear part $\vec{\phi} : \vec{\text{Aff}}(S) \rightarrow \mathbb{R}^{\kappa(S)}$ verifying:

- $\vec{\phi}(\vec{\text{Aff}}(S)) = \mathbb{Z}^{\kappa(S)}$
- $\phi\left(\frac{1}{m}S_m^{\text{reg}}\right) = \frac{1}{m}\mathbb{Z}^{\kappa(S)} \cap \Delta$ where $\Delta = \phi(\Delta(S))$, $\forall m \in \mathbb{N}(S)$.

Proof. Let $\rho : \mathbb{Z}S \rightarrow \mathbb{Z}$ be the restriction to $\mathbb{Z}S$ of the projection onto the first component $\mathbb{Z} \times V \rightarrow \mathbb{Z}$. We have that $\text{Im}(\rho) = \mathbb{Z}m(S)$. Since every short exact sequence of free \mathbb{Z} -modules splits, we can construct one such as

$$0 \rightarrow \ker \rho \rightarrow \mathbb{Z}S \xrightarrow{\rho} \mathbb{Z}m(S) \rightarrow 0$$

meaning that there is a \mathbb{Z} -basis of $\mathbb{Z}S$ $u_0, u_1, \dots, u_{\kappa}$ such that $\rho(u_0) = m(S)$ and $p(u_1, \dots, u_{\kappa})$ generates $\ker \rho = \mathbb{Z}S \cap (\{0\} \times V)$. We define ϕ appropriately as the composition

$$\text{Aff}(S) \xrightarrow{\sim} \mathbb{R}S \cap (\{1\} \times V) \hookrightarrow \mathbb{R}S = \bigoplus_{i=0}^{\kappa} \mathbb{R}u_i \xrightarrow{\rho} \bigoplus_{i=1}^{\kappa} \mathbb{R}u_i \simeq \mathbb{R}^{\kappa(S)}$$

with linear part:

$$\vec{\text{Aff}}(S) \xrightarrow{\sim} \mathbb{R}S \cap (\{0\} \times V) = \bigoplus_{i=1}^{\kappa} \mathbb{R}u_i \simeq \mathbb{R}^{\kappa(S)}$$

Having proved the first part of the lemma. Regarding the latter, notice that $\frac{1}{m}S_m^{\text{reg}}$ is sent by ϕ to

$$\frac{1}{m}(\{m\} \times S_m^{\text{reg}}) = \frac{1}{m}\overline{C}(S) \cap \mathbb{Z}S \cap (\{m\} \times V) = \frac{1}{m}(\overline{C}(S) \cap \rho^{-1}(\{m\}))$$

which coincides with

$$\frac{1}{m} \overline{C}(S) \cap \left(\sum_{i=1}^k \mathbb{Z}u_i + \frac{m}{m(S)} u_{\kappa+1} \right)$$

where by hypothesis m is a multiple of $m(S)$ or otherwise it would be empty. One can quickly see that this is indeed $\frac{1}{m} \mathbb{Z}^{\kappa(S)} \cap \Delta$. \square

We can state now an equidistribution result as the following theorem:

Theorem 3.6. *Let $S \subset \mathbb{N} \times V$ a discrete graded semi-group. For any continuous function with compact support $f : V \rightarrow \mathbb{R}$ we have that*

$$\lim_{m \in \mathbb{N}(S), m \rightarrow \infty} \frac{1}{m^{\kappa(S)}} \sum_{x \in \frac{1}{m} S_m} f(x) = \int_{\Delta(S)} f d\mu_S$$

Proof. For the proof of this theorem we will need an auxiliary lemma, whose proof can be found in Boucksom's article.

Lemma 3.7. *Let $S \subset \mathbb{N} \times V$ be a discrete graded semi-group, $K \subset V$ a convex compact set contained in $\Delta(S)$ (relative to V). Then, for $m \gg 0$ we have*

$$K \cap \frac{1}{m} S_m = K \cap \frac{1}{m} S_m^{reg}$$

Using this result, we will first prove the desired statement for S^{reg} and then we will go back to the general case.

Take the affine isomorphism $\phi : \text{Aff}(S) \rightarrow \mathbb{R}^{\kappa(S)}$ and denote $g := f|_{\text{Aff}(S)} \circ \phi^{-1}$. The second statement of Lemma 3.5 gives us the equality

$$\frac{1}{m^{\kappa(S)}} \sum_{x \in \frac{1}{m} S_m^{reg}} f(x) = \frac{1}{m^{\kappa(S)}} \sum_{y \in \frac{1}{m} \mathbb{Z}^{\kappa(S)} \cap \Delta} g(y), \quad \forall m \in \mathbb{N}m(S)$$

Since Δ is a convex set, its border $\partial\Delta$ is of null measure in $\mathbb{R}^{\kappa(S)}$. Thus, $\mathbb{1}_{\Delta} g$ is Riemann-integrable and so

$$\lim_{m \rightarrow \infty} \frac{1}{m^{\kappa(S)}} \sum_{y \in \frac{1}{m} \mathbb{Z}^{\kappa(S)} \cap \Delta} g(y) = \int_{\Delta} g d\lambda$$

with λ the Lebesgue measure over $\mathbb{R}^{\kappa(S)}$. After composing with ϕ one gets $\int_{\Delta} g d\lambda = \int_{\Delta(S)} f d\mu_S$.

Now, in order to extend to the general case, it suffices to see that

$$\frac{1}{m^{\kappa(S)}} \sum_{x \in \frac{1}{m} S_m^{reg} \setminus \frac{1}{m} S_m} f(x) = o(m^{\kappa(S)})$$

Let $\text{supp } f \subset B$ be a closed ball for a certain norm $\|\cdot\|$ of V , and $\varepsilon > 0$. Since $\Delta(S) \cap B$ is a convex compact set, we can find a convex compact set $K \subset V$ inside the interior of $\Delta(S)$ and a function $\chi \in C_c^0(V)$ such that $0 \leq \chi \leq 1$ with $\chi \equiv 1$ in $(\Delta(S) \cap B) \setminus K$ and $\int_{\Delta(S)} \chi d\mu_S \leq \varepsilon$.

Lemma 3.7 implies that $\forall m \gg 1$ we have $\frac{1}{m} S_m^{reg} \setminus \frac{1}{m} S_m \subset K^c$, so $\chi \equiv 1$ on $B \cap (\frac{1}{m} S_m^{reg} \setminus \frac{1}{m} S_m)$. Setting $\|f\| = \sup |f|$ we get

$$\sum_{x \in \frac{1}{m} S_m^{reg} \setminus \frac{1}{m} S_m} |f(x)| \leq \|f\| \cdot \sum_{x \in \frac{1}{m} S_m^{reg}} \chi(x)$$

Since by the general case we have

$$\frac{1}{m^{\kappa(S)}} \sum_{x \in \frac{1}{m} S_m^{reg}} \chi(x) \xrightarrow{m \rightarrow \infty} \int_{\Delta(S)} \chi d\mu_S \leq \varepsilon$$

we get $\sum_{x \in \frac{1}{m} S_m^{reg} \setminus \frac{1}{m} S_m} |f(x)| \leq 2\|f\|\varepsilon$ for $m \gg 0$, from where we get what we need for the general case. \square

As immediate consequences of our theorem characterising $\Delta(S)$ we have the next corollary and a final approximation theorem.

