Quantization of the radiation electromagnetic field

Author: Aniol Sala
Facultat de Física, Universitat de Barcelona, Diagonal 645, 08028 Barcelona, Spain.

Advisor: Josep Taron

Abstract: This essay studies a method to quantize the radiation electromagnetic field in the Coulomb gauge, starting off classical electromagnetism of Maxwell's equations and imposing the commutation rules. It also studies the concept of photon as an excitation of the field and some of its properties. Finally, the results are applied to construct states of definite total angular momentum and the study of coherent states of the electromagnetic field.

I. INTRODUCTION

The first person to address the problem of a quantum description of the free electromagnetic field was P. A. M. Dirac in 1927 [1], where he constructed a quantum theory of radiation and applied to the emission and absorption of light.

This present work focuses in the quantization of the electromagnetic field in the Coulomb gauge and the study of some properties of photons, that emerge naturally of the quantized E. M. field. It is also studied how to describe this particles in terms of its definite total angular momentum. Finally, through the introduction of coherent states, some differences about the expressions of classical electromagnetic waves and the quantum fields are discussed.

II. CLASSICAL TREATMENT

A. Gauge choice

For our purpose of deriving the expressions for the radiation electromagnetic field in the quantum formalism we will work in the Coulomb gauge. This gauge is a natural option since it provides automatically a transverse character for the potential vector $\boldsymbol{A}(\boldsymbol{x},t)$, which satisfies the homogeneous wave equation in absence of transverse current [3]

$$\nabla^2 \boldsymbol{A}(\boldsymbol{x},t) - \frac{\partial^2}{c^2\partial^2 t}\boldsymbol{A}(\boldsymbol{x},t) = 0. \tag{1}$$

The first step to quantize the radiation electromagnetic field is expand the vector potential (which we know from our gauge choice that will be transverse) in plane waves. For simplicity, we will work in a box of volume $V = L^3$ and we will impose periodic boundary conditions to obtain travelling waves. This conditions impose a specific form for the wave vectors in the lattices, $k_i = 2\pi n_i/L\,(i = x,y,z)$, with $n_i = 0, \pm 1, \pm 2, ...$.

The expression for the vector potential $\boldsymbol{A}(\boldsymbol{x},t)$ is then

$$\boldsymbol{A}(\boldsymbol{x},t) = \frac{1}{V^{1/2}}\sum_{\boldsymbol{k}}\left[\boldsymbol{A}_{\boldsymbol{k}}(t)e^{i\boldsymbol{k}\boldsymbol{x}} + \boldsymbol{A}^*_{\boldsymbol{k}}(t)e^{-i\boldsymbol{k}\boldsymbol{x}}\right], \tag{2}$$

where we added the second term (which is just the complex conjugate of the first one) to make $\boldsymbol{A}(\boldsymbol{x},t)$ real.

As the waves form a complete orthonormal set, they will follow the expressions

$$\int dV e^{i(\boldsymbol{k}-\boldsymbol{k}')\boldsymbol{x}} = V\delta_{\boldsymbol{k}\boldsymbol{k}'},\ \sum_{\boldsymbol{k}} e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}')} = V\delta(\boldsymbol{x}-\boldsymbol{x}'). \tag{3}$$

Taking the limit $V \to \infty$, as the volume in $\boldsymbol{k}$ space of one mode is $(2\pi)^3/V$, the relations between the discrete modes and the continuous spectrum will be

$$\frac{1}{V}\sum_{\boldsymbol{k}} \xrightarrow{V\to\infty} \int \frac{d\boldsymbol{k}}{(2\pi)^3}\,. \tag{4}$$

From expression (2) we see that the gauge condition $\boldsymbol{\nabla}\cdot\boldsymbol{A}(\boldsymbol{x},t) = 0$ gives $\boldsymbol{k}\cdot\boldsymbol{A}_{\boldsymbol{k}}(t) = 0$, and so $\boldsymbol{A}_{\boldsymbol{k}}(t)$ is perpendicular to the direction of propagation of the waves. Furthermore, from (1), we see that the $\boldsymbol{A}_{\boldsymbol{k}}(t)$ will follow the same equation, and so $\boldsymbol{A}_{\boldsymbol{k}}(t) = \boldsymbol{A}_{\boldsymbol{k}}(0)e^{-i\omega_{\boldsymbol{k}}t}$, where $\omega_{\boldsymbol{k}} \equiv c\,|\boldsymbol{k}|$. Now, using the expressions for $\boldsymbol{E}$ and $\boldsymbol{B}$ from Maxwell equations, we find

$$\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} = \frac{i}{cV^{1/2}}\sum_{\boldsymbol{k}}\omega_k\left[\boldsymbol{A}_{\boldsymbol{k}}(t)e^{i\boldsymbol{k}\boldsymbol{x}} - c.c.\right], \tag{5}$$

$$\boldsymbol{B} = \nabla\times\boldsymbol{A} = \frac{i}{V^{1/2}}\sum_{\boldsymbol{k}}\left[\boldsymbol{k}\times\boldsymbol{A}_{\boldsymbol{k}}(t)e^{i\boldsymbol{k}\boldsymbol{x}} - c.c.\right]. \tag{6}$$

