The Penrose Process

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Abstract: The Penrose process is a theoretical mechanism by which energy can be extracted from a rotating black hole. This possibility arises in the Kerr spacetime, where the existence of an ergoregion allows for asymptotic negative-energy states to exist. In this work, we study how these states allow part of the rotational energy of the black hole to be extracted when an infalling particle splits inside the ergoregion. We show that the maximum relative energy gain of the process (with respect to the infalling particle) is $\Delta E \approx 0.207$. The process is shown to be possible only under specific physical restrictions. The area theorem shows that this extraction cannot continue indefinitely, but only until the black hole has lost its rotational energy, while the horizon area does not decrease. The Wald and Bardeen-Press-Teukolsky inequalities further show that the process requires highly relativistic conditions: the decay products must have velocity $|v| \geq \frac{1}{2}c$ and relative velocity $|w| \geq \frac{1}{2}c$.

Keywords: Black holes, Kerr spacetime, Penrose process
SDGs: Quality education

I. INTRODUCTION

General Relativity describes gravity not as a force acting at a distance, but as a manifestation of the curvature of spacetime. The Einstein field equations relate the geometry of spacetime to the distribution of matter-energy within it, and determine how the distribution of energy and momentum shapes the spacetime metric [1]. Due to their highly nonlinear character, however, there are few exact analytical solutions [2].

Among these solutions, we find the Kerr metric, which describes the spacetime of a rotating, stationary, and axially symmetric black hole [3]. While the Schwarzschild solution represents the idealized case of a non-rotating black hole, the Kerr metric is more relevant from an astrophysical point of view, since real black holes are expected to possess angular momentum. As a result of the black hole's rotation, the Kerr black hole has features that are absent in the Schwarzschild case. In particular, it contains a region outside the event horizon where no observer can remain at rest with respect to infinity, called the ergoregion [4–6].

Remarkably, the existence of the ergoregion allows for the possibility of negative-energy states, as measured by an observer at infinity [4, 5]. This is essential for the so-called "Penrose process", proposed by Roger Penrose as a mechanism for extracting energy from a rotating black hole [7].

This work aims to study the Penrose process in the Kerr spacetime, focusing on the role of the ergoregion, the existence of negative-energy states and the conditions under which energy extraction becomes possible. It is organized as follows. In Section II we introduce the Kerr metric and discuss its main geometrical features. We describe the structure of the spacetime, including the ring singularity, the event horizons, the stationary limit surface, the ergoregion and how negative-energy states arise inside it. This is the background needed for the analysis of energy extraction. In Section III, we study the Penrose process itself. We explain how the decay of a particle into two fragments can lead to a net gain of energy at infinity. We also discuss the physical conditions required for the process to occur, including the restrictions imposed by the area theorem and by two relevant inequalities which implement the kinematical constraints that make the Penrose process possible. We conclude in section IV.

II. THE KERR BLACK HOLE

In this section, we describe the main features of the Kerr solution, which is one of the most important exact solutions of the Einstein field equations [3]. This solution describes a rotating, stationary, and axially symmetric spacetime. The Kerr metric reads, in Boyer and Lindquist coordinates (r, θ, ϕ, t), as follows [6]:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \Sigma\left(\frac{\mathrm{d}r^2}{\Delta} + \mathrm{d}\theta^2\right) + (r^2 + a^2)\sin^2\theta\mathrm{d}\phi^2 + \frac{2Mr}{\Sigma}(\sin^2\theta\mathrm{d}\phi - \mathrm{d}t)^2\,, \quad (1)$$

where

$$\Delta(r) \equiv r^2 - 2Mr + a^2\,, \quad (2)$$

$$\Sigma(r,\theta) \equiv r^2 + a^2\cos^2\theta\,, \quad (3)$$

and a is the angular momentum per unit mass.

This metric is invariant under simultaneous transformations $t \to -t$ and $\phi \to -\phi$. Thus, it is stationary but

not static. In the limit $r \to \infty$, we recover the asymptotically flat spacetime, and in the limit $a \to 0$, we recover the Schwarzschild metric [4, 6].