Corollary 3.8. *For any $S \subset \mathbb{N} \times V$ graded semi-group*

$$\lim_{m \in \mathbb{N}(S), m \rightarrow \infty} \frac{\#S_m}{m^{\kappa(S)}} = \mu_S(\Delta(S)) \in (0, +\infty]$$

being finite if and only if $\Delta(S)$ is compact.

Theorem 3.9. *Let $S \subset \mathbb{N} \times V$ be a discrete graded semi-group, $T(m) \subset S$ a sequence of sub-semi-groups verifying $S_m \subset T(m)_m \forall m$. Then we have*

- $\text{Aff}(T(m)) = \text{Aff}(S)$ and $\vec{\text{Aff}}(T(m))_{\mathbb{Z}} = \vec{\text{Aff}}(S)_{\mathbb{Z}}$ for $m \gg 0$.
- $\lim_{m \in \mathbb{N}(S), m \rightarrow \infty} \mu_{T(m)}(\Delta T(m)) = \mu_S(\Delta(S)) \in [0, +\infty]$

Proof. Let $\langle S_m \rangle \subset S$ denote the sub-semi-group generated by $S_m \times \{m\}$. Its base corresponds with

$$\Delta(\langle S_m \rangle) = \pi_V(\overline{C}(S_m) \cap (\{1\} \times V)) = \text{Conv}(\frac{1}{m}S_m)$$

and our initial hypothesis translates in this context to $\langle S_m \rangle \subset T(m) \subset S$.

We will begin by showing $\lim_{m \in \mathbb{N}(S), m \rightarrow \infty} \mu_S(\Delta(\langle S_m \rangle)) = \mu_S(\Delta(S))$, which will imply both that $\Delta(\langle S_m \rangle)$ is of nonempty interior inside $\text{Aff}(S)$ and $\text{Aff}(\langle S_m \rangle) = \text{Aff}(S)$ for all $m \gg 0$, which will imply itself $\text{Aff}(T(m)) = \text{Aff}(S)$ by sandwich.

The inclusion $\Delta(\langle S_m \rangle) \subset \Delta(S)$ for all S_m gives us immediately that $\limsup_{m \in \mathbb{N}(S), m \rightarrow \infty} \mu_S(\Delta(\langle S_m \rangle)) \leq \mu_S(\Delta(S))$.

On the other hand, we need the following auxiliary lemma:

Lemma 3.10. *Let W be a \mathbb{R} -vector space of finite dimension and (C_α) an increasing family of convex sets. Then, the interior of the union coincides with the union of interiors $\bigcup_\alpha \overset{\circ}{C}_\alpha$*

Proof. (of Lemma). Going right to left is trivial by definition of interior. For going left to right, pick x in the interior of the union. We can find $x_1, x_2, \dots, x_N \in \bigcup_\alpha C_\alpha$ such that $x \in \text{Conv}\{x_1, x_2, \dots, x_N\}$. But there is a C_α containing all x_i , thus its convex hull and by necessity $x \in \overset{\circ}{C}_\alpha$. \square

Knowing this, we have that the characterization of the base

$$\Delta(S) = \overline{\bigcup_{m \geq 1} \frac{1}{m}S_m} = \overline{\bigcup_{m \geq 1} \text{Conv}(\frac{1}{m}S_m)} = \overline{\bigcup_{m \geq 1} \Delta(\langle S_m \rangle)}$$

with $\Delta(\langle S_m \rangle) \subseteq \Delta(\langle S_{m+1} \rangle)$.

The lemma gives us that for any compact set $K \subset \Delta(S)$ we will have $K \subset \Delta(\langle S_m \rangle)$ for m sufficiently large.

Therefore $\liminf_{m \in \mathbb{N}(S), m \rightarrow \infty} \mu_S(\Delta(\langle S_m \rangle)) \geq \mu_S(K)$ for all compact subset in the interior of $\Delta(S)$, and since μ_S is a Radon measure, we obtain $\liminf_{m \in \mathbb{N}(S), m \rightarrow \infty} \mu_S(\Delta(\langle S_m \rangle)) \geq \mu_S(\Delta(S))$, getting the desired equality.

Now, we will show $\vec{\text{Aff}}(\langle S_m \rangle)_{\mathbb{Z}} = \vec{\text{Aff}}(S)_{\mathbb{Z}}$ in order to get $\vec{\text{Aff}}(T(m))_{\mathbb{Z}} = \vec{\text{Aff}}(S)_{\mathbb{Z}}$ and $\mu_{T(m)} = \mu_S$.

In order to do so, we will consider $\phi : \text{Aff}(S) \xrightarrow{\sim} \mathbb{R}^{\kappa(S)}$ the affine morphism from Lemma 3.5, and we will denote $S_m - S_m = \mathbb{Z}S_m$ the group generated by S_m . To show that $S_m - S_m$ generates $\text{Aff}(S)_{\mathbb{Z}}$, we will see instead that through the isomorphism, i.e., that

$$\vec{\phi}(S_m - S_m) = m \cdot (\vec{\phi}(\frac{1}{m}S_m) - \vec{\phi}(\frac{1}{m}S_m))$$

generates $\mathbb{Z}^{\kappa(S)}$.

Let $K \subset \Delta(S)$ be a compact set, by Lemma 3.7 we know $\frac{1}{m}S_m$ contains $\frac{1}{m}S_{reg} \cap K$ for $m \gg 0$.

This means that, by Lemma 3.5, $m \cdot (\vec{\phi}(\frac{1}{m}S_m) - \vec{\phi}(\frac{1}{m}S_m))$ contains the set of all differences between points in $\mathbb{Z}^{\kappa(S)} \cap (m \cdot \phi(K))$. Taking m large enough so that $m \cdot \phi(K)$ contains $[-1, 1]^{\kappa(S)}$, we have that the image set $m \cdot (\vec{\phi}(\frac{1}{m}S_m) - \vec{\phi}(\frac{1}{m}S_m))$ generates $\mathbb{Z}^{\kappa(S)}$, ending the proof. \square

3.2 Construction of Okounkov bodies

From now on we will use the following conventions: X will denote a proper algebraic variety over $k = \bar{k}$ a field of arbitrary characteristic. X will have as function field $k(X) = K$. In regards to notation, n will be the dimension of X , and the tensor product of line bundles will be expressed as $mL = L^{\otimes m}$.