Using (3) we are now able to calculate the energy of the field, which takes the form:

$$H_\gamma = \frac{1}{2}\int dV(|\boldsymbol{E}|^2 + |\boldsymbol{B}|^2) = 2\sum_{\boldsymbol{k}} k^2|\boldsymbol{A}_{\boldsymbol{k}}|^2\,. \tag{7}$$

As the potential vector obeys equation (1), we can read the expression (7) for the energy of the radiation field as a set of linear independent oscillators with frequency $\omega_k = c\,|\boldsymbol{k}|$. This is the first remarkable result, because this expression will be our bridge to obtain the expressions for the quantum fields.

B. Helicity

Helicity is a degree of freedom of photons and as will be seen later it is analogous to the spin of an electron.

As we said, the different modes $\boldsymbol{A_k}(t)$ are perpendicular to the direction of propagation $\boldsymbol{k}$. We can therefore expand $\boldsymbol{A_k}(t)$ in terms of an orthonormal set of three independent vectors $\{\boldsymbol{\epsilon_{k,i}}\}$, $i = 1,2,3$, with $\boldsymbol{\epsilon_{k,3}} = \hat{\boldsymbol{k}}$, called linear polarization vectors, to remark this. As the modes $\boldsymbol{A_k}(t)$ are perpendicular to $\boldsymbol{k}$, they will not have a component in the direction of $\boldsymbol{\epsilon_{k,3}}$, and just the first two vectors are needed. From these vectors we define the circular polarization vectors

$$\boldsymbol{e}_{k\pm1} = \mp\frac{1}{\sqrt{2}}\left(\boldsymbol{\epsilon_{k1}} \pm i\boldsymbol{\epsilon_{k2}}\right), \tag{8}$$

from which the relations follow

$$\boldsymbol{e}^*_{\boldsymbol{k}\lambda}\cdot\boldsymbol{e}_{\boldsymbol{k}\lambda'} = \delta_{\lambda\lambda'};\ \boldsymbol{e}^*_{\boldsymbol{k}\lambda}\times\boldsymbol{e}_{\boldsymbol{k}\lambda'} = i\lambda\hat{\boldsymbol{k}}\delta_{\lambda\lambda'};\ i\hat{\boldsymbol{k}}\times\boldsymbol{e}_{\boldsymbol{k}\lambda} = \lambda\boldsymbol{e}_{\boldsymbol{k}\lambda} \tag{9}$$

with λ, $\lambda' = \pm1$.

C. Canonical variables

We have seen that the energy of the field can be expressed as an infinite set of independent oscillators. To make this result explicit, we write equation (7) in terms of the canonical variables of the harmonic oscillator, which we define as follows:

$$\boldsymbol{P_k}(t) = -ik\left[\boldsymbol{A_k}(t) - \boldsymbol{A}^*_{\boldsymbol{k}}(t)\right],\ \boldsymbol{Q_k}(t) = \frac{1}{c}\left[\boldsymbol{A_k}(t) + \boldsymbol{A}^*_{\boldsymbol{k}}(t)\right]. \tag{10}$$

With this definition, we see that the expression for (7) becomes

$$H_\gamma = \frac{1}{2}\sum_{\boldsymbol{k}}\left(\omega^2_{\boldsymbol{k}}Q^2_{\boldsymbol{k}} + P^2_{\boldsymbol{k}}\right), \tag{11}$$

with $\dot{P}_{\boldsymbol{k},i} = -\frac{\partial H_\gamma}{\partial Q_{\boldsymbol{k},i}}$, $\dot{Q}_{\boldsymbol{k},i} = \frac{\partial H_\gamma}{\partial P_{\boldsymbol{k},i}}$ $\quad (i = x,y,z)$.

III. QUANTUM TREATMENT

On the basis of the expressions (10), we postulate that this variables obey the commutation rules of the canonical variables for the harmonic oscillator:

$$\begin{aligned}&[Q_{\boldsymbol{k}\lambda}, P_{\boldsymbol{k}'\lambda'}] = i\hbar\delta_{\boldsymbol{k}\boldsymbol{k}'}\delta_{\lambda\lambda'},\\ &[Q_{\boldsymbol{k}\lambda}, Q_{\boldsymbol{k}'\lambda'}] = 0,\ [P_{\boldsymbol{k}\lambda}, P_{\boldsymbol{k}'\lambda'}] = 0,\end{aligned} \tag{12}$$

where now the subscript λ stands for helicity. With expressions (12) and (11) we can now write

