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ORIGINAL PAPER

Future teachers' reflections on mathematical errors made in their teaching practice

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Abstract
This study answers the following research questions: 1) What types of mathematical errors do future teachers identify when they reflect on their practice? and 2) Which levels of development of the didactic suitability assessment competence for the "errors" component can be inferred when they reflect on their practice? To answer these questions, we explain the Didactic Suitability Criteria construct and describe the associated training cycle structure in the theoretical and methodological framework sections. We followed a qualitative research methodology that mainly consists of thematic analysis. The study conducted allows finding inductive categories of types of mathematical errors, such as error in the task instructions, error of proposition, procedural error, error in the representation, error in the definition and error in the argument. It also enables establishing levels of development of the didactic suitability assessment competence of future teachers for the "errors" component. The main conclusion of this research is the importance of the context to decide what a mathematical error is. The need to further examine the notion of mathematical error in the training of future mathematics teachers is also stressed. Another conclusion is the development of a rubric that allows for more accurate and deeper reflections of future teachers on the errors made.

Keywords Didactic suitability criteria · Errors · Future teachers · Mathematics · Secondary education

1 Introduction

The aim of researching learners' errors was twofold: eliminating them, or exploring their potentialities. Errors are hence identified as opportunities for teaching and learning rather than as problems to avoid (e.g., Borasi, 1996). Research on mathematics teachers' competences and knowledge has also produced studies that explore the assessment of teachers' knowledge as one of the most critical parameters of teaching quality (e.g., Blömeke & Delaney, 2012).

In the latter, assessing and analysing misconceptions of students is an important approach to acquiring knowledge of student thinking. For this purpose, future teachers need to have the necessary knowledge and competences to identify errors, analysing reasons for the errors, and taking action for correction (e.g., Brodie, 2014). More recently, key teacher competences required to engage with learners' errors in written work and classroom interactions have been identified. For instance, Wuttke and Seifried (2017) propose professional error competence (teachers have to be able to diagnose typical student errors, know potential causes for errors, and have strategies to handle them adequately). Teachers' competence to diagnose typical student errors is considered a sub-competence of diagnostic competence (Pankow et al., 2018). Leuders et al. (2018) define teachers' diagnostic competence as a broad construct that combines the interpretation of errors, the performance of diagnostic tasks and judgements, knowledge, the use of assessment procedures, and the consideration of motivational and affective aspects.

This research assumes that teachers can access students' thinking if they are competent to do so. In the aforementioned studies, it is clear that the teacher does not access students' thinking with empathy, but implicitly supposes the students do a mathematical activity subject to mathematical rules that are public and supposedly known by the teacher. This theoretically allows the teacher to evaluate whether students have followed them correctly or not. In other words, it is assumed that the teacher has the competence to analyse

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the mathematical activity. According to the Didactic-Mathematical Knowledge and Competences (DMKC) model of the mathematics teacher (Pino-Fan et al., 2023), this competence is an ability teachers should have.

Numerous investigations on student errors have been conducted studying the role and responsibility of teachers regarding making errors. While errors are seldom taught directly by teachers, there are cases in which teachers do make errors. For instance, Hacisalihoğlu-Karadeniz et al. (2017) identified errors made by future teachers when asked to explain student errors. Research has therefore also been conducted on the errors made by teachers and future teachers in tasks proposed by researchers (e.g., Işik & Kar, 2012).

It can be agreed that the quality criterion of an instructional process is that teachers do not make mathematical errors (Hill et al., 2008). However, if teachers do make errors and are aware of them, they use them in a didactic way as an opportunity to facilitate student learning, something we value positively. Research on the mathematical quality of instruction analyses, amongst other aspects, the errors teachers make in their classrooms (e.g., Lee & Santagata, 2020).

In the literature, there is no consensus on what is considered a mathematical error, and the idea of error is mixed up with other related notions like difficulty, obstacle, or ambiguity (e.g., Fernández Palop, 2013). We mainly encountered studies that show different types of mathematical errors made by students or teachers. For example, in Idris and Narayanan (2011), three types of errors are considered: systematic error, random error and careless error.

In various studies, errors are grouped into procedural and conceptual errors, or into classifications that include this duality (e.g., Delastri & Lolang, 2023). These investigations also analyse other aspects such as the origin of errors, and, in the case of teachers and future teachers, their impact on teaching, or strategies to correct these errors.

Galligan and Hobohm (2015) analysed the errors of future teachers, and asked them to explain why they thought they made them. Despite the importance given to reflection on practice in teacher training, a research gap is identified in the literature review, as future teachers' reflection on their errors is a line of research that has hardly been developed.

In most training programmes for future mathematics teachers, teaching practice (its planning, implementation, and reflecting on it) is key because the knowledge needed to teach mathematics is situational professional knowledge. Teaching practice also allows relating theory to practice (Kieran et al., 2013).