Consequently, we will denote the **algebra of sections of L over X** as

$$R(X, L) := \bigoplus_{m \in \mathbb{N}} H^0(X, mL)$$

If we consider a valuation ν over K with center ξ in X , we can define ν over the algebra of sections the following way:

Consider $\sigma \in H^0(X, L) \setminus \{0\}$, in a neighbourhood of ξ we can take a trivialization τ of L so that $\sigma = \tau \cdot f$ for $f \in \mathcal{O}_{X, \xi}$. Since any other trivialization is of the form $\tau \cdot u$ with $u \in U(\mathcal{O}_{X, \xi})$ we have that $\nu(\sigma) = \nu(f)$ is well-defined.

Notice that by construction $\nu(\sigma) > 0$ if and only if σ is null on ξ . Also, for L' another line bundle on X , $\sigma' \in H^0(X, L')$ we have $\nu(\sigma \otimes \sigma') = \nu(\sigma) + \nu(\sigma')$.

This opens the path for us to define the concepts introduced in the previous sections in the context of line bundles:

Definition 3.11. *Let ν be a valuation over K , L a line bundle over X such that $R(X, L) \neq k$. Then we define the **graded semi-group of values of ν over L** to be:*

$$S_\nu(X, L) := \left\{ (m, \nu(\sigma)) : m \in \mathbb{N}, \sigma \in H^0(X, mL) \setminus \{0\} \right\}$$

$S_\nu(X, L)$ is a sub-semi-group of $\mathbb{N} \times V_\nu$.

More generally, for any graded sub-algebra $A \subseteq R(X, L)$ we have

$$S_\nu(A) := \left\{ (m, \nu(\sigma)) : m \in \mathbb{N}, \sigma \in A_m \setminus \{0\} \right\}$$

with $S_\nu(A)_m = \nu(A_m \setminus \{0\})$, having that $\mathbb{N}(S_\nu(A)) = \mathbb{N}(A)$.

The next proposition brings over to valuation over the algebra of section the necessary properties of graded semi-groups of values obtained from *MRR* valuations.

Proposition 3.12. *Let ν be a MRR valuation over K . For all line bundles L over K , $k \neq A \subseteq R(X, L)$ graded sub-algebra we have:*

- i. $S_\nu(A) \subset \mathbb{N} \times V_\nu$ is a discrete graded semi-group.*
- ii. $\#S_\nu(A)_m = \dim_k A_m$ for all $m \in \mathbb{N}$.*
- iii. $\kappa(S_\nu(A)) = \kappa(A)$.*
- iv. If L is big, then $S_\nu(X, L)$ generates $\mathbb{Z} \times \Lambda_\nu$.*

Proof. The first point comes as a result of the fact that Λ_ν is a lattice of V_ν , by Proposition 2.16. The semi-group of values $S_\nu(A)$ being contained in $\mathbb{Z} \times \Lambda_\nu$ generates a discrete subgroup of $\mathbb{R} \times V_\nu$, which is itself the definition of discrete semi-group.

The second point is a direct consequence of $\nu(A_m \setminus \{0\}) = S_\nu(A)_m$ and Proposition 2.16.ii.

Point *iii.* comes working with the extension $\hat{\nu}_L$ of ν to $K(L)$ the field of functions of the total space of L^\vee , which in particular contains $R(X, L) = \mathcal{O}(L^\vee)$. We have that this extension is also *MRR*, of $\text{rat. rk } \hat{\nu}_L = \text{rat. rk } \nu + 1$ since L is trivial for an open (Zariski) neighbourhood of ξ center of ν . By Proposition 2.16.iii. we know its restriction to $K(A)$ is also *MRR* and if we pair this with the fact that $\hat{\nu}_L(K(A)^*) = \mathbb{Z}S_\nu(A)$, which comes as a straightforward consequence of the definition, we get the desired result.

The final point is a refinement of the result, given as a consequence of *iii.*, that $\mathbb{Z}S_\nu(X, L)$ is of finite index in $\mathbb{Z} \times \Lambda_\nu$. Since Λ_ν is a free abelian

group of rank n , we can find $f_1, \dots, f_n \in K^*$ such that their images $\nu(f_i)$ generate Λ_ν .

We consider the conductor sheaf $\mathfrak{c} = \{g \in \mathcal{O}_X : g \cdot f_i \in \mathcal{O}_X\}$. It is a coherent sheaf of ideals, therefore there is an ample line bundle H such that $\mathcal{O}_X(H) \otimes \mathfrak{c}$ admits a nontrivial global section τ . For $i = 1, \dots, n$ we have that $\tau \cdot f_i$ defines a global section $\sigma_i \in H^0(X, H)$ such that $\nu(f_i) = \nu(\sigma_i) - \nu(\tau)$.

Since L is big, we can choose $m \in \mathbb{N}^*$ such that $mL - H$ admits a nonzero section s . We have then that $S_\nu(X, L)$ contains both $(m, \nu(s) + \nu(\tau))$ and $(m, \nu(s) + \nu(\sigma_i))$ for all $i = 1, \dots, n$, and so $\mathbb{Z}S_\nu(X, L)$ contains all $(0, \nu(f_i))$ and $\{0\} \times \Lambda_\nu$ as a consequence.

To see that it also contains $(1, 0)$, we take H very ample such that $H' := H + L$ is also very ample. A nonzero section $s \in H^0(X, mL - H)$ for m large enough can be seen also as a section of $(m + 1)L - H'$. Because H, H' are very ample, each admit a nonvanishing section at ξ center of ν . This makes $(m, \nu(s)), (m + 1, \nu(s)) \in S_\nu(X, L)$ and thus $(1, 0) \in \mathbb{Z}S_\nu(X, L)$. \square

From now on ν will be a fixed *MRR* valuation and L a line bundle on X , A a nontrivial graded sub-algebra of $R(X, L)$. Without further build-ups, let us now dive into the main definition, that which gives name to this section.

Definition 3.13. *The **Okounkov body of A relative to ν** is defined as the base, in the sense of Definition 3.3, of the graded semi-group $S_\nu(A) \subset \mathbb{N} \times V_\nu$. We will denote it by $\Delta_\nu(A)$.*

For $A = R(X, L)$ we abuse notation and write $\Delta_\nu(X, L)$. Same as with any base we can characterize the Okounkov body as

$$\Delta_\nu(A) = \overline{\bigcup_{m \geq 1} \left\{ \frac{1}{m} \nu(s) : s \in A_m \setminus \{0\} \right\}} \subset V_\nu$$

$\Delta_\nu(A)$ is a closed convex set. Even more:

Lemma 3.14. *$\Delta_\nu(A)$ is compact.*

Proof. It is enough to prove this for the largest case $A = R(X, L)$. Chow's lemma tells us that by X being proper there exists Y projective variety and $\pi : Y \dashrightarrow X$ birational map. Thus, we can find H ample line bundle on Y such that $H - \pi^*L$ admits a nonzero global section $s \in H^0(Y, H - \pi^*L)$.

Thinking of H and π^*L as divisors we have that $H = \text{div}(s) + \pi^*L$. If we take ν as a valuation over $k(Y) \simeq k(X)$ we have that $\nu(s) + \Delta_\nu(X, L) \subset \Delta_\nu(Y, H)$, leading us to show instead the compactness of $\Delta_\nu(Y, H)$.