$$H_\gamma = \frac{1}{2}\sum_{\boldsymbol{k}}\sum_{\lambda}\left(\omega^2_{\boldsymbol{k}}Q^2_{\boldsymbol{k}\lambda} + P^2_{\boldsymbol{k}\lambda}\right), \tag{13}$$

with H_γ being the Hamiltonian of the radiation field. Now, in analogy for the quantum harmonic oscillator we define creation and annihilation operators as follows:

$$a_{\boldsymbol{k}\lambda} = \frac{1}{\sqrt{2\hbar\omega}}\left(\omega_{\boldsymbol{k}}Q_{\boldsymbol{k}\lambda} + iP_{\boldsymbol{k}\lambda}\right),\ a^\dagger_{\boldsymbol{k}\lambda} = \frac{1}{\sqrt{2\hbar\omega}}\left(\omega_{\boldsymbol{k}}Q_{\boldsymbol{k}\lambda} - iP_{\boldsymbol{k}\lambda}\right), \tag{14}$$

$$\left[a_{\boldsymbol{k}\lambda}, a^\dagger_{\boldsymbol{k}\lambda}\right] = \delta_{\boldsymbol{k}\boldsymbol{k}'}\delta_{\lambda\lambda'}. \tag{15}$$

As we know from the theory of the quantum harmonic oscillator, we define a state $|\boldsymbol{k}\lambda\rangle$, which now we call one-photon state, as $a^\dagger_{\boldsymbol{k}\lambda}|0\rangle$, where $|0\rangle$ is the *vacuum state*, and so the operators of (14) act over states of this Fock space. An arbitrary vector of the Fock space of n photons with momentum k and helicity λ is written as

$$|n_{\boldsymbol{k}\lambda}\rangle = \frac{1}{\sqrt{n_{\boldsymbol{k}\lambda}!}}\left(a^\dagger_{\boldsymbol{k}\lambda}\right)^{n_{\boldsymbol{k}\lambda}}|0\rangle\ . \tag{16}$$

If we keep proceeding in analogy with the theory of harmonic oscillator, we can define the number operator $N_{\boldsymbol{k}\lambda} = a^\dagger_{\boldsymbol{k}\lambda}a_{\boldsymbol{k}\lambda}$, with eigenvalues $0,1,2....$ Writing the operators $Q_{\boldsymbol{k}\lambda}$ and $P_{\boldsymbol{k}\lambda}$ in terms of the $a_{\boldsymbol{k}\lambda}$ and $a^\dagger_{\boldsymbol{k}\lambda}$ from expression (14)

$$H_\gamma = \sum_{\boldsymbol{k},\lambda}\hbar\omega_{\boldsymbol{k}}\left(a^\dagger_{\boldsymbol{k}\lambda}a_{\boldsymbol{k}\lambda} + \frac{1}{2}\right) = \sum_{\boldsymbol{k},\lambda}\hbar\omega_{\boldsymbol{k}}\left(N_{\boldsymbol{k}\lambda} + \frac{1}{2}\right). \tag{17}$$

The term $\sum_{\boldsymbol{k},\lambda}\frac{1}{2}$ is infinite, but as the origin of energy is not an observable, we assign zero energy to the vacuum state. The Hamiltonian of the radiation electromagnetic field is then

$$H_\gamma = \sum_{\boldsymbol{k},\lambda}\hbar\omega_{\boldsymbol{k}}\,a^\dagger_{\boldsymbol{k}\lambda}a_{\boldsymbol{k}\lambda}\,. \tag{18}$$

Now, with the same procedure, we can find the quantum expressions for $\boldsymbol{A}(\boldsymbol{x},t)$, $\boldsymbol{E}(\boldsymbol{x},t)$ and $\boldsymbol{B}(\boldsymbol{x},t)$:

$$\begin{aligned}\boldsymbol{A}(\boldsymbol{x},t) &= \sum_{\boldsymbol{k}\lambda}\sqrt{\frac{\hbar c}{2Vk}}\left[a_{\boldsymbol{k}\lambda}(t)\boldsymbol{e}_{\boldsymbol{k}\lambda}e^{i\boldsymbol{kx}} + h.c.\right],\\ \boldsymbol{E}(\boldsymbol{x},t) &= i\sum_{\boldsymbol{k}\lambda}\sqrt{\frac{\hbar ck}{2V}}\left[a_{\boldsymbol{k}\lambda}(t)\boldsymbol{e}_{\boldsymbol{k}\lambda}e^{i\boldsymbol{kx}} - h.c.\right],\\ \boldsymbol{B}(\boldsymbol{x},t) &= i\sum_{\boldsymbol{k}\lambda}\sqrt{\frac{\hbar c}{2Vk}}\left[a_{\boldsymbol{k}\lambda}\boldsymbol{k}\times\boldsymbol{e}_{\boldsymbol{k}\lambda}e^{i\boldsymbol{kx}} - h.c.\right],\end{aligned} \tag{19}$$

where we stress that as we proceed from the classical expressions (which depend explicitly on time), the operator fields obtained are in the Heisenberg picture. As $\dot{a}_{\boldsymbol{k}\lambda}(t) = [a_{\boldsymbol{k}\lambda}(t), H_\gamma] = -i\omega_{\boldsymbol{k}}a_{\boldsymbol{k}\lambda}(t)$, one can check that quantum fields obey the Maxwell equations.