There are two singularities, one at $\Delta = 0$ and the other at $\Sigma = 0$; the former is a coordinate singularity and the latter is a true singularity of the manifold. We can see this by noting that the curvature invariants are regular on $\Delta = 0$ and singular on $\Sigma = 0$. If we change to Kerr-Schild coordinates, it is possible to characterize the structure of the singularity and determine its ring-shaped geometry. Inside the ring, the metric is regular. In Kerr coordinates, we can extend the manifold to include the submanifold $\Delta = 0$ [4, 6].

When $a^2 > M^2$, $\Delta = 0$ has no real solution and the Kerr metric does not describe a black hole. In this case, there is no horizon concealing the singularity at $\Sigma = 0$ and the singularity is said to be "naked". According to roger Penrose's cosmic censorship conjecture, naked singularities cannot form in nature [8]. From now on, we restrict our analysis to the case $a^2 \leq M^2$. The limiting case $a = M$ corresponds to a so-called "extremal black hole", which possesses a single horizon. We shall rewrite $\Delta = r^2 + a^2 - 2Mr = (r - r_-)(r - r_+)$, where $r_- \equiv M - \sqrt{M^2 - a^2}$ and $r_+ \equiv M + \sqrt{M^2 - a^2}$ are the solutions to $\Delta = 0$ [4, 6].

The surfaces $r = r_+$ and $r = r_-$ are null hypersurfaces as their normal vector is a null vector. Null hypersurfaces separate regions of spacetime where $r = constant$ are timelike hypersurfaces from regions where $r = constant$ are spacelike hypersurfaces. Therefore, an object crossing a null hypersurface $r = constant$ can never go back. For this reason, null hypersurfaces are called horizons [4, 6]. Therefore, we have two horizons, the outer one (r_+) and the inner one (r_-), which separate 3 spacetime regions—see Fig. 1. Of these two horizons, only the outer one ($r = r_+$) is an "event horizon", which determines the boundary of causal contact with future null infinity. The inner horizon ($r = r_-$) lies inside the black hole region and is a Cauchy horizon [4, 6].

A. Killing vectors and conserved quantities

For general geodesics on this background there are two Killing vectors, $\vec{K} = \partial_t$ and $\vec{R} = \partial_\phi$, and hence, there are two conserved quantities: $E = -u^\mu K_\mu = -u_t$ and $L = u^\mu R_\mu = u_\phi$, respectively. For massive particles, we have the four-momentum $P^\mu = mu^\mu$, and the conserved quantities are the energy E and the angular momentum per unit mass L, measured at radial infinity. For massless particles, we choose the affine parameter so that $P^\mu = u^\mu$, and E and L are the energy and the angular momentum at infinity [4, 6].

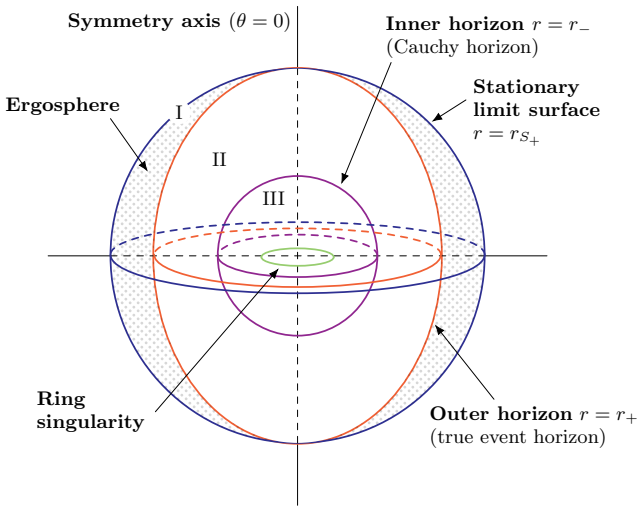


FIG. 1: Schematic diagram of the Kerr black hole geometry [6]. In blue, the Stationary limit surface ($r = r_{S_+}$ surface). In red, the outer horizon or event horizon ($r = r_+$ surface). In purple, the inner horizon or Cauchy horizon ($r = r_-$ surface). In green, the ring singularity. Region I corresponds to $r > r_+$ and it is the black hole horizon; the $r = constant$ hypersurfaces are timelike. Region II corresponds to $r_- < r < r_+$ the $r = constant$ hypersurfaces are spacelike; when $a^2 = M^2$, $r_+ = r_-$ and this region disappears. An object that falls inside $r < r_+$ can only access decreasing values of r until it reaches the inner horizon and region III. Region III corresponds to $r < r_-$ and it contains the singularity; the $r = constant$ hypersurfaces are timelike. The stationary limit surface, or ergosphere, is the outer boundary of the ergoregion. The ergoregion is the dotted area, where $r_{S_+} > r > r_+$; here, an observer cannot stay still, they are forced to move in the direction of the black hole's rotation.