We can take now advantage of working with a projective variety, since we have that by Hilbert-Serre's theorem $\dim H^0(Y, mH) = O(m^n)$, which by Proposition 3.12, *ii*. we have $\#S_\nu(Y, H)_m = O(m^n)$, and the same proposition gives us $\kappa(S_\nu(Y, H)) = n$.

If we plug our values in the expression of Corollary 3.8 we get:

$$\lim_{m \rightarrow \infty} \frac{\#S_m}{m^{\kappa(S)}} = \lim_{m \rightarrow \infty} \frac{O(m^n)}{m^n} < +\infty$$

concluding that $\Delta_\nu(Y, H)$ is compact, and so are all others. \square

As we saw when we worked with bases in Section 1, the affine subspace of V_ν generated by $\Delta_\nu(A)$ has a natural integral structure that allows to normalize its Lebesgue measure $\mu_{S_\nu(A)}$. In particular it is of interest the following case:

Remark 3.15. *For L big, we have that μ_ν , the Lebesgue measure of V_ν normalized by its lattice Λ_ν coincides with the measure given by the semi-group of values, that is, $\mu_\nu = \mu_{S_\nu(X, L)}$.*

Proof. Since the affine hull of $\Delta_\nu(X, L)$ has an integral structure dependent only on $\mathbb{Z}S_\nu(X, L)$, Proposition 3.12. *iv*. tells us that $\mathbb{Z}S_\nu(X, L) = \mathbb{Z} \times \Lambda_\nu$, \square

This leads us to state the main result concerning Okounkov bodies, that which makes them so useful.

Theorem 3.16. *Let X be a proper variety over k , L a line bundle on X . For any non-trivial graded sub-algebra $A \subset R(X, L)$, the limit*

$$\lim_{m \in \mathbb{N}(A), m \rightarrow +\infty} \frac{\dim_k A_k}{m^{\kappa(A)}} \in (0, +\infty)$$

Moreover, for any ν MRR valuation over $k(X)$ we have

$$\lim_{m \in \mathbb{N}(A), m \rightarrow +\infty} \frac{\dim_k A_k}{m^{\kappa(A)}} = \mu_{S_\nu(A)}(\Delta_\nu(A))$$

This fundamental result is a consequence of Corollary 3.8, and for A finitely generated it can be seen as a consequence of Hilbert-Serre's theorem. It is interesting to us then that Theorem 3.16 allows for more generality as neither A nor $S_\nu(A)$ need to be finitely generated. This leads us to define the following notion:

Definition 3.17. *The **multiplicity of A** is defined as*

$$e(A) := \lim_{m \in \mathbb{N}(A), m \rightarrow +\infty} \frac{\dim_k A_m}{m^{\kappa(A)} / \kappa(A)!}$$

This multiplicity doesn't need to be an integer, not even a rational, number. This notion relates back to big bundles as reflected in the next corollary, which is a straightforward conclusion of Theorem 3.16 and Definition 2.40

Corollary 3.18. *For L big line bundle of $\text{vol}(L)$ we have*

$$e(R(X, L)) = \text{vol}(L) = n! \mu_\nu(\Delta_\nu(X, L))$$

where μ_ν is the Lebesgue measure of V_ν normalized by the lattice Λ_ν .

That is, in the concept of Okounkov bodies we find a way to measure the volume of big line bundles in terms of the Euclidean volume of a convex body. This in essence is the main result on which lies the interest of Okounkov bodies. There are, however, other results regarding the shape of Okounkov bodies that hint at a deeper relation between them.

To put all this information in context, we will now consider a practical simple example of an Okounkov body.

Example 3.19 (The case of curves). *Consider C a smooth projective curve of genus g . For this case any valuation over $k(C) = K$ corresponds with a discrete valuation $\nu : K \rightarrow \mathbb{Z}$ multiple of the valuation measuring the order of vanishing at a point $p \in C$.*

A line bundle L on C with nontrivial algebra of sections $R(C, L) \neq k$ must necessarily be ample, since for the case $n = 1$ ampleness becomes equivalent with positive degree. Let us say then that L is of degree $d > 0$.

This means that, as a result of Riemann-Roch Theorem (Theorem 2.48) for all $m \gg 1$ $S_{\nu_p}(C, L)_m$ consists of $md + 1 - g$ integers between 0 and md , corresponding to the possible vanishing orders of mL at p .

In fact, as it happens that $md > 2g - 2$, we have that $h^1(C, mL) = h^0(C, K_C - mL) = 0$ by Serre's duality, which when put back in Riemann-Roch's formula gives us that:

$$h^0(C, mL) - h^1(C, mL) = h^0(C, mL) = md + 1 - g$$

Together with Proposition 3.12.ii. we get that $\#S_{\nu_p}(C, L)_m = md + 1 - g$. Moreover we have $\nu(\sigma) \leq md$ for all $\sigma \in H^0(C, mL) \setminus \{0\}$. The Okounkov body in this case clearly amounts to

$$\Delta_{\nu_p}(C, L) = [0, d]$$

It is worth to note that such a simple case can still make $S_{\nu_p}(C, L)$ not be of finite type. In fact, it can be seen that $S_{\nu_p}(C, L)$ will be of finite type if and only if $\exists \sigma \in H^0(C, mL)$ such that $\nu_p(\sigma) = md$.

Even more, we can see that

$$S_{\nu_p}(C, L) \text{ is of finite type} \iff L - dp \text{ is of torsion in } \text{Pic}^0(C).$$

Multiplicity holds geometrical meaning with respect to the induced mapping into a projective space by linear series of L , but we will focus instead in other properties and refer to [4] §3.1 for more in this regard.

3.3 Numerical Okounkov bodies

As well as the fundamental result for Okounkov bodies, there are some further results that relate Okounkov bodies with numerical classes of \mathbb{R} -divisors inside the \mathbb{R} -vector space spanned by the Neron-Severi group $N^1(X)$.

We can interpret any valuation over $k(X)$ as a \mathbb{Z} -linear map $\nu : \text{Div}(X) \rightarrow \Lambda_\nu$ where $\nu(D) = \nu(f)$ for f a local equation of the divisor D at a neighborhood of ξ the center of ν . For this map we have that $\nu(\text{Div}_{\geq 0}) \subset S_\nu(X)$. By linearity, we may extend the valuation into an \mathbb{R} -linear operator $\nu : \text{Div}(X)_\mathbb{R} \rightarrow V_\nu$.

If L is a line bundle we have the equivalence

$$R(X, L) \neq k \iff \{D \in \text{Div}_{\geq 0}(X)_\mathbb{Q} : D \sim_\mathbb{Q} L\} \neq \emptyset$$

where $D \sim_\mathbb{Q} L$ is a generalization of linear equivalence for \mathbb{Q} -divisors. Two divisors will be equivalent if there exists $m \in \mathbb{N}^*$ such that $m(D-L) = \text{div}(f)$ for $f \in k(X)^*$, i.e, if $mD \sim_{lin.} mL$ in $\text{Div}(X)$.