Let's see now how commutation relations are in agreement with the gauge fixing condition $\nabla\cdot\boldsymbol{A} = 0$. From (19) $\boldsymbol{E}(\boldsymbol{x},t) = (-1/c)\dot{\boldsymbol{A}}(\boldsymbol{x},t)$, and recalling that the Lagrangian density is $\mathcal{L} = (|\boldsymbol{B}(x)|^2 - |\boldsymbol{E}(x)|^2)/2$, the expression for the conjugate momentum is

$$\pi_i(x) = \frac{\partial\mathcal{L}}{\partial\dot{A}_i(x)} = \frac{-1}{c}\frac{\partial\mathcal{L}}{\partial E_i(x)} = \frac{1}{c}\boldsymbol{E}_i(x), \tag{20}$$

and so, using (15) and that $\sum_\lambda e^*_{\lambda,i}e_{\lambda,j} = \delta ij - k_ik_j/k^2$ $(i,j = x,y,z)$ [2], the commutation relation between the components of $\boldsymbol{A}(\boldsymbol{x},t)$ and $\boldsymbol{\pi}(\boldsymbol{x},t)$ at equal time will be:

$$
\begin{aligned}
[A_i(\boldsymbol{x},t),\pi_j(\boldsymbol{x}',t)] =& -i\hbar\,\frac{1}{V}\sum_{\boldsymbol{k}}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}')}\\
=& -i\hbar\,\delta_{Tij}(\boldsymbol{x}-\boldsymbol{x}'),
\end{aligned}
\tag{21}
$$

where $\delta_{Tij}(\boldsymbol{x}-\boldsymbol{x}')$ is called the transverse delta. From this last expression we see that the commutation relations are effectively compatible with the constrains:

$$
\begin{aligned}
[\partial_i A_i(\boldsymbol{x},t),\pi_j(\boldsymbol{x}',t)] =& -i\hbar\sum_{\boldsymbol{k}}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)(ik_i)e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}')}\\
=&0\,.
\end{aligned}
\tag{22}
$$

A. Linear and angular momentum of the field

Recalling the classical expressions for the linear and angular momentum [3], and using the expressions for the quantum fields in (19), we obtain

$$
\begin{aligned}
\boldsymbol{P} =&\frac{1}{c}\int dV(\boldsymbol{E}\times\boldsymbol{B}) = \sum_{\boldsymbol{k}}\hbar\boldsymbol{k}a^\dagger_{\boldsymbol{k}\lambda}a_{\boldsymbol{k}\lambda},\\
\boldsymbol{J} =&\frac{1}{c}\int dV\left[\boldsymbol{r}\times(\boldsymbol{E}\times\boldsymbol{B})\right]\\
=&\frac{1}{c}\int dV\{E_n(\boldsymbol{r}\times\boldsymbol{\nabla})A_n+\boldsymbol{E}\times\boldsymbol{A})\}\ (l=x,y,z)
\end{aligned}
\tag{23}
$$

where repeated index are summed. The first term of $\boldsymbol{J}$ clearly changes the result under a change of the origin $\boldsymbol{r}\to\boldsymbol{r}+\boldsymbol{r}_0$, while the second term does not. The first term is identified as orbital angular momentum, and the second term as the intrinsic angular momentum or angular momentum of spin. Moreover, if we explicit the expression for the second term, we obtain:

$$
\boldsymbol{J}_{spin} = \frac{1}{c}\int dV(\boldsymbol{E}\times\boldsymbol{A}) = \sum_{\boldsymbol{k}\lambda}\hbar\lambda\,a^\dagger_{\boldsymbol{k}\lambda}a_{\boldsymbol{k}\lambda}\hat{\boldsymbol{k}}\,. \tag{24}
$$

We see that one-photon state of momentum $\hbar\boldsymbol{k}$ has intrinsic angular momentum $\pm\hbar$ along $\hat{\boldsymbol{k}}$. Recalling the expression for $\boldsymbol{A}(\boldsymbol{x},t)$ in (19), the orbital part of $\boldsymbol{J}$ can be written as