Research on reflective practice has sought to answer different questions, among others, about what aspects (future) mathematics teachers reflect on (e.g., Chamoso et al., 2012). Schön (1983) distinguished between reflection-in-action and reflection-on-action. This paper focuses on the latter. Pollard and Tann (1987) pointed out that teachers' reflection on their practice is a cyclical process including the following phases: 1) teachers need to understand what has happened in their practice, and describe it; 2) they need to interpret what has happened, and be able to give some kind of explanation; 3) they need to assess what has happened; 4) they need to guide their future practice in such a manner that the aspects considered in the assessment as improvable are improved. This paper hence considers future teachers' reflections should include an assessment, amongst other aspects.

Once the importance of reflection on teaching practice is assumed, the question is if it should be guided or if offering teachers the opportunity to reflect on their practice is enough. The review of the literature on teacher reflection shows the second option is mostly opted for. Considering the limitations and differences teachers may have when reflecting on their practice (Breda et al., 2021), giving teachers the opportunity to reflect on their practice is insufficient since they need tools to direct their attention to salient aspects of teaching episodes. For instance, Perrenoud (2001) maintains that, to accomplish reflective teaching, teachers need to have a method for reflection. Conceptual frameworks specific to each discipline that serve to analyse the information on which to reflect is also necessary. There are different models of reflective cycles, one of the most used being the model proposed by Korthagen et al. (2001). Artzt et al. (2015) propose a model for teachers to observe other teachers or themselves as a tool for assessment or self-assessment and reflection, and for supervisors to observe and assess future or novice teachers.

Research agendas on teacher training formulate proposals for competences that have either been directly called reflective competence (e. g., Knievel et al., 2015; Seckel, 2016), or include many of the aspects related to teacher reflection. Examples are the professional noticing competence, which involves the ability to reflect on instruction as it occurs (Mason, 2002), and the didactic suitability assessment competence considered in the DMKC model (Pino-Fan et al., 2023). Moreover, these research agendas propose observation and assessment tools to develop teacher reflection. They are supposed to be taught as part of teacher education. An example is the didactic suitability criteria (DSC) construct in the DMKC model.

In Spain, future teachers have to do a Master's degree in which they have to submit a Master's dissertation (MD) to be able to teach mathematics in secondary and upper-secondary education. In the Master's degree programme considered in this study, future teachers learn how to use the DSC construct to assess the implementation of a learning sequence they have to teach. They then make a redesign proposal improving the worst valued aspects of the implementation. The DSC tool is also used to observe and organise future teachers' reflection, amongst other things, on the errors they made during their teaching practice. The comments of the reflections on their practice provide the necessary data to answer the research questions developed in this study:

1) What types of mathematical errors do future teachers identify when they reflect on their practice in their Master's dissertation?
2) Which levels of development of the didactic suitability assessment competence for the "errors" component can be inferred from the future teachers' comments in their Master's dissertation?

2 Theoretical framework

2.1 Didactic suitability criteria

Didactic suitability is the degree to which a mathematical teaching process has certain characteristics considered optimal or appropriate to succeed in the adaptation between the personal meanings attained by the students (learning) and the intended or implemented institutional meanings (teaching), considering the circumstances and resources available (environment). Didactic suitability consists of six criteria (DSC) (Breda et al., 2017): the *epistemic suitability* criterion to assess whether the mathematics taught are "good quality mathematics" or not; the *cognitive suitability* criterion to assess, before starting the instructional process, if the content to be taught adapts to what students already know and, after the process, if the learning achieved is close to what was intended to be taught; the *interactive suitability* criterion to assess whether the interactions resolve doubts and difficulties of the students or not; the *media suitability* criterion to assess the adequacy of the material and temporal resources used in the instructional process; the *affective suitability* criterion to assess student involvement (interests and motivations) during the instructional process; and the *ecological suitability* criterion to assess the adaptation of the instructional process to the school's educational project, curricular guidelines, and conditions of the social and professional environment, amongst others.

For DSC to be operational, they are organised into components and indicators (Breda et al., 2017). Epistemic suitability considers a component that may affect the quality of mathematics taught: mathematical errors made by teachers.

2.2 Professional competences within the framework of the DMKC model

The DMKC model suggests two key competences for the mathematics teacher's professional activity: mathematical competence, and competence of analysis and didactic intervention. This second competence refers to "designing, applying, and assessing learning sequences through didactic analysis techniques and status criteria to establish planning, implementation, assessment, and proposals for improvement cycles" (Breda et al., 2017, p. 1897).

The didactic intervention and analysis competence is a general competence formed by different sub-competences: (1) competence in the analysis of the mathematical activity; (2) competence in the analysis and management of interactions; (3) competence in the use and management of resources; and (4) competence in the assessment of didactic suitability. The kinds of didactic analyses that facilitate the first three competences provide a description (and in some cases an explanation) of what has happened. This serves as a basis for the assessment of didactic suitability, the fourth competence. Of these four competences, the last one is the most related to teacher reflection. It is developed by teaching teachers how to use the DSC construct as a tool to assess the suitability of the instructional process implemented.