If $R(X, L) \neq k$, then there is a power mL such that there is a non-zero global section $s \in H^0(X, mL)$, which as such defines an effective divisor $D = \text{div}(s)$ linearly equivalent to mL . If we consider the \mathbb{Q} -divisor $D' = \frac{1}{m}D$, we have $D' \sim_\mathbb{Q} L$ and the set is not empty.

On the other hand, if we have D \mathbb{Q} -divisor effective such that $D \sim_\mathbb{Q} L$, then mL is linearly equivalent to an effective divisor, and therefore it must have a global section, making $R(X, L)$ non-trivial.

This fact, paired together with the characterization of the base given in Section ??, implies that the Okounkov body of the algebra of sections can be characterized as:

$$\Delta_\nu(X, L) = \overline{\nu\left(\{D \in \text{Div}_{\geq 0}(X)_{\mathbb{Q}} : D \sim_{\mathbb{Q}} L\}\right)}$$

where the correspondence comes when one considers $\nu(s) = \nu(mD) = m \cdot \nu(D)$ for $s \in H^0(X, mL)$. Even more:

Proposition 3.20. *For L a big line bundle on X we have*

$$\Delta_\nu(X, L) = \overline{\nu\left(\{D \in \text{Div}_{\geq 0}(X)_{\mathbb{R}} : D \sim_{num.} L\}\right)}$$

In particular, Okounkov bodies are invariant under numerical equivalence of big line bundles.

Proof. The inclusion " \subseteq " is clear, since numerical equivalence is coarser than $\sim_{\mathbb{Q}}$. For the other inclusion, we have to see that $\nu(D)$ is in $\Delta_\nu(X, L)$.

In order to reason for D effective \mathbb{R} -divisor, we will generalize the case D a \mathbb{Q} -divisor in the following manner. Consider $W \subset W \text{Div}(X)_{\mathbb{R}}$ the \mathbb{R} -vector subspace generated by the irreducible components of D . Note that any divisor in W sufficiently close to D will be effective.

We consider the affine space defined over \mathbb{Q} :

$$N := \{D \in \text{Div}(X)_{\mathbb{R}} : D \in W, D \sim_{num.} L\}$$

We have that $D \in N$ and is limit of a sequence $\{D_j\}_j \subset N$. We have that $\nu(D) = \lim_{j \rightarrow +\infty} \nu(D_j)$ and D_j are \mathbb{Q} -divisors equivalent to L and eventually effective for $j \gg 1$.

This way we can reason for D \mathbb{Q} -Cartier, \mathbb{Q} -effective numerically equivalent to L and see whether $\nu(D) \in \Delta_\nu(X, L)$. We have that, as a consequence of Fujita's vanishing theorem (Theorem 2.39), that we can always find a line bundle H on X such that $H + P$ is very ample for all P nef line bundles.

Since D is big by hypothesis, we can choose $m_0 \in \mathbb{N}$ such that $m_0 D$ is Cartier and $m_0 D - H$ has a nontrivial section $\sigma \in H^0(X, m_0 D - H) \setminus \{0\}$. For all $m \in \mathbb{N}$, let us write $P_m = (m + m_0)(L - D)$. We can see that:

$$(m + m_0)L = mD + m_0(D - H) + (H + P_m)$$

If m is sufficiently divisible, then P_m is numerically trivial. $H + P_m$ is thus very ample and admits a section σ non-vanishing at ξ center of ν in X . This way

$$\frac{m\nu(D) + \nu(\sigma)}{m + m_0} \in \Delta_\nu(X, L)$$

Since it is true for m sufficiently divisible, we have it is true for the limit which means $\nu(D) \in \Delta_\nu(X, L)$. \square

This proposition admits a reciprocal result, due to Jow (See [21]): For two big line bundles L_1, L_2 with the same Okounkov body $\Delta_\nu(X, L_1) = \Delta_\nu(X, L_2)$ for all MRR valuations over $k(X)$, then $L_1 \sim_{num.} L_2$.

Definition 3.21. Let $\delta \in N^1(X)_\mathbb{Q}$ be a big class. We define the **numerical Okounkov body of δ** as

$$\Delta_\nu^{num}(X, \delta) = \frac{1}{m} \Delta_\nu(X, L)$$

for L big line bundle such that $c_1(L) = m\delta$ for $m \in \mathbb{N}^*$.

We can extend this definition to the pseudoeffective case the following way:

Definition 3.22. The **global numerical Okounkov body of X relative to ν** $\Delta_\nu^{num}(X)$ is defined to be the closure inside $V_\nu \times N^1(X)$ of

$$\bigcup_{\delta \in N^1(X)_\mathbb{Q} \cap \text{Big}(X)} \Delta_\nu(X, \delta) \times \{\delta\}$$

If $\delta \in \overline{\text{Big}(X)}$ is only a pseudoeffective class, then we define its numerical Okounkov body as the fiber

$$\Delta_\nu^{num}(X, \delta) := \{x \in V_\nu : (x, \delta) \in \Delta_\nu^{num}(X)\}$$

This global Okounkov body is a closed convex cone, it verifies that:

$$\Delta_\nu^{num}(X, \delta) + \Delta_\nu^{num}(X, \delta') \subset \Delta_\nu^{num}(X, \delta + \delta')$$

$$\Delta_\nu^{num}(X, t\delta) = t\Delta_\nu^{num}(X, \delta)$$

for all $\delta, \delta' \in N^1(X)$ pseudoeffectives and $t \in \mathbb{R}_+$.

Numerical Okounkov bodies admit as well the next description:

Proposition 3.23. *For any $\delta \in N^1(X)$ big, $\Delta_\nu^{num}(X, \delta)$ is a convex compact set of non-empty interior with*

$$\Delta_\nu^{num}(X, \delta) = \overline{\nu\left(\{D \in \text{Div}_{\geq 0}(X)_{\mathbb{R}} : D \sim_{num.} \delta\}\right)}$$

If δ is only pseudoeffective, we have that $\Delta_\nu^{num}(X, \delta)$ is a non-empty convex compact set and

$$\Delta_\nu^{num}(X, \delta) = \bigcap_{\varepsilon > 0} \Delta_\nu^{num}(X, \delta + \varepsilon h)$$

for all ample classes $h \in N^1(X)$.

We leave here Okounkov bodies in order to delve into the theory of prevaluations. If one needs further motivation to study the relation between Okounkov bodies coming from different valuations, think that although the volume of said bodies will be invariant under numerical equivalence, its shape will vary wildly from valuation to valuation.

Examples of this sadly fall out of reach for us, since we decided to state everything in a more general setting and getting into the details of certain standard valuations used in this context, mainly **flag valuations**, is out of our time budget. We leave the sources [14] and [15] for this purpose.

4 Prevaluations

This section will cover in a first instance the concept of prevaluation and the definition of a product of prevaluations. Later, we will do a preliminary approach on how one may relate Okounkov bodies with prevaluation product, namely if we can define such a thing as the Okounkov body of the product of two *MRR* valuations, and whether or not we may recover information from this construction.