$$
\begin{aligned}
\boldsymbol{J}_{orb} =& \frac{1}{c}\int dV\,E_n\left(\epsilon_{jkl}\,x_k\,\partial_l\right)A_n\hat{\boldsymbol{x}}_j\\
=&\sum_{\boldsymbol{k}\lambda}\sqrt{\frac{\hbar}{2Vck}}\int dV E_n\left[\epsilon_{jkl}\,x_k(ik_l)\hat{\boldsymbol{x}}_j\,a_{\boldsymbol{k}\lambda}e_{\boldsymbol{k}\lambda,n}e^{i\boldsymbol{k}\boldsymbol{x}}\right.\\
&\left.-\,h.c.\right]\\
=&\sum_{\boldsymbol{k}\lambda}\sqrt{\frac{\hbar}{2Vck}}\int dV\,E_n\cdot(\boldsymbol{r}\times\boldsymbol{k})\left[ia_{\boldsymbol{k}\lambda}e_{\boldsymbol{k}\lambda,n}e^{i\boldsymbol{k}\boldsymbol{x}}-h.c.\right].
\end{aligned}
\tag{25}
$$

and so, for each mode in the sum, $\boldsymbol{J}_{orb}$ is perpendicular to $\boldsymbol{k}$. From this last expression and (24), if we apply the helicity operator (this is, the projection of the total angular momentum in the direction of $\boldsymbol{k}$) to one-photon state we obtain

$$
\boldsymbol{J}\cdot\hat{\boldsymbol{k}}\,|\boldsymbol{k}\lambda\rangle = \boldsymbol{J}_{spin}\cdot\hat{\boldsymbol{k}}\,|\boldsymbol{k}\lambda\rangle = \hbar\lambda\,|\boldsymbol{k}\lambda\rangle\,. \tag{26}
$$

Thus, the photon has spin 1 but has not the zero projection of spin. These results are in agreement with the study of the Poincare group representations for zero mass particles [4]. The reason is that the photon has null mass and there is not a rest frame where it can be studied using the general theory of spin. Classically, the null component of spin corresponds to the absence of longitudinal modes of an electromagnetic wave.

Finally, using the expressions (18) and (23) we can deduce the momentum and energy of a single photon state with momentum $\boldsymbol{k}$ and helicity λ:

$$
\begin{aligned}
H_\gamma\,|\boldsymbol{k}\lambda\rangle =& \hbar\omega_{\boldsymbol{k}}\,|\boldsymbol{k}\lambda\rangle\,,\\
\boldsymbol{P}\,|\boldsymbol{k}\lambda\rangle =& \hbar\boldsymbol{k}\,|\boldsymbol{k}\lambda\rangle\,,
\end{aligned}
\tag{27}
$$

and so, for the photon, defining $E_{\boldsymbol{k}\lambda}\equiv\hbar\omega$, $P_{\boldsymbol{k}\lambda}=\hbar\boldsymbol{k}$:

$$
m_\gamma^2\,c^2 = \frac{E^2_{\boldsymbol{k}\lambda}}{c^2}-\boldsymbol{P}^2_{\boldsymbol{k}\lambda} = \frac{\hbar^2(c|\boldsymbol{k}|)^2}{c^2}-\hbar^2|\boldsymbol{k}|^2 = 0. \tag{28}
$$

That is, *the photon has null mass.*

IV. IMPORTANT RESULTS

A. Eigentstates of total angular momentum

One result that can be achieved from (23) and the theory of rotations is the expression of the fields in terms of operators with definite total angular momentum and helicity. For this purpose we will use the helicity operator $h\equiv\boldsymbol{J}\cdot\boldsymbol{k}/|\boldsymbol{k}| = \boldsymbol{J}_{spin}\cdot\hat{\boldsymbol{k}}$, with eigenvalues $\lambda=-1,1$ (in units of $\hbar$) as it has been deduced in section III A. The states with definite total angular momentum $\boldsymbol{J}$ and helicity λ will be written $|kjm\lambda\rangle$, with k being the magnitude of the momentum, j the total angular momentum quantum number and m the eigenvalue of J_z. Using the isotropy of the space, we will take a single photon state $|\boldsymbol{k}_z\,\lambda\rangle$ with momentum in the $\boldsymbol{n}_z$ direction, and then we will rotate it to $|\boldsymbol{k}\,\lambda\rangle = D(\boldsymbol{n})\,|\boldsymbol{k}_z\,\lambda\rangle$, a state with momentum in a general direction $\hat{\boldsymbol{n}}$. Both states will have the same helicity as it is not affected under rotations. Here $D(\boldsymbol{n})$ represents the rotation matrix which performs the rotation $\boldsymbol{n}_z\to\boldsymbol{n}$.

The objective is to find the change of basis relation from one-photon states $|\boldsymbol{k}\,\lambda\rangle$ to $|kjm\lambda\rangle$ states. First of all we notice that a state $|\boldsymbol{k}_z\,\lambda\rangle$ is an eigenstate of $\boldsymbol{J}_z$ with eigenvalue $m=\lambda$. Then the change of basis relation from states with momentum in the z-direction $|\boldsymbol{k}_z\,\lambda\rangle$ to states with total angular momentum j and $m=\lambda$, which we will write $|kjm\lambda\rangle_{m=\lambda}$, is simply:

$$|\boldsymbol{k}_z\,\lambda\rangle = \sum_{j=|\lambda|}^{\infty} |kjm\lambda\rangle_{m=\lambda} \langle kjm\lambda|\boldsymbol{k}_z\,\lambda\rangle_{m=\lambda}\,, \qquad (29)$$

and so:

$$\begin{aligned}|\boldsymbol{k}\,\lambda\rangle &= \sum_{j=|\lambda|}^{\infty} D(\boldsymbol{n})\,|kjm\lambda\rangle_{m=\lambda} \langle kjm\lambda|\boldsymbol{k}_z\,\lambda\rangle_{m=\lambda} \\ &= \sum_{j=|\lambda|}^{\infty}\sum_{m'=-j}^{j} |kjm'\lambda\rangle\, D^{j}_{m'\lambda}(\boldsymbol{n})\,\langle kjm\lambda|\boldsymbol{k}_z\,\lambda\rangle_{m=\lambda}\,,\end{aligned} \qquad (30)$$

where we have used the change of basis relation:

$$\begin{aligned}D(\boldsymbol{n})\,|kjm\lambda\rangle &= \sum_{j'm'\lambda} |kj'm'\lambda\rangle\,\langle kj'm'\lambda|D(\boldsymbol{n})|kjm\lambda\rangle \\ &\equiv \sum_{m'} |kjm'\lambda\rangle\, D^{j}_{m'm}(\boldsymbol{n})\end{aligned} \qquad (31)$$

The matrix elements $D^{j}_{mm'}(\boldsymbol{n})$ follows (as can be seen, for example, in [2]) the orthogonality relations

$$\int d\boldsymbol{n}\, D^{j_1}_{m_1m_1'}(R) D^{j_2}_{m_2m_2'}(R)^* = \frac{4\pi}{2j_1+1}\delta_{j_1j_2}\delta_{m_1m_2}\delta_{m_1'm_2'} \qquad (32)$$

which leads us to obtain from (30) the state $|kjm\lambda\rangle$, multiplying both sides for $D^{j}_{m\lambda}(\boldsymbol{n})^*$ and then integrate over all solid angle $\boldsymbol{n}$:

$$\begin{aligned}\int d\boldsymbol{n}\, D^{j}_{m\lambda}(\boldsymbol{n})^*\,|\boldsymbol{k}\,\lambda\rangle &= \langle kjm\lambda|\boldsymbol{k}_z\lambda\rangle_{m=\lambda}\,\frac{4\pi}{2j+1}\;|kjm\lambda\rangle \\ &\equiv \frac{1}{N(j)}\,|kjm\lambda\rangle\,,\end{aligned} \qquad (33)$$

so the problem to obtain an expression for the states $|kjm\lambda\rangle$ reduces to finding the constant $N(j)$. Taking the orthogonality relations

$$\begin{aligned}\langle kjm\lambda|k'j'm'\lambda'\rangle &= \frac{\delta(k-k')}{k^2}\delta_{jj'}\delta_{mm'}\delta_{\lambda\lambda'}\,, \\ \langle \boldsymbol{k}\lambda|\boldsymbol{k}'\lambda'\rangle &= \delta^3(\boldsymbol{k}-\boldsymbol{k}')\delta_{jj'}\delta_{mm'}\delta_{\lambda\lambda'}\,,\end{aligned} \qquad (34)$$

and equations (32) and (33), we obtain

$$\langle kjm\lambda|k'j'm'\lambda'\rangle = |N(j)|^2\frac{4\pi}{2j+1}\frac{\delta(k-k')}{k^2}\delta_{jj'}\delta_{mm'}\delta_{\lambda\lambda'} \qquad (35)$$

and so $N(j) = \sqrt{\frac{2j+1}{4\pi}}$. As one-photon state is defined as $|\boldsymbol{k}\lambda\rangle = a^\dagger_{\boldsymbol{k}\lambda}|0\rangle$, we now define a new operator $|kjm\lambda\rangle \equiv a^\dagger_{kjm\lambda}|0\rangle$. With this definition, we see from (30) the relation between the operator $a^\dagger_{jm\lambda}$ and $a^\dagger_{\boldsymbol{k}\lambda}$:

$$a^\dagger_\lambda(\boldsymbol{k}) = \sum_{jm}\sqrt{\frac{2j+1}{4\pi}}D^{j}_{m\lambda}(\boldsymbol{n})a^\dagger_{jm\lambda}(k), \qquad (36)$$

where we write the operators considering $\boldsymbol{k}$ continuous, that is, in the limit $V\to\infty$. Recalling (4), and writing the vector potential in terms of the operators of definite total angular momentum with the relation (36), we finally obtain:

$$\boldsymbol{A}(\boldsymbol{x},t) = \frac{\sqrt{\hbar c}}{8\pi^2}\sum_{jm\lambda}\sqrt{2j+1}\int\frac{d\boldsymbol{k}}{\sqrt{k}}[a^\dagger_{jm\lambda}(k)e^{i\omega t}\boldsymbol{f}_{jm\lambda}(\boldsymbol{k},\boldsymbol{x}) + h.c.] \qquad (37)$$