B. Ergoregion

In the present subsection we comment on the ergoregion of a Kerr black hole, a spacetime region that allows for the existence of negative-energy states and therefore makes the Penrose process possible.

In the Kerr spacetime, the horizon does not coincide with the surface where g_{tt} changes sign. We can write g_{tt} as $g_{tt} = -\frac{1}{\Sigma}(r - r_{S_+})(r - r_{S_-}) = 0$, so that $g_{tt} > 0$ inside the interval $[r_{S_-}, r_{S_+}]$ and $g_{tt} < 0$ outside it. We have $r_{S_-} \leq r_- \leq r_+ \leq r_{S_+}$. Therefore, there is a region outside the outer horizon ($r_+ < r < r_{S_+}$) where $g_{tt} > 0$, which means that $\vec{K}$ becomes spacelike [4–6]:

$$K^\mu K_\nu = g_{\mu\nu} K^\mu K^\nu = g_{tt} > 0 \implies K^\mu \text{ spacelike} \quad (4)$$

This is the ergoregion. The surface r_{S_+} is called ergosphere or stationary limit surface, and it is the outer boundary of the ergoregion—see Fig. 2; here, $\vec{K}$ is a null vector [4–6].

Since $\vec{K}$ is spacelike in the ergoregion, an observer cannot stay still in this region, i.e., a static observer cannot exist. They are forced to move in the direction of the ro-

tation of the black hole but can still move toward or away from the event horizon and exit the ergosphere [4, 6].

Another important consequence of $\vec{K}$ being spacelike in the ergoregion is that it is possible for the energy to be negative. This is essential for the Penrose process to be possible. Inside the event horizon, energy can also be negative, but as we cannot observe it, it has no interest for our purposes here.

III. PENROSE PROCESS

Penrose noticed that the existence of negative energy in the ergosphere gives rise to the possibility of extracting energy and angular momentum from the black hole [7]. In this section we show how this can be achieved.

Consider a particle, A, with energy E^A, being dropped into the ergosphere and splitting into two particles, B and C [4, 5, 7]:

$$E^A = E^B + E^C \,. \tag{5}$$

It is possible to arrange for particle A to follow a geodesic trajectory, which conserves the energy throughout. The decay can be such that particle B has negative energy ($E^B < 0$) and crosses the event horizon, whereas particle C escapes back to infinity. Energy conservation, expressed in eq. (5), implies that, at the end of this process, $E^C > E^A$. Thus, the escaping particle carries away more energy than the original particle brought into the ergoregion. The gained energy would come from the black hole, and we would be extracting it by decreasing its angular momentum [4, 5, 7]. This argument—which is clearly reminiscent of the Hawking effect responsible for the evaporation of black holes—shows the qualitative idea behind the Penrose process. Let us now describe this mechanism more quantitatively by introducing a particular kinematic configuration.

Let A be a particle at rest at infinity (hence, with energy $E^A = 1$), and let particles B and C be photons. Particle A decays into the two photons at a point $r < 2M$ such that $\dot{r} = 0$ (that is to say, a turning point). We denote the energies and the angular momenta, respectively, by

$$\{E^A = 1, L^A = \alpha^A\}, \{E^B, L^B\}, \{E^C, L^C\} \tag{6}$$

where

$$\alpha^A = \frac{1}{r - 2M}\left(-2aM + \sqrt{2Mr\,\Delta}\right), \tag{7}$$

follows from the radial equations [5].

Analogously for L^B and L^C, we obtain the following relations

$$L^B = \alpha^B E^B \,, \quad L^C = \alpha^C E^C \,, \tag{8}$$

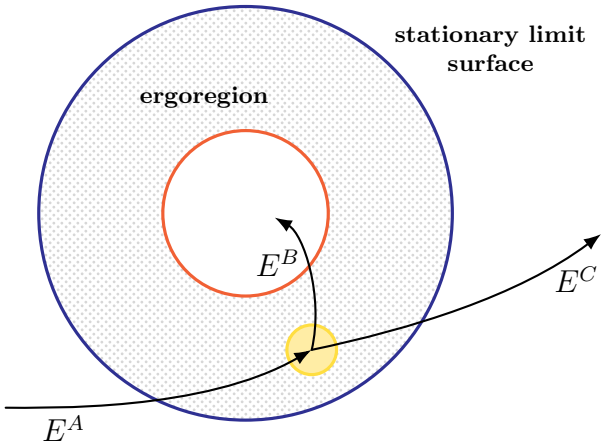


FIG. 2: The Penrose process. The dotted region represents the ergoregion, bounded externally by the stationary limit surface or ergosphere (blue), and internally by the outer event horizon (red). An object falls with energy E^A toward the Kerr black hole and splits into two pieces while in the ergosphere (yellow circle). One piece falls into the horizon with a negative energy E^B, while the other escapes to infinity with a larger energy than that of the original object, $E^C > E^A$.

where [5]

$$\alpha^B = \frac{1}{r - 2M}\left(-2aM - r\sqrt{\Delta}\right), \tag{9}$$

$$\alpha^C = \frac{1}{r - 2M}\left(-2aM + r\sqrt{\Delta}\right). \tag{10}$$

We now impose the conservation of energy and angular momentum

$$E^B + E^C = E^A = 1\,, \tag{11}$$

$$L^B + L^C = \alpha^B E^B + \alpha^C E^C = L^A = \alpha^A\,, \tag{12}$$

with which we find

$$E^B = \frac{\alpha^A - \alpha^C}{\alpha^B - \alpha^C} = -\frac{1}{2}\left(\sqrt{\frac{2M}{r}} - 1\right), \tag{13}$$

$$E^C = \frac{\alpha^B - \alpha^A}{\alpha^B - \alpha^C} = +\frac{1}{2}\left(\sqrt{\frac{2M}{r}} + 1\right). \tag{14}$$

Notice that $E^B < 0$ and $E^C > 1$. Namely, the energy of particle B, as measured by an asymptotic observer, is negative. Let us denote the relative energy gain with respect to the initial energy of the infalling particle by ΔE. Since we have normalized the initial energy to $E^A = 1$, ΔE is a dimensionless quantity and reads

$$\Delta E = \frac{1}{2}\left(\sqrt{\frac{2M}{r}} - 1\right) = -E^B > 0\,. \tag{15}$$

Hence, throughout this process, the black hole has absorbed negative energy (equivalently, it has lost it) with

absolute value $|\Delta E|$, and the particle escaping to infinity (C) has gained an energy ΔE [5].

From equation (15) we can see that the gained energy will reach its maximum when the particle arriving from infinity has a turning point at the event horizon. Thus

$$\Delta E \leq \frac{1}{2}\left(\sqrt{\frac{2M}{r_+}} - 1\right). \tag{16}$$

In turn, r_+ reaches its minimum value $r_+ = M$ when $a^2 = M^2$, that is to say, in the extremal black hole limit. Therefore [5]

$$\Delta E \leq \frac{1}{2}\left(\sqrt{2} - 1\right) \approx 0.207 \tag{17}$$

The inequality is saturated by an extremal black hole and a particle that has a turning point at the event horizon.

The result above shows that the Penrose process can, in principle, extract energy from a rotating black hole, with the maximum efficiency obtained in the extremal limit($a^2 = M^2$). In the next two subsections, we discuss the general constraints that limit this extraction mechanism. First, we review the area theorem, and then the Wald and Bardeen-Press-Teukolsky inequalities, respectively.

A. Area theorem

As the black hole absorbs a particle with negative energy, its mass changes according to $\delta M = E^B < 0$, implying a decrease in its total mass. At first sight, this appears to contradict Hawking's area theorem [9], which states that the horizon area of a black hole cannot decrease, $\mathrm{d}A/\mathrm{d}t \geq 0$.

This apparent paradox is resolved by noticing that the total mass of the black hole has two contributions: an irreducible part and a rotational energy contribution [4, 5],

$$M = M_{\mathrm{irr}} + (\text{rotational energy contribution}) \tag{18}$$

The irreducible mass is defined by [10]

$$M_{\mathrm{irr}} \equiv \frac{1}{2}\sqrt{r_+^2 + a^2} = \sqrt{\frac{1}{2}Mr_+}\,. \tag{19}$$

The irreducible mass is directly related to the area of the event horizon according to $A = 16\pi M_{\mathrm{irr}}^2$. This implies, on account of Hawking's area theorem,

$$\delta A \geq 0 \Leftrightarrow \delta M_{\mathrm{irr}} \geq 0 \tag{20}$$

In other words, the irreducible mass does not decrease, and hence the name.

The Penrose process is compatible with this result because it does not extract energy from the irreducible mass of the black hole. Instead, it extracts energy from its rotation energy. During the process, the total mass M may decrease, but the horizon area, and hence M_{irr}, does not decrease [4, 5].

We can now rewrite equation (16) in terms of the irreducible mass

$$\Delta E \leq \frac{1}{2}\left(\frac{M}{M_{\mathrm{irr}}} - 1\right) \tag{21}$$

This expression shows that energy extraction can proceed only until the black hole has lost all of its rotational energy. In that final state, $M \to M_{\mathrm{irr}}$, and no further energy can be extracted by the Penrose process.

B. Inequalities

We now look at the kinematical constraints that make the Penrose process possible. It is necessary that one of the fragments produced inside the ergoregion can acquire negative conserved energy and be absorbed by the black hole. This requirement is constrained by inequalities relating the locally measured energy, the conserved energy at infinity and the angular momentum of the particle [5]. In this section, we present the Wald inequality [11] and the Bardeen-Press-Teukolsky inequality [12].

1. The Wald inequality

Consider a particle with dimensionless specific energy $E = \mathcal{E}/mc^2$ (where $\mathcal{E}$ is the total energy of the particle), that breaks into fragments, one of which has specific energy ϵ and three-velocity v. The Wald inequality sets limits on ϵ. Since our purpose is to understand its role in the Penrose process, we will not reproduce the full calculation here, but simply state the final result. A full derivation of this result can be found in [5]. The Wald inequality can be written as follows [11]

$$\gamma E - \gamma|v|(E^2 - g_{00})^{1/2} \leq \epsilon \leq \gamma E + \gamma|v|(E^2 - g_{00})^{1/2} \tag{22}$$

In the Kerr geometry, $g_{00} = 1 - 2Mr/\Sigma < 0$ in the ergosphere, and its lower bound, $g_{00} = -1$, is attained at the event horizon for $a = M$ and $\theta = \pi/2$. Hence, the Wald inequality can be expressed as [5]

$$\gamma E - \gamma|v|(E^2 + 1)^{1/2} \leq \epsilon \leq \gamma E + \gamma|v|(E^2 + 1)^{1/2} \tag{23}$$

The maximum energy that a particle describing a stable circular orbit can have is $E_{\max} = 3^{-1/2}$ (it is reached in the extremal case, $a = M$, for the innermost stable circular orbit). Thus, for ϵ to be negative, it is necessary that [5]

$$|v| > \frac{E}{\sqrt{E^2 + 1}} = c/2\,. \tag{24}$$

This result shows that the fragments must have relativistic energies before any extraction of energy by a Penrose process becomes possible.

2. The Bardeen-Press-Teukolsky inequality

The Bardeen-Press-Teukolsky inequality [12] seeks for a bound on the magnitude of the relative three-velocity, $|w|$, between two particles that collide at some point. Let these particles have specific energies E_+ and E_-, respectively. Choosing a local orthonormal tetrad frame at the collision point, in which the two particles have equal and opposite spatial velocities, we can write [5]

$$|v| \geq \frac{1}{E_+ + E_-} |\sqrt{(E_+^2 - g_{00}} \pm \sqrt{(E_-^2 - g_{00}}| \tag{25}$$

and

$$|w| = \frac{2|v|}{1+|v|^2}\,. \tag{26}$$

Substituting (25) in (26), we obtain the required lower bound on $|w|$. This inequality is that of Bardeen, Press and Teukosky [12]. If $E_+ = E_{\max} = 3^{-1/2}$, $E_- = 0$ and $g_{00} = -1$, the inequality (25) gives

$$|v| > \left(\sqrt{1+1/3} - 1\right)\sqrt{3} = 2 - \sqrt{3}\,. \tag{27}$$

The corresponding inequality for $|w|$ is

$$|w| \geq \frac{c}{2} \tag{28}$$

in agreement with (24), derived from Wald's inequality.

The main implication of these two inequalities is that an efficient Penrose process cannot occur unless the decay products possess relativistic velocities: the fragments must reach speeds greater than half of the speed of light.

IV. CONCLUSIONS

In this work we have studied the Penrose process as a theoretical mechanism for extracting energy from a rotating black hole. To this end, we first introduced the Kerr metric and its main features relevant to the process. We have seen that this process is not arbitrary: the area theorem and the Wald and Bardeen-Press-Teukolsky inequalities impose constraints on the amount of energy that can be extracted and on the velocities of the decay products, respectively.

In particular, the constraints on the split velocity imply that the fragments must be produced with highly relativistic relative velocities (exceeding $c/2$) for any net energy gain to occur. This highlights that the maximum efficiency of roughly 20.7% is an upper bound that is extremely difficult to saturate in realistic astrophysical environments. This raises doubts on the physical feasibility of the Penrose process, given the highly relativistic and fine-tuned conditions required. Still, the question remains: could we dream of using black holes as batteries in the far future?

This study also points to several possible directions for future work. A first question is whether analogous processes can occur in black holes with electric charge (the Reissner–Nordström black hole or the Kerr-Newman black hole), where interaction between gravitational and electromagnetic effects may modify the conditions for energy extraction. In particular, this could potentially allow for maximum efficiencies beyond the one obtained in the Kerr case.

Finally, the Penrose process shows that it is theoretically possible to extract energy from a black hole, opening the door to other classical and quantum mechanisms. Whether we look at continuous fields or quantum effects, the ability to tap into a black hole's energy highlights a deep connection between spacetime geometry and thermodynamics. Exploring these broader scenarios is a natural next step to better understand the fundamental limits of gravity.

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El procés de Penrose

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Resum: El procés de Penrose és un mecanisme teòric mitjançant el qual es pot extreure energia d'un forat negre en rotació. Aquesta possibilitat apareix en l'espaitemps de Kerr, on la presència d'una ergoregió permet l'existència d'estats amb energia negativa respecte de l'infinit. En aquest treball estudiem com aquests estats permeten extreure part de l'energia rotacional del forat negre quan una partícula incident es fragmenta dins de l'ergoregió. Mostrem que el guany relatiu màxim d'energia del procés, respecte de la partícula incident, és $\Delta E \approx 0.207$. També es mostra que el procés només és possible sota certes restriccions físiques. El teorema de l'àrea indica que aquesta extracció no pot continuar indefinidament, sinó només fins que el forat negre ha perdut la seva energia rotacional, mentre que l'àrea de l'horitzó no disminueix. A més, les desigualtats de Wald i Bardeen-Press-Teukolsky mostren que el procés requereix condicions altament relativistes: els productes de la desintegració han de tenir una velocitat $|v| \geq \frac{1}{2}c$ i una velocitat relativa $|w| \geq \frac{1}{2}c$.
Paraules clau: Forats negres, espaitemps de Kerr, procés de Penrose
ODSs: Aquest TFG està relacionat amb l'ODS4

Objectius de Desenvolupament Sostenible (ODSs o SDGs)

1. Fi de la es desigualtats		10. Reducció de les desigualtats	
2. Fam zero		11. Ciutats i comunitats sostenibles	
3. Salut i benestar		12. Consum i producció responsables	
4. Educació de qualitat	X	13. Acció climàtica	
5. Igualtat de gènere		14. Vida submarina	
6. Aigua neta i sanejament		15. Vida terrestre	
7. Energia neta i sostenible		16. Pau, justícia i institucions sòlides	
8. Treball digne i creixement econòmic		17. Aliança pels objectius	
9. Indústria, innovació, infraestructures			