4.1 Prevaluations on idempotent semi-rings

We will present now the main definitions and results that relate prevaluations with Krull valuations. The original concept of prevaluations is due to the Giansiracusa brothers and can be found in [16], but we will use as a reference for the definitions [17], where the concepts are directly stated in a more intuitive way for our purposes.

Definition 4.1. A *commutative idempotent semi-ring* $(S, +, \star, 0_S, 1_S)$ is a tuple where S is a set, $+$, \star are binary operations such that:

- (S, \star) is a commutative monoid with identity element 1_S
- $(S, +)$ is a commutative **monoid** (i.e, we drop the opposite element assumption) with identity element 0_S and such that $a + a = a$ for all $a \in S$.
- \star is distributive with respect to $+$
- 0_S annihilates S with respect to \star .

Idempotent semi-rings are, in lay man terms, rings where the existence of negatives $-a$ is not assumed and such that addition creates singleton subgroups $\{a\}$. Semi-ring morphisms are analog to ring homomorphisms with these relaxed conditions.

Example 4.2. Perhaps the most famous example of idempotent semi-ring is the **tropical semi-ring** $\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \min, +_{\mathbb{R}}, \infty, 0)$. Here one has as a set the extended real numbers and as operations

$$a +_{\mathbb{T}} b = \min\{a, b\} \quad , \quad a \star_{\mathbb{T}} b = a +_{\mathbb{R}} b$$

This is very well-studied object and in fact gives name to a whole branch of study, that of tropical geometry.

Example 4.3. Another well-known example, in fact the only idempotent semi-ring of order 2, is the **Boolean semi-ring** $\mathbb{B} = (\{\perp, \top\}, \vee, \wedge, \perp, \top)$

These idempotent semi-rings carry in themselves an implicit partial order, which we will use to build prevaluations out of them. Two elements $a, b \in S$ are ordered in the following manner:

$$a \leq b \iff a + b = a$$

This way, in the tropical semi-ring case one has that the partial order coincides with the usual ordering of elements of \mathbb{R} . Even more, in this case the induced order is total. This defined partial order is consistent with the structure defined by semi-ring morphisms.

Proposition 4.4. *Semi-ring morphisms preserve partial orders.*

Proof. For $\varphi : S \rightarrow S'$ semi-ring morphism we have that, if $a + b = a$, then

$$\varphi(a) + \varphi(b) = \varphi(a + b) = \varphi(a)$$

and therefore $\varphi(a) \leq \varphi(b)$. □

We may now present the central definition to this section:

Definition 4.5. A *Giansiracusa-Giansiracusa valuation*, or **prevaluation** is a map $\nu : R \rightarrow S$ from R an integral domain, S a commutative idempotent semi-ring such that it verifies:

- i. $\nu(ab) = \nu(a) \star \nu(b)$
- ii. $\nu(0_R) = 0_S$
- iii. $\nu(1_R) = \nu(-1_R) = 1_S$
- iv. $\nu(a + b) + \nu(a) + \nu(b) = \nu(a) + \nu(b)$

If $\nu^{-1}(0_S) = \{0_R\}$, then we say the prevaluation is **non-degenerate**. Unless stated otherwise, we will always assume this is the case.

Remark 4.6. *Note that property iv. is equivalent to $\nu(a + b) \geq \nu(a) + \nu(b)$ with respect to the induced partial order.*

Since the defining conditions of prevaluation mirror that of classical valuations, it is naturally interesting to see how these are related with Krull valuations. We address this concern in the following theorem:

Theorem 4.7. *Any classical valuation (Krull valuation) is equivalent to a prevaluation on a totally ordered semi-group.*

Proof. Let us see how a Krull valuation gives naturally a prevaluation whose associated idempotent semi-ring is totally ordered.

Suppose $\nu : R \rightarrow (\Lambda, +_\lambda, \leq)$ is a Krull valuation, defined at 0_R by setting $\nu(0_R) = +\infty$ as in Definition 2.1. We construct from Λ an idempotent semi-ring of the form:

$$(S = \Lambda, +_S = \min, \star_S = +_\lambda, 0_S = +\infty, 1_S = 0_\lambda)$$

We get then $\nu' : R \rightarrow S$ defined by the Krull valuation ν as $\nu'(a) = \nu(a)$. Note as well that this semi-ring is totally ordered, since the induced partial order coincides with the total order of the group.

It is immediate to see that it verifies conditions *i.* to *iii.* from the definition. As for condition *iv.*, we have that $\nu'(a+b) = \nu(a+b) \geq \min \{ \nu(a), \nu(b) \}$, which is equivalent to saying $\min \{ \nu(a+b), \nu(a), \nu(b) \} = \min \{ \nu(a), \nu(b) \}$.

If we take into account our notation $+_S = \min$, then condition *iv.* is fulfilled. This identification is even stronger, since the equality $(S, \star_S) = (\Lambda, +_\lambda)$ tells us that the monoidal structure of the semi-ring is in fact that of a group, making S an idempotent semi-field. \square

This inclusion of Krull valuations into the set of all prevaluations motivates a generalization of the property stated at Lemma 2.2.

Lemma 4.8. *For $X \subseteq S$ finite one has:*

$$\inf(X) = \sum_{x \in X} x \quad , \quad \inf\{a, b\} = a + b$$

In the case of Krull valuations we have that this infimum values become minimums, at least for $\nu(x) \neq \nu(y) \forall x, y \in X$. We will however soon stray away from classical valuations in the next section, so this generalization of the lemma will become useful.

In any case, as a result of this lemma and the remark at the end of the proof of Theorem 4.7 comes an interesting characterization of idempotent semi-rings:

Corollary 4.9. *To define an idempotent semi-ring is equivalent to define $(S, \star, 1_S)$ a monoid together with a meet-semi-lattice, that is, a lattice with a greatest lower bound for all nonempty finite subsets of S .*

Back to Krull valuations, Theorem 4.7 motivates us to translate into the language of prevaluations those concepts classically associated to valuations. To refer to them, from now on we will talk about **Krull prevaluations** whenever we talk about prevaluations coming from valuations.

Definition 4.10. *For a prevaluation ν , we define the **prevaluation ring** of ν $\mathcal{O}_\nu \subseteq K = \text{Frac}(R)$ defined by*

$$\mathcal{O}_\nu = \{f \in K : \nu(f) + 1_S = 1_S\}$$

Remark 4.11. *For Krull prevaluations it coincides with the valuation ring associated to the valuation. In particular, we can identify the maximal ideal of the local ring as*

$$\mathfrak{m}_\nu = \{f \in K : \nu(f) + 1_S = 1_S, \nu(f) \neq 1_S\}$$

Let us see now what happens for a regular prevaluation:

Proposition 4.12. *Let ν be a general prevaluation. Then, \mathcal{O}_ν is a ring, but \mathfrak{m}_ν as above fails to be an ideal in general.*

Proof. Let us see first that \mathcal{O}_ν defines a sub-ring of K . To do so, we will see that it is closed under sum and product.