where

$$\boldsymbol{f}_{jm\lambda}(\boldsymbol{k},\boldsymbol{x}) \equiv \int d\boldsymbol{n}\;\boldsymbol{e}^*_\lambda(\boldsymbol{k})e^{-i\boldsymbol{k}\boldsymbol{x}}D^{j}_{m\lambda}(\boldsymbol{n}) \qquad (38)$$

are the *spherical harmonics*. This expression takes special importance, for example, in the study of transition probabilities and the emission and absorption of light by atoms. Moreover, as the sum in (37) starts with $j=|\lambda|=1$, two systems interacting via emission or absorption of light can't have zero total angular momentum.

B. Coherent states

If we take the expression of the electric field from (19), as it is linear with $a^\dagger_{\boldsymbol{k}\lambda}$ and $a_{\boldsymbol{k}\lambda}$, the expectation value of $\boldsymbol{E}(\boldsymbol{x},t)$ in any photon state $|\boldsymbol{k}\lambda\rangle$ will be 0. Moreover, the vacuum fluctuations of the electric field diverge, since

$$\begin{aligned}\langle 0|\boldsymbol{E}^2|0\rangle &= \sum_{\boldsymbol{k}\boldsymbol{k}'\lambda\lambda'}\frac{\hbar c}{2V}\sqrt{k\,k'}\,\langle 0|\left(a_{\boldsymbol{k}\lambda}a^\dagger_{\boldsymbol{k}'\lambda'}e^{i(\boldsymbol{k}-\boldsymbol{k}')\boldsymbol{x}}\boldsymbol{e}_{\boldsymbol{k}\lambda}\boldsymbol{e}^*_{\boldsymbol{k}'\lambda'} + h.c\right)|0\rangle \\ &= \sum_{\boldsymbol{k}\boldsymbol{k}'\lambda\lambda'}\frac{\hbar c}{2V}\sqrt{k\,k'}\,\langle 0|\left(\delta_{\boldsymbol{k}\boldsymbol{k}'}\delta_{\lambda\lambda'} + 2N_{\boldsymbol{k}\lambda}\delta_{\boldsymbol{k}\boldsymbol{k}'}\delta_{\lambda\lambda'}\right)|0\rangle \\ &= \frac{1}{V}\sum_{\boldsymbol{k}}\hbar c|\boldsymbol{k}| \to \int\frac{d\boldsymbol{k}}{(2\pi)^{3/2}}\,\hbar\omega\to\infty\,.\end{aligned} \qquad (39)$$

These results differ considerably from the classical expressions of an electromagnetic wave. So now the question would be what states are the closest to the classical picture of light. This states are called *coherent states* [2]. We first define a coherent state $|\alpha\rangle$ as an eigenstate of the operator $a_{\boldsymbol{k}\lambda}$, $a_{\boldsymbol{k}\lambda}|\alpha\rangle = \alpha\,|\alpha\rangle$ (from now we will omit the subscripts $\boldsymbol{k},\lambda$ as we will assume that a coherent state is definite in one arbitrary momentum $\boldsymbol{k}$ and one arbitrary helicity λ). We should note here that as a is not hermitician, in general α will be complex. Expanding the state $|\alpha\rangle$ in a basis of the Fock space $\{|n\rangle\}_{n=0,1,2...}$, with $a\,|n\rangle = \sqrt{n}\,|n-1\rangle$:

$$|\alpha\rangle = \sum_{n=0}^{\infty} C_n\,|n\rangle\,, \qquad (40)$$

and imposing that $a\,|\alpha\rangle = \alpha\,|\alpha\rangle$, it's easy to see that $C_n = C_0\alpha^n/\sqrt{n!}$. If we now demand that $\langle\alpha|\alpha\rangle = 1$, then $C_0^2 = \sum_n |\alpha|^{2n}/n! = exp(-|\alpha|^2)$, and

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|n\rangle\,. \qquad (41)$$

From this expression it's easy to see that:

$$\begin{aligned}\langle N\rangle_\alpha &= \langle\alpha|a^\dagger a|\alpha\rangle = \alpha^*\alpha = |\alpha|^2,\\ \left\langle N^2\right\rangle_\alpha &= \langle\alpha|\big(a^\dagger a + (a^\dagger)^2a^2\big)|\alpha\rangle = |\alpha|^2 + |\alpha|^4,\\ \Delta_\alpha^2 N &= \left\langle N^2\right\rangle - \langle N\rangle^2 = |\alpha|^2 = \langle N\rangle\,.\end{aligned} \qquad (42)$$

From (42), the probability of finding n photons in a coherent state will be:

$$p(n) = |\,\langle n|\alpha\rangle\,|^2 = e^{-|\alpha|^2}\frac{(|\alpha|^2)^n}{n!} = e^{-\langle N\rangle}\frac{\langle N\rangle^n}{n!}, \qquad (43)$$

which is a Poisson distribution.