Regarding the levels of development of the didactic suitability assessment competence, based on the levels of development for this competence proposed in Pino-Fan et al. (2023), three levels of development were established in this research. At level L1, the teachers do not use an explicit guide to reflect; at level L2, they use a guide they have been taught to use to organise reflection; at level L3, they make improvements to the aspects the reflection indicates as being improvable. At all three levels, the reflection includes an assessment.

In this study, four sub-levels of level L2 of the didactic suitability assessment competence were established for the "errors" component (Table 1):

2.3 Characterisation of mathematical error

In mathematics education, the notion of mathematical error has different interpretations, and appears linked to other concepts such as difficulty or obstacle, which may lead to confusion. An error is considered to be a mathematical practice (a sequence of actions subject to mathematical rules) that, from the perspective of the mathematical institution, is not valid (Godino, 2004). This practice can be defined as "invalid" since it does not meet the mathematical reference standard it can be compared with.

In this paper, we analyse the errors made and detected by future teachers during their practicum period when those errors are not part of a planned strategy to improve student learning.

Identifying their mathematical errors is one of the skills teachers need to effectively address student errors, and to reframe them as valuable for future learning processes. However, in the research studies consulted seeking to identify the knowledge and competences teachers need to productively work with student errors (Hoth et al., 2022), this skill does not appear in a specific manner. It is included

Table 1 Sub-levels of level L2

Sub-levels	Characterisation	Examples
L2.1	None of the errors in the future teachers' comments are considered to be mathematical errors by the researchers	The future teacher comments they explained the Pythagorean theorem without using manipulatives
L2.2	The future teachers provide examples of mathematical errors and other errors not considered as errors by the researchers	He explains that when defining the sine, he defined the cosine, and made the error of using the metaphor of "a function is a machine."
L2.3	The errors commented on are considered mathematical errors by the researchers, but few and superficial aspects are reflected on	Only the error made is commented on.
L2.4	The researchers consider the errors commented on to be mathematical errors and the reflection analyses several aspects in some depth	The error, who detected it, how it was managed, etc. are commented on.

in the content knowledge of the correct solution and possible causes of the error. In this study, for future teachers to identify their mathematical errors is considered a skill that enables them to reflect on the quality of the mathematics taught.

Future teachers' "mathematical errors" refer to the errors they make, and to the ones they validate, that is, errors their students make, or errors that appear in textbooks and work sheets used.

3 Methodology

In this study, of a naturalistic nature, we followed a qualitative research methodology, which mainly consists of thematic analysis. The study was conducted for four consecutive years, but it is not a longitudinal study, as the individuals observed were different every academic year.

3.1 Research context

The context of this research is the Master's degree programme in Teacher Training for compulsory secondary and upper-secondary education (specialisation of mathematics), offered at a public university in Spain. In the practicum module of the Master's degree, the participants need to carry out educational practices at secondary schools. During their practicum, they are asked to design and implement a teaching-learning sequence. This educational intervention is determined by the school, the level of the students, and the time of the school year in which the intervention takes place. To prepare for their MD, the future teachers are introduced to DSC following a training cycle explained below. In the Master's degree, the future teachers were taught a rubric of DSC (including criteria, components, and indicators) to help them assess their practice and guide the redesign of a learning sequence they implement.

3.2 Basic structure of the training cycle

This section briefly describes the steps of the 15-h training cycle whose objective is to promote future teachers' didactic suitability assessment competence. The instruction process starts with the participants' reflection without any guidelines, the first level of the didactic suitability assessment competence. The second level is then developed through a training cycle in which the future teachers are taught how to use the DSC tool, and in the third level of development, the participants make justified proposals on how to improve the adequacy of their instructional process.

Step 1: The participants are asked to analyse classroom episodes to perform an analysis based on their prior knowledge, beliefs and values, without being given any guidelines.

Step 2: The discussion generated in this step allows the teacher trainers and participants to observe the following: 1) the participants use certain criteria about how classes should be taught to be quality classes, and how to improve them (criteria that guide the practice); 2) these criteria are similar, regardless of the teachers' country, culture, religion, or educational level (a certain consensus has been reached in an important part of the mathematics education community); 3) these criteria are related to current trends in mathematics teaching (they are hence related to the results and theoretical constructs generated in mathematics education). In this step, time is dedicated to explicitly stating these trends.

Step 3: After observing the three points described in step 2, which shows evidence of a relationship between the criteria guiding the teaching practice (when oriented towards improvement) and the results and theoretical constructs of mathematics education, the teacher trainers recommend considering the following questions: What criteria should be used to design and redesign a sequence of tasks to improve it? What is the role of mathematics education in generating these criteria? Participants are encouraged to observe these questions are related to how to determine the quality of an instructional process.

Concerning these questions, two extreme solutions are put forward by the trainers: 1) considering that the research conducted in the field of mathematics education provides teachers with "practices that work", since evidence supports them, and 2) considering how guiding the improvement of teaching processes and mathematics learning should arise from the argumentative discourse of the entire educational community when it seeks to achieve a consensus on "what can be considered as the best."

From this perspective, the participants should agree on principles thatfirst serve to guide the teaching and learning processes of mathematics, and then to assess their implementations.

Step 4: Following this consensual perspective, the group of participants is asked to establish a local consensus (within the group) on the criteria to be considered to regard a teaching and learning process as appropriate.

Step 5: The trainer points out to the participants that, in mathematics education, several authors have made attempts to compile criteria to guide teaching practice for it to be high quality (e.g., Charalambous & Praetorius, 2018; Hill et al., 2008). Although the criteria achieve a broad consensus in mathematics education, these compilations are merely lists of criteria within a given historical and cultural context outside of which they may not generate any consensus.

Step 6: The notion of suitability is explained. Next, the group of participants reflects on the criteria agreed upon in step 4 and their relationship with DSC. The trainer encourages them to check they have implicitly used criteria that can be reinterpreted in terms of DSC before knowing them.

Step 7: The DSC construct is made operational considering components and indicators. The teaching and learning process of the DSC construct with its components and indicators is the core of the training cycle described, and takes up the most time. Through different tasks, the group agrees on several components and indicators of the criteria. The aim is for the group to generate a rubric to help future teachers assess their practice and guide the redesign of their learning sequence. The trainer leads this rubric-creation process for the core components and indicators agreed upon by the group to fit with those proposed in Breda et al. (2017).

How the four components of the epistemic suitability criterion, the one this research focuses on, are generated, is explained below. The components are: 1) mathematics planned and taught should be error-free; 2) in mathematics taught the use of ambiguities should be controlled; 3) richness of processes, and 4) representative sample of the complexity of the mathematical object to be taught.

The group is first shown several videos in which some teachers make errors. The aim is for the group to identify the errors, and agree that a basic criterion to ensure the quality of planned or taught mathematics is that it should not contain mathematical errors. The component "mathematical error" thus appears. The trainer then defines an error as a mathematical practice that, from the viewpoint of the mathematical institution, is not considered valid.

The group is then shown other videos of classrooms in which teachers use ambiguous language such as metaphors. The participants observe this kind of language generates confusion and errors in students. The component "in mathematics taught the use of ambiguities should be controlled" hence arises.

Two more components subsequently emerge that "improve" mathematics teaching, the "richness of processes" component, and the component called "representative sample of the complexity of the mathematical object to be taught."

Once these four components are generated, as well as others that may have emerged in the process described above, the participants are encouraged to note that some of the components generated to define the epistemic suitability criterion are like those that appear in other instruments used to assess the quality of instruction provided. For example, the Mathematical Quality of Instruction instrument includes the richness of mathematics, errors, and accuracy (Hill et al., 2008).

This step ends by stressing that: 1) the DSC construct aims to offer teachers instructions to guide the design and redesign of their learning sequence, 2) teachers can use it as a guideline to organise their teaching practice, and 3) suitability can be understood as quality relativised and conditioned by the context and the teacher's judgment because there are numerous criteria, components, and indicators. Some of them enter into conflict with each other, and future teachers should decide which criterion is the most important to consider in the context the lesson is implemented in.

Step 8: This step is for the participants to use DSC as a guideline to reflect on the implementation of a learning sequence, and to make proposals for improvement in its redesign.

In the training cycle, mathematical errors are defined, and the future teachers are taught how to use the DSC tool to reflect on the mathematical errors they made in their practice Furthermore, it is assumed future teachers have competence in analysing the mathematical activity, allowing them to detect their errors.

3.3 Thematic analysis

We considered 297 MDs of four academic years, from 2014–15 to 2017–18. For the thematic analysis of errors, we followed several *steps*. In the *first step*, the authors of this paper became acquainted with the data. They read the future teachers' MDs to become familiar with their justifications for mathematical errors. The identifying data of each MD (author, title, educational level, mathematical content) were recorded, and tables were drawn up using text excerpts and phrases or words related to types of error. This allowed establishing initial codes for grouping those texts, phrases, and words into themes. A file was created for each MD, and, when a paragraph was reproduced, the number of the file in which it was found was indicated. Only paragraphs in which an error was specifically commented on were considered. Paragraphs in which errors made were reflected without any further specification were not considered. The first step finished with a binary classification (they recognise errors, they do not recognise errors).

In the *second step*, each table was coded by the authors, and, when necessary, the MD that generated the table was consulted again. By using the excerpts, phrases, or words of the MDs that, according to the participants, explained their errors, the authors made a second distinction between "mathematical errors" (the ones the authors indeed considered as errors), "ambiguities" (those the authors considered to be ambiguities, not errors), and "others" (when they did not fit into any of the other two categories), which was further divided into subcategories. The authors of this manuscript met regularly to discuss the generation and application of these codes until a minimum agreement was reached between three of the four authors.

One type of mathematical error on which the four authors agreed were errors in the procedure (see File 207 in Sect. 4.1.6).

Knowing what could be considered a mathematical error was not so obvious in numerous other cases. The authors often had to reread the future teachers' MDs to decide if there was an error or not, depending on several contextual factors. For example:

> *I proposed some exercises on systems of linear equations, the solution of which was two numbers (x, y) without decimals (…). However, in some cases this was not the case because I had made an error solving the system (File 164).*

In this case, the authors agreed, after some discussion, that it was not a procedural error since proposing the resolution of systems of equations whose solutions are decimal numbers cannot be considered a mathematical error. This type of system was proposed because of the error the future teacher made when solving the system, but it was not made during the implementation of the learning sequence.

Another example that generated discussion was the following comment from a future teacher:

> *An error was detected (…) the distance between Barcelona and Lesbos had to be calculated using some indications given as a linear combination of vectors (…). The result provided was totally unreal, as the distance was much shorter than the real one. The students noticed the error (File 179).*

In this case, it was agreed it was a mathematical error since the instructions of the problem provided a solution that was not realistic and not in accordance with the extra-mathematical context of the problem.

An example that, in line with what is mentioned in Sect. 2.3, was not considered an error by the four authors of this paper was found in File 166. The future teachers explained they used a mathematical error in the textbook to discuss it with the students. In this case, the error was not theirs, nor did they validate the error in the textbook. They did the opposite because they pointed it out.

In some cases, the future teachers considered possible ambiguities as mathematical errors (see error 2 in file 19).

In the category of "others", i.e., those errors detected by future teachers that were not considered mathematical errors or ambiguities by the authors, the subcategory of "inappropriate didactic decision" emerged. An inappropriate didactic decision is an action that, according to the description of future teachers, does not benefit, and even hinders the achievement of the objectives of the teaching and learning process in a class. For instance, the omission of the definition of fundamental content:

> *(…) but there was one concept that I did not work on: what is movement in the plane? I consider this to be a serious error since the whole unit dealt with movement in the plane (File 8).*

Cases in which the future teachers considered a correct definition to be an error because it was too abstract or difficult for the students to understand were classified as inappropriate didactic decisions.

Those errors the authors agreed were not mathematical errors, ambiguities, or inappropriate didactic decisions were grouped in the subcategory "without categorising." (e. g., File 164).

In the *third step*, only the errors considered as such by the authors of this article were considered. The errors were regarded as ambiguities and those considered as "others" were excluded. In this step, initial error codes were established (for instance, an initial code was "interchanging definitions of related concepts" that came from a comment in which a future teacher explained he had confused the definition of sine with that of cosine and vice versa). The method was the same as in the previous step, to reach a minimum agreement between three of the four authors.

Although the error of procedure was the type of error that the authors agreed on the most, there were also cases in which it was difficult to reach an agreement. The following extract is an example (File 5):

I wanted to solve the following function on the board to work on the quotient rule: (x) = (3x−2) / (√(3x). When I arrived home and reviewed what I had explained that day, I realised that instead of taking the derivative of this function (which was the statement written on the board), I derived the following: f (x) = (3x−2) / (3x^2). None of the students noticed at the time. I decided to use this error, and the next day, I asked a student to write the development of the derivative that we had worked out the day before on the board. I told the students that there was an error and that they should look for it. Finally, after reviewing the calculations for a while, a student found that the derivative of the root did not add up to what I had written on the board (File 5).

In this case, as in others, it was difficult to decide if the case studied corresponded to an error in the procedure, or to an error in another category (an error in the proposition since a function was given as a result that was not the derivative of the function that appeared in the task instructions). In the end, the authors agreed it was an error made in the proposition.

In the *fourth step*, the initial codes were organised into groups (for instance, "errors in the definition" and "interchanging definitions of related concepts" were put together) to define themes and classify the errors. It was ensured that the themes were consistent with the coded excerpts and, if deemed appropriate, the themes were redefined. They were analysed because of their ability to both answer the research questions, and meaningfully describe the dataset.

When the error was identified, some of the future teachers offered more information. This led us, in a *fifth step*, to analyse their comments in more depth in order to assign levels of development of the didactic suitability assessment competence for the "errors" component. In this step, all the comments in every MD on those errors were analysed together.

Of the three levels of this competence discussed in Sect. 2.2, level L1 was not applicable, as the future teachers reflected on their practice using a guide they had been taught how to use. This paper only focuses on the reflection they make on their errors, but it turns out it is a naturalistic study, and the future teachers reflected on numerous other aspects. Moreover, the space was limited, which may be why they did not think it was important to explain improvements related to errors, and put forward proposals on other components they considered more important. We therefore believe we lack information to assign level L3 to them. Only the sub-levels of level L2 can be assigned to them.

As shown in the previous comment (File 5), the future teacher does not limit herself to detecting the error since she begins the comment by explaining the task proposed and her objective in proposing it. She then explains what her error was, when it was detected and who detected it (first she did and then a student). She also explains how she managed it, although she does not give much information about this. Furthermore, on a scale of 1 to 5, she rates the mathematical errors she has made as a 4. In other words, the number of errors and the type of error she has made do not lead her to make a negative assessment of the impact her errors have had on the quality of her instruction. We consider this reflection as evidence that this future teacher has reached level L2.4 of the didactic suitability assessment competence.

In the example below, the future teacher only explains the mistake she made. On a scale of 1 to 5, she rates the impact of the mathematical errors she has made as a 3. We consider this reflection as evidence that this future teacher has reached level L2.3 of the didactic suitability assessment competence:

(…) I showed the students an image of an octahedron, and asked them which two figures formed the one shown in the image. The students answered "two tetrahedra", and I said "yes", which is not true (…) (File 65).

In the following example, level L2.2 was assigned because the future teacher commented on three errors. The first one was considered an error by the researchers, the second one was an ambiguity (it is considered an error to place the comma that separates the integer part of the decimal of a component of a vector of the plane at the bottom instead of at the top), and the third one was regarded as an inappropriate didactic decision.

Error 1: (…) at some moments, I became nervous (…) I wrote Ax + By +C = 0, and said that the value of "A" was the slope, while in reality it was: m= -A/B. In the next session, I realised I had explained erroneous concepts, I corrected and justified them. The error was probably motivated by the fact that the coefficient of "x" is indeed the slope when the line is expressed explicitly.
Error 2: Not explaining the notation in detail (…)
Error 3: Not using GeoGebra to its full potential (…) If I had explained the concepts using this tool, most of the geometric concepts would have been easier to assimilate (File 19).

File 8, discussed above, is an example of level L2.1 since the researchers considered the error was an inappropriate didactic decision.

Regarding the participants that said they did not make any errors, we believe we lack information to assign any level of competence (amongst other reasons because in some cases there is evidence they did make errors).

Table 2 Number of errors, ambiguities, and "others"

Recognises having made a specific mathematical error		Number of MDs	(%)
Considered mathematical error		166	67%
Considered ambiguity		23	9%
Others	Considered inappropriate didactic decision	40	24%
	Without categorising	18	
Total		247	100%

Table 3 Types of errors

Types of errors	Number of errors	(%)
1) Definition	43	26%
2) Representation	38	23%
3) Task instructions	39	24%
4) Argument	3	2%
5) Proposition	29	17%
6) Procedure	13	8%
Total	166	100%

4 Results

4.1 Types of mathematical errors identified by the future teachers

Of the 297 MDs of future teachers analysed, 158 participants commented they made and detected specific errors during the implementation of their learning sequence. The other 139 participants stated they did not make any errors, or recognised some errors, but without specifying them.

The 158 participants who said they identified their errors in the implementation of their learning sequence pointed out a total of 247 errors. However, as shown in Table 2, in the second step, of those 247 errors, only 166 were considered mathematical errors. The number of errors obtained (166) does not coincide with the number of MDs (158) in which future teachers admit having made errors because, in some of the MDs, several errors were recognised, and some of the errors found were not considered as errors by the authors. In 19 cases, the comments in which they explained their errors were ambiguities, and in 4 cases, they were ambiguities the context disambiguated. In the remaining 58 cases, other subcategories such as "inappropriate didactic decisions" emerged in 40 of them, and in the other 18 cases mathematical errors were not considered, either because they had nothing to do with mathematics (when a link did not work in the task instructions of a problem), or because they were not considered mathematical errors (e.g., identifying a function using its symbolic expression).

The first result to highlight is that approximately 33% of the errors the participants considered as mathematical were not according to the authors of this paper.

In the fourth step of the methodology, the codes assigned in the third step were grouped into themes, and, whenever necessary, the themes were refined.

In the comments the participants made about the errors they made, they usually gave explicit or implicit information about the type of error made, which allowed classifying them in the following Table (Table 3):

4.1.1 Error in the definition

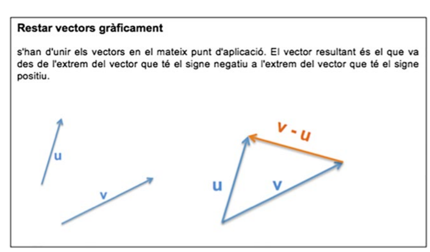
This category includes errors made when defining mathematical concepts (for example, when a definition was given that did not include all the cases). It also contains errors due to confusion between notions, incorrectly explaining a notion to students, or incorrectly applying a definition. For example:

> *I made the error of confusing sine for cosine on the blackboard. The students immediately detected the error and made me see it (File 31).*

4.1.2 Error in the representation

Errors made when representing were included in this category (e.g., errors in notations, incorrect use of mathematical language, and representations of mathematical objects in general). The following excerpt exemplifies this type of error:

> *The most important error was found in the graphic explanation of vector subtraction. Initially, it was as follows (File 58):*



The vectors must meet at the same point. The resulting vector is the one that goes from the head of the vector that has a negative sign to the head of the vector that has a positive sign.

4.1.3 Error in the task instructions

This category includes errors that appeared in the task instructions in the learning sequence, or in the task instructions that contained errors in a textbook used. An example is shown below:

> *A right triangle with a hypotenuse of 10 cm is rotated about the hypotenuse to generate a cone. Find the length of the sides of the triangle for the cone to have the maximum volume. This is a clear error of expression, as I wanted to say that the triangle is rotated about its height (File 25).*

4.1.4 Error in the proof or argument

This category includes situations in which erroneous or not very accurate arguments were put forward, reasons for which they could not be considered as mathematical proof. For example:

> *(…) in the proof of the surface area of a cone, I confused the radii of the two circles involved, r being the radius of the base and g of the generatrix, which acts as the radius of the circular sector. I made the error when I presented the argument, and I did not notice it till after the class was over. As the final formula for the area of the circular sector is πrg, it did not affect the result, but the radii were interchanged during the proof (File 95).*

4.1.5 Error in the proposition

Errors in stating a false proposition were included in this category. Wrong solutions provided to a problem, and not giving information about the process of solving the problem, were also considered in this category. For example:

> *(…) I would like to comment on an error in one of the solutions on the list of combinatorics problems (…) (File 13).*

4.1.6 Procedural error

This category includes errors that occur when applying a procedure incorrectly. For example:

> *The error consisted of initially validating the placement and result of the change of units requested, from 5 m^3 to km^3, where the student forgot to write the two zeros preceding the number 5 to complete the m^3 (File 207).*

Table 4 Sub-levels of L2 of the didactic suitability assessment competence

Sub-levels	Number of future teachers
L 2.4	31
L 2.3	60
L 2.2	24
L 2.1	43

4.2 Levels of didactical suitability assessment competence used to identify errors

The comments on errors made by participants show a dissimilar level of reflection, which led us to analyse the comments in more depth to see if they could be considered evidence of a certain level of development of the future teachers' didactic suitability assessment competence.

The 139 participants that said they made no errors were not assigned any level of development of the didactic suitability assessment competence because we lacked information. For level 2, the following results were obtained (Table 4):

We also consider we lack information to assign level 3 to any of the future teachers.

Although the participants were taught how to reflect on their errors following the training cycle explained in Sect. 3.2, only 20% reached level 2.4.

4.3 Other aspects that emerge from the reflection on errors

In this section, the statements in comments at any level that refer to who made the error, how the error was managed, the possible implications of the error, etc. are analysed.

4.3.1 Aspects related to who makes the error

In most of the comments considered as errors (157 comments), it was the future teacher who made the error (in their classroom management, or related to the material used), but in some cases it was made by the students (for example, when they solved a problem on the blackboard), and the future teacher validated what the student did (9).

4.3.2 Aspects related to who detects the error

In most of the MDs, in addition to commenting on the error, information was given about who detected it. In most cases (120), it was the future teacher who detected the errors. However, at times (19), the students helped the future teacher realise the errors, and at others (20), it was the mentor teacher at the school or the university tutor who detected it. In still other cases (7), no clear information was provided on who detected the error.

4.3.3 Aspects related to when the error is corrected

In addition to commenting on the error, most future teachers provided information on when they corrected it. In numerous cases (104), it was explicitly explained (or enough implicit information was offered) that errors could be corrected during the implementation, either at the moment they were detected, or in a later session (because the mentor told the future teachers after class, or because the future teachers realised they had made an error when reflecting on the class taught). In some cases (8), the participants commented the errors were detected too late and could not be corrected. In other words, they were detected while writing their MDs, when the internship period had already ended. In other cases (54), no information was provided on when the error was corrected.

4.3.4 Aspects related to error management

Another aspect observed was that the future teachers often commented how they didactically managed the error. In most cases (55), the error was managed by the future teachers, and they corrected their errors (for example, correcting the instructions of the problems that contained errors) in the same session or in the subsequent session. In one case (1), the future teacher asked the students to find the error. In other cases (83), no information was provided about how it was managed. In yet other cases (27), the future teachers commented they used the error to better explain some content, or to make their students see that is normal to be wrong, and that everybody makes errors.

4.3.5 Aspects related to the subjective assessment of the importance of the error

In addition to commenting on the error, in a few cases, future teachers explicitly assessed the importance of the error, and either considered it important (20) or not (11).

4.3.6 Aspects related to the cause of the error

In some cases, they commented on the cause of the error. In 36 cases, no information was given. When they did comment on the cause, in several cases (27), the cause was a distraction. In most cases (65), it was not checking the material and/or not solving the problems before implementing them. In a few cases (3), the cause was the lack of time that made the future teachers do things fast, while in others (5), they did not check the calculations of the students and validated their resolution. In a few cases, they said the cause was related to nerves and insecurity (3), or to being over-confident and to improvising the explanation (4). In one case (1), the reason was that it was not considered important (for instance, showing the students). In several cases (11), it was due to a lack of accuracy and/ or wanting to be intuitive, while in yet other cases (5), the cause was a lack of concentration produced by student behaviour. For example:

> *My interpretation of this error is that I trusted the student who corrected the exercise since he/she usually answers correctly. Furthermore, I had to check on the rest of the students who were not behaving properly, and I was resolving the doubts of other students about how to do the next exercises (File 29).*

It is worth noting that in only six cases (6) the future teachers considered the cause of the error was due to their lack of knowledge. In five of them (5), it was because of a lack of mathematical knowledge (5), and in one case (1) it was because of a lack of knowledge related to an aspect of the problem contextualised.

5 Conclusions

Research studies on quality teaching of mathematics (e.g., Charalambous & Praetorius, 2018; Hill et al., 2008) are an important reference for both initial and continuous teacher training programmes. This is the case of the DSC guideline that originated in Spain, but has been widely disseminated in Latin America, showing it is a tool that can be applied in different cultural environments. There are multiple research studies that use the DSC tool (e.g., Beltrán-Pellicer & Giacomone, 2018; Seckel et al., 2023). Two types of tools to improve the reflective process are found in the literature review: tools for observation and tools for assessment. DSC is a tool for assessment. The participants in this study, however, also used it to observe and organise their reflection.

Regarding the first question, 53% of the future teachers comment they make errors when reflecting on their practice. Some confusion between error, ambiguity, and an inappropriate didactic decision is observed since 33% of the errors the future teachers admit they make are not considered mathematical errors by the authors of this article. One of the reasons why the participants consider an inappropriate didactic decision or an ambiguity to be a mathematical error is that they caused a mathematical error in the students.

In numerous cases, it was necessary to resort to contextual information (the school context, the realistic context of the problem the didactic contract, etc.) to decide whether it was a mathematical error or not. This result seems especially relevant because it shows that mathematical errors in the

Table 5 Guideline for reflection on the "errors" component

Which is the mathematical error you made?

Explain it:

Is what you considered to be an error a real error?

☐ Mathematical error ☐ Ambiguity ☐ Inappropriate didactic decision ☐ Others

Who detected the error? ☐ The future teacher ☐ The students ☐ The mentor or the tutor

When?

- ☐ Before the implementation when checking the material
- ☐ During the implementation
- ☐ After the implementation when reflecting on the errors made

Who made the error?

☐ The future teacher ☐ It was found in the material used, and the future teacher validated it ☐ A student, and the future teacher validated it

Was it an important error?

☐ Hardly important ☐ Important ☐ Very important

Which was the assessment criterion used?

Explain it

¿What type of error was it?

- ☐ Error in the definition
- ☐ Error in the representation
- ☐ Error in the task instructions
- ☐ Error in the proof or argument
- ☐ Error in the proposition
- ☐ Procedural error

What was the cause of the error?

- ☐ Distraction
- ☐ Not checking the material and/or solving the task before the implementation
- ☐ Lack of concentration in trying to control the classroom
- ☐ Lack of knowledge
- ☐ Other

How was the error managed?

- ☐ It was corrected during the implementation

Explain how:

- ☐ It was corrected in the redesign

Explain how:

training of future mathematics teachers are a topic that needs to be addressed in greater depth.

The types of errors that emerged in this study refine the dual classification: conceptual errors and procedural errors. To classify errors, a broader classification is required: 1) error in the definition, 2) error in the representation, 3) error in the task instructions, 4) error in the proof or argument, 5) error in the proposition, and 6) procedural error.

It is also concluded that most of the errors are related to the instructions of the tasks proposed, to the representations used, and to the definitions.

The study conducted allows finding inductive categories for deeper reflection on the errors of future teachers. As far as the causes of the error are concerned, those that are specific to future teachers are stressed. We refer to those errors that, according to the future teachers, are due to being nervous because of their lack of experience, or the ones they make or validate because they are concerned about maintaining order in the classroom.

With respect to who detects the error, in the Master's degree where this research is developed, it is assumed that future teachers have competence to analyse the mathematical activity, which allows them to detect their errors. However, the analysis of the data shows that they do not always do it directly. On certain occasions, the students, the mentor teacher at the school, or the university tutor detected them.

Regarding the second question, Table 4 shows that the current training cycle (see Sect. 4.2) does not ensure that the majority of future teachers reach level 2.4 of development of the didactic suitability assessment competence for the "errors" component.

Based on the results of this study, the "errors" component of epistemic suitability can be analysed more in-depth using the following guideline (Table 5):

When the DSC guideline is taught in teacher training, it can be expanded using Table 5 to guide the future teachers' reflection on the type of mathematical error made during their practice. We hypothesise that if future teachers use this tool for the "errors" component, they will be able to reach level 2.4 of the didactic suitability assessment competence more easily.

Since distinguishing between error and ambiguity often depends on the context, another line of development of this research is to extend this study on errors to a joint study on the errors and ambiguities that future teachers detect in their teaching practice when they reflect on it using DSC.

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Declarations

Conflicts of Interest The authors have no competing interests to declare that are relevant to the content of this paper.

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