Take $f, g \in \mathcal{O}_\nu$, we have to check that $f + g$ verifies $\nu(f + g) + 1 = 1$. To do so, we play a bit with the operations, using that by hypothesis $1 = 1 + \nu(f) = 1 + \nu(g)$. We get:

$$\nu(f + g) + 1 = \nu(f + g) + \nu(f) + \nu(g) + 1 = \nu(f) + \nu(g) + 1 = 1$$

where the third equality comes from Definition 4.5.iv. Now, for the product, note that $\nu(f) + 1 = 1 \implies \nu(f)\nu(g) + \nu(g) = \nu(g)$ by distributivity of the product. Then:

$$\nu(fg) + 1 = \nu(fg) + \nu(g) + 1 = (\nu(f) + 1)\nu(g) + 1 = \nu(g) + 1 = 1$$

Therefore \mathcal{O}_ν is a sub-ring of R . To see now when \mathfrak{m}_ν is an ideal of the ring, we will check first it is closed under addition. Let $f, g \in \mathfrak{m}_\nu$. Suppose that $\nu(f + g) = 1$, then:

$$\nu(f + g) + \nu(f) + \nu(g) = 1 + \nu(f) + \nu(g) = \nu(f) + \nu(g)$$

while at the same time

$$\nu(f + g) + \nu(f) + \nu(g) = 1 + \nu(f) + \nu(g) = 1$$

Thus, $1 = \nu(f) + \nu(g)$. If the order were total, then this would imply either $\nu(f) = 1$ or $\nu(g) = 1$, leading to contradiction. However, this is not guaranteed when the order is partial and \mathfrak{m}_ν is thus not an ideal generally. \square

Remark 4.13. *While \mathfrak{m}_ν may not be an ideal always, it always absorbs multiplication by elements of \mathcal{O}_ν .*

Proof. Suppose $f \in \mathfrak{m}_\nu$, $g \in \mathcal{O}_\nu$. We have that $\nu(f) + 1 = 1$, $\nu(g) + 1 = 1$ with $\nu(f) \neq 1$. If it were the case that $\nu(fg) = \nu(f)\nu(g) = 1$, then by distributivity we would get

$$\nu(fg) + \nu(f) = \nu(f) \implies \nu(f) + 1 = \nu(f) \implies \nu(f) = 1$$

arriving at a contradiction, and proving the desired result. \square

4.2 Product of prevaluations

We proceed now to show that a product of prevaluations is well-defined and gives rise to a prevaluation.

We define, for $\nu_1 : R \rightarrow S_1$, $\nu_2 : R \rightarrow S_2$ two prevaluations over the same commutative ring R the prevaluation

$$\nu_{1,2} \equiv \nu_1 \times \nu_2 : R \rightarrow S_1 \times S_2 \quad , \quad \nu_{1,2}(a) = (\nu_1(a), \nu_2(a))$$

The product of idempotent semi-rings is an idempotent semi-ring with operations defined component-wise. Let us see now that this is indeed a prevaluation.

Proposition 4.14. $\nu_{1,2} : R \rightarrow S_1 \times S_2$ is a prevaluation.

Proof. It comes directly from checking that the properties of prevaluations hold true component-wise, and therefore for the product itself. \square

Ideally, one would like the product of Krull prevaluations to be again a Krull prevaluation, but that is not the case. Turns out that the order induced becomes partial, not total, and the locality of the valuation ring is lost. The best we can say is the following:

Proposition 4.15. Let $\nu \equiv \nu_{1,2} : R \rightarrow S = S_1 \times S_2$ be the product of two Krull prevaluations ν_1, ν_2 . Then its prevaluation ring is of the form

$$\mathcal{O}_\nu = \mathcal{O}_{\nu_1} \cap \mathcal{O}_{\nu_2}$$

and fails to be a local ring.

Proof. The shape of the prevaluation ring follows straightforward if we go by definition:

$$\begin{aligned} \mathcal{O}_\nu &= \{f \in K : \nu(f) + 1_S = 1_S\} = \\ &= \{f \in K : \nu(f) + (1_{S_1}, 1_{S_2}) = (1_{S_1}, 1_{S_2})\} \\ &= \{f \in K : \nu(f) + 1_{S_1} = 1_{S_1}\} \cap \{f \in K : \nu(f) + 1_{S_2} = 1_{S_2}\} \end{aligned}$$

Considering $\mathfrak{m}_\nu = \{f \in K : \nu(f) + 1_S = 1_S, \nu(f) \neq 1_S\}$, let us see for $f, g \in K$ such that $\nu(f) + 1_S = 1_S$, $\nu(g) + 1_S = 1_S$ and both different than 1_S what can we say about $\nu(f + g)$.

Of course $\nu(f + g) + 1_S = 1_S$ by definition of the ring. We would need that $\nu(f + g) \neq 1_S$. However, the condition $\nu(f) \neq 1_S$ is fulfilled for $f \in \mathfrak{m}_{\nu_1} \cap \mathcal{O}_{\nu_2}$ as well as for $f \in \mathfrak{m}_{\nu_2} \cap \mathcal{O}_{\nu_1}$. This means that $\mathfrak{m}_\nu = (\mathfrak{m}_{\nu_1} \cup \mathfrak{m}_{\nu_2}) \cap \mathcal{O}_\nu$, which is not an ideal in general. \square

We have then that two Krull valuations over the same field of fractions give rise to another prevaluation, this one however, may not come from a Krull valuation since it does not define a maximal ideal by the definition.

There is an exception to this case, whenever $\mathcal{O}_{\nu_1} \subseteq \mathcal{O}_{\nu_2}$. For this case to occur ν_1 would need to be **composite with** ν_2 , i.e ν_2 can be seen as ν_1 composed with a quotient group homomorphism.

For such a case $\nu_{1,2}$ would become a valuation equivalent to ν_1 in the sense of Lemma 2.5. This case is not of interest to us, mainly because if ν is composite with ν' their rational ranks are related such that

$$\text{rat. rk } \nu = \text{rat. rk } \nu' + \text{rat. rk } \bar{\nu}$$

For a certain valuation $\bar{\nu}$ (See [18] Proposition 1.11 for further details). Since we will be interested in the product of *MRR* valuations, we will from now on assume that ν_i are independent, i.e, non-composite with the same non-trivial valuation.

We will abuse notation and call $\mathfrak{m}_i = \mathfrak{m}_{\nu_i} \cap \mathcal{O}_\nu$, which still is a prime ideal of the ring \mathcal{O}_ν .

Lemma 4.16. *In our context, $\mathfrak{m}_i \setminus \mathfrak{m}_j$ is always non-empty.*

Proof. Take without loss of generality $\mathfrak{m}_1 \setminus \mathfrak{m}_2$ and suppose it were empty. Looking back at the definition of \mathcal{O}_ν , this would imply that $\nexists f$ such that $\nu_1(f) = 1_{S_1}$ when $\nu_2(f) = 1_{S_2}$.