Let's analyse the time evolution of coherent states we turn in Schrödinger picture. The states $|n\rangle$ are eigenstates of $H = \hbar\omega a^\dagger a$, and so $|n\rangle_t = e^{-in\omega t}\,|n\rangle$. Then the expression for time evolution of a coherent state will be:

$$|\alpha\rangle_t = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n(t)}{\sqrt{n!}}\,|n\rangle \equiv |\alpha(t)\rangle\,, \qquad (44)$$

with $\alpha(t) \equiv \alpha e^{-i\omega t}$. We can see that, for a coherent state, the expectation value and the fluctuations of N are time-independent:

$$\langle N\rangle_{\alpha(t)} = \Delta^2_{\alpha(t)}N = |\alpha(t)|^2 = |\alpha|^2\,. \qquad (45)$$

Now we are able to calculate the expectation value of $\boldsymbol{E}$ and its vacuum fluctuations in a coherent state. For this we write $\alpha(t) = |\alpha|e^{i\phi}e^{-i\omega t}$ and we take the expression of $\boldsymbol{E}(\boldsymbol{x},t)$ in $t = 0$ from (19). The expectation values of $\boldsymbol{E}$ and $\boldsymbol{E}^2$ are then:

$$\begin{aligned}\langle\boldsymbol{E}(\boldsymbol{x})\rangle_\alpha &= \sqrt{\frac{\hbar\omega}{2V}}\,i\left(|\alpha|e^{i(\phi-\omega t)}e^{i\boldsymbol{kx}} - c.c.\right)\boldsymbol{\epsilon}\\ &= \sqrt{2\hbar\omega\bar{n}}\quad sin(\omega t - \boldsymbol{kx} - \phi)\boldsymbol{\epsilon}\,.\end{aligned} \qquad (46)$$

$$\begin{aligned}\left\langle\boldsymbol{E}^2(\boldsymbol{x})\right\rangle_\alpha &= \frac{\hbar\omega}{2V}\left(1 - \big(\alpha(t)e^{i\cdot\boldsymbol{x}} - c.c.\big)^2\right)\\ &= \frac{\hbar\omega}{2V}\left(1 + 4|\alpha|^2 sin^2(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t + \phi)\right)\,.\end{aligned} \qquad (47)$$

where we defined $\bar{n} = |\alpha|^2/V = \langle N\rangle/V$. Recalling the classical description of a plane wave [3] we see that eq. (46) takes the same form as a classical electromagnetic wave relating its amplitude $E_o = \sqrt{2\hbar\omega\bar{n}}$, and as the time average density is, classically, $\bar{u} = |E_0|^2/2$, we see that the classical and the quantum picture take the same form with the expected correspondence $\bar{u} = \bar{n}\hbar\omega$. Moreover, from (46) and (47), the fluctuations of $\boldsymbol{E}(\boldsymbol{x},t)$ are such that:

$$\Delta_{\alpha(t)}\boldsymbol{E}(\boldsymbol{x}) = \left\langle\boldsymbol{E}^2(\boldsymbol{x})\right\rangle - \langle\boldsymbol{E}(\boldsymbol{x})\rangle^2 = \frac{\hbar\omega}{2V}\,. \qquad (48)$$

From (45) and (48) we stress that the dispersions of the electric field and the number of photons N are $\boldsymbol{x},t$ independent. Hence these states are called coherent states.

V. CONCLUSIONS

When the radiation electromagnetic fields are quantized imposing the commutation rules of the harmonic oscillator, they become field operators which act over Fock states. One-photon state can be read as an excitation with zero mass of such fields with definite angular momentum and helicity. Moreover, one-photon state has spin 1 but its projections take only values $\pm\hbar$.

It has also been shown that the picture of one-photon state is rather different than the classical expressions of an electromagnetic wave. Indeed, the expectation value of the electric field in one-photon state is exactly 0, and the fluctuations of the vacuum become infinity. Coherent states have been defined and it has been shown that they provide the most similar image to classical expressions for an electromagnetic wave.

Acknowledgments

I am deeply grateful to my advisor Josep Taron for his constant patience and guidance, and for all knowledge I have acquired beyond this project. I am also grateful to my family and Gemma, whose support during all these years has been of enormous importance for me.

[1] Dirac, P. A. M. "The Quantum Theory of the Emission and Absorption of Radiation". Proc. Royal Soc. (London) **A114** (1927).

[2] K. Gottfried, T. Yan, *Quantum Mechanics: Fundamentals*, (Springer Science+Business Media New York, New York 2003, 2nd. ed.).

[3] J.D Jackson, *Classical Electrodynamics*, (John-Wiley & Sons, New York 2003, 2nd. ed.).

[4] P. Ramond, *Field Theory A Modern Premier*, (The Benjamin/Cummings Publishig Company, INC, Massachusetts 1981, 2nd. ed.).