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Kähler Geometry

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Abstract

The main goal of this work is to provide an introductory dive into the subject of Complex Geometry by giving three different characterizations of Kähler manifolds and proving their equivalence. We define complex, Hermitian, Kähler and symplectic manifolds and we briefly study their properties. We present the Hodge conjecture and define the holonomy group. Finally, we present a brief glimpse into other types of spaces, namely Calabi-Yau and Hyperkähler manifolds.

1 Introduction

It would not be a stretch to say that everybody is at least a bit familiarized with geometry. Studied for millennia, starting by the ancient Babylonians, then the Egyptians and the Greek, its story is as old as mathematics itself.

Nowadays, topology and geometry are the branches that mathematics enthusiasts are most likely to be aware of. Indeed, the Internet is full of pages and videos directed at non-mathematicians on the subjects and its various interesting results, and books intended to bring the subject of mathematics to "regular people" most certainly focus on the subject whenever they want to get away from the typical probability conundrums. It is certainly the subject I was most familiar with before taking a university course about it. Perhaps this popularity of geometry and topology is due to the fact that they are easy to visualise, and the flexibility and generality of topology makes it easy to understand its concepts.

But sometimes we want more rigid structures than the ones describe rd by topological spaces. We then turn to smooth and Riemannian manifolds, which incorporate concepts such as differentiability, integration and distance, all characteristic traits of real numbers $\mathbb{R}$. Today we will go further, asking for more rigid manifolds that exhibit more structure; we will talk about complex manifolds and see how their structures relate to typical properties of $\mathbb{C}$, which will lead to Hermitian and Kähler manifolds. We will not stop there, as we will keep piling conditions on our manifolds in order to get an even more rigid space, leading to symplectic, Einstein, Calabi-Yau and Hyperkähler manifolds.

My interest in studying these structures is born from my interest in physics, since such manifolds have surprising applications in very different areas of physics, and this is sometimes the reason behind their definitions and their study. For instance, symplectic manifolds provide a very natural description of classical mechanics (a description I wished I were taught on the classical mechanics courses I took). If the points of our manifolds represent the possible values of the coordinates of the particles in our system, the phase space is then the cotangent and it has a natural symplectic structure, which is used to easily obtain the differential equations that dictate the evolution of the system.

Einstein manifolds appear in general relativity in a natural way: Einstein's equations (in natural units) say that $R_{ij} = -\frac{1}{2}g_{ij}R + \Lambda g_{ij} = 8\pi T_{ij}$ where g_{ij} are the components of the metric of a Riemannian manifold (M, g), R is the curvature of M, R_{ij} are the components of the Ricci curvature, Λ is a constant (called the cosmological constant), and T_{ij} is the stress-energy tensor,

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which describes the distribution of mass and energy in space. For an empty universe, $T_{ij} = 0$ so the equation says $R_{ij} = (R - \Lambda)g_{ij}$ which is the condition for an Einstein metric.

Complex and Kähler manifolds have applications in quantum mechanics and in the field of supersymmetry. We consider the L^2-completion space of the forms of the manifold as our space of quantum states, and we consider the Laplacian as the Hamiltonian. This gives a space where one can perform quantum mechanics. The space of states is automatically $\mathbb{Z}/2\mathbb{Z}$-graded as it can be separated into forms of even and odd degree. One can then introduce an algebra of operators that commutes with the Hamiltonian and that act on this graded space of states which means that it is a superalgebra (the particular choice of operators will depend on whether our manifold is a complex, Hermitian or Kähler one). The study of such spaces belongs to supersymmetry.

Calabi-Yau manifolds are central to the study of superstring theory. This theory works best when it is used to describe a universe of 10 dimensions, which is a problem since the space we live in is just 4 dimensional (if we count time). This is solved by proposing that we can locally describe the universe as $M \times X$ where M is the usual 4-dimensional Minkowski space and X has to be a Calabi-Yau manifold (due to arguments relating to supersymmetry) of complex dimension 3 (and hence real dimension 6) giving us a decomposition $10 = 4 + 2 \cdot 3$. The Calabi-Yau manifold X is assumed to have a very small radius (of the order of 10^{-34} meters) which is why we can't observe them in our everyday lives.

Calabi-Yau manifolds are also the central objects of the conjectures of Mirror Symmetry. Modern physics is inseparable from Quantum Field Theory (QFT), but it is also inseparable of the lack of mathematical rigor that accompanies QFT. It seems that we can always quantize a given Calabi-Yau manifold X by giving it what is known as a super conformal field theory, or SCFT. One then notes that there is a very natural automorphism of a SCFT structure (a change of sign of an action of the group U(1) on the SCFT). We say that two Calabi-Yau manifolds X and X' are mirror to each other if their SCFT structures