Looking at the definitions of $\mathfrak{m}_1, \mathfrak{m}_2$, this is equivalent to saying $f \notin \mathfrak{m}_2$ implies $f \notin \mathfrak{m}_1$. But in turn this means $\mathfrak{m}_1 \subseteq \mathfrak{m}_2$, which we have discarded initially. \square

Proposition 4.17. *The ring \mathcal{O}_ν is a **semi-local ring**, that is, it has a finite number of maximal ideals. More precisely, the maximal ideals of \mathcal{O}_ν are \mathfrak{m}_1 and \mathfrak{m}_2 .*

Proof. We will first show that \mathfrak{m}_1 is maximal. Let $f \in \mathcal{O}_\nu \setminus \mathfrak{m}_1$, we need to prove that $\mathfrak{m}_1 + (f) = \mathcal{O}_\nu$. For this we can assume f is not invertible in \mathcal{O}_ν , otherwise it is obvious. By definition of \mathfrak{m}_1 we have that $\nu_1(f) = 1_{S_1}$. Since f is not invertible, when we look at its image for ν_2 we must get $\nu_2(f) \neq 1_{S_2}$, which implies $f \in \mathfrak{m}_2$. Even more, $f \in \mathfrak{m}_2 \setminus \mathfrak{m}_1$. By Lemma 4.16, $\mathfrak{m}_i \setminus \mathfrak{m}_j$, so we can do the same thing the other way around and have $g \in \mathfrak{m}_1 \setminus \mathfrak{m}_2$.

Then, $\nu(f + g) = 1_S$ so $f + g$ invertible, thus $\mathfrak{m}_1 + (f) = \mathcal{O}_\nu$. Reasoning analogously, \mathfrak{m}_2 is also maximal.

Let us see now that there are no other maximal ideals in \mathcal{O}_ν . Every element not in any \mathfrak{m}_i is invertible immediately by checking its valuation. By the argument in the proof of Proposition 4.15, every ideal, including maximal ones, is necessarily included in $\mathfrak{m}_1 \cup \mathfrak{m}_2$.

Then, since any ideal contained in a union of prime ideals is itself contained in one of them (See [22].Proposition 1.11) we have that there are no other maximal ideals in \mathcal{O}_ν . \square

4.3 Okounkov body with respect to the product of MRR valuations

Having now discussed the result of multiplying Krull prevaluations, it remains only to check what exactly can we recover from the important results in valuation theory in order for such a thing as an Okounkov body to even make sense.

We aim now to replicate the results from Section 2 in order to be able to propose a definition of Okounkov body of a product of MRR valuations in such a way that we preserve the most important properties.

The first result, essential to even be able to propose the definition gives us no problem.

Proposition 4.18. *Let $\nu \equiv \nu_1 \times \nu_2$ be the prevaluation product of two MRR valuations over $K = k(X)$, L a line bundle on X with $k \neq R(X, L)$ and $k \neq A \subseteq R(X, L)$ a graded sub-algebra. Then, the semi-group of values*

$$S_\nu(A) = \left\{ \left(m, (\nu_1(\sigma), \nu_2(\sigma)) \right) : m \in \mathbb{N}, \sigma \in A_m \setminus \{0\} \right\}$$

is a discrete semi-group of $\mathbb{N} \times V_\nu$.

Proof. The proof coincides to the one in Proposition 3.12.i., since the product of spaces will not interfere with the properties of Λ_{ν_1} and Λ_{ν_2} stated in Proposition 2.16. \square

In this first proposition we have already made an implicit decision, by which we will take as group of values the product of abelian groups $\Lambda_{\nu_1} \times \Lambda_{\nu_2}$ and the generated \mathbb{R} -vector space $V_\nu = V_{\nu_1} \times V_{\nu_2}$.

This way, we can give a tentative definition of Okounkov body with respect to prevaluation product, mirroring the one in Definition 3.13.

Definition 4.19. *The **Okounkov body of A relative to the prevaluation ν** is defined as the base, in the sense of Definition 3.3, of the graded semi-group $S_\nu(A) \subset \mathbb{N} \times V_\nu$. We will denote it by $\Delta_\nu(A)$.*

Reading this definition one may feel tempted to define the Okounkov body of the prevaluation product directly as the product of Okounkov bodies. This is however not true, at least in general, and can be seen to fail in the simplest of cases.

Example 4.20 (Two points on a curve). *Let D be a divisor of degree $d > 0$ on C a smooth projective curve of genus g , in the same setting as the one in Example 3.19.*

Consider ν_i the valuations associated to the order of zeroes at different points $p_i \in C$. For this case, we already showed that $\Delta_{\nu_i} = [0, d]$ for any MRR valuation, although the semigroup generally will not be of finite type.

If s is a section in kD , it will be of degree kd , and as such the sum of its orders of zeroes at p_1, p_2 will be lesser or equal than kd . Note that in no case this means that the valuations ν_1, ν_2 are composite to each other.

It follows then that necessarily it is the case that the point inside the Okounkov body $\Delta_\nu(C, \mathcal{O}_C(D))$ given by s is $(\frac{\nu_1(s)}{k}, \frac{\nu_2(s)}{k})$ which verifies the equation $x + y \leq d$.

It is deduced then that $\Delta_\nu(C, \mathcal{O}_C(D))$ lies inside the triangle of vertices $(0, 0), (0, d), (d, 0)$. And so, it is not the square $[0, d]^2$.

From this example, one can deduce the following lemma:

Lemma 4.21. $\Delta_\nu(A) \subseteq \Delta_{\nu_1}(A) \times \Delta_{\nu_2}(A)$

Proposition 4.22. Δ_ν equals the triangle described in Example 4.20.

Proof. We have that the characterization of the Okounkov body of the prevaluation still holds, having being defined straight from the notion of base of a semi-group. this way, we can write:

$$\Delta_\nu(C, \mathcal{O}_C(D)) = \overline{\bigcup_{m \geq 1} \left\{ \frac{1}{m} \nu(s) : s \in H^0(C, \mathcal{O}_C(mD)) \setminus \{0\} \right\}}$$

This, combined with the fact that for degree md , we have that at most the orderings can add up to md makes that for each m we have $md - g - 1$ points spanned inside the triangle $(0, 0), (0, md), (md, 0)$.

As we take $m \gg 0$ we have that, in a similar way as in Example 3.19, the set of points reverted back to the section given by intersecting with the plane $\{1\} \times V_\nu$ will have as closure the triangle, getting the desired coincidence. \square

We have constructed then a new convex set out of the prevaluation product of two MRR valuations. This result aims the right way, but is far from the universality that one would desire. It, however, marks the end of what we have been able to achieve in the scope of this Master's Final Project.

4.4 Conclusions and hopes

We will end the text giving some directions on which the work made here could be further developed. Having that the construction of an Okounkov body out of a prevaluation product does not stray that far away from the original one, the immediate next step would be to see what happens with the results regarding volume.

Even further away, as a star on the horizon, one could perhaps imagine what the universality results regarding prevaluations over the same ring could entail for Okounkov bodies, even dreaming of a universal Okounkov body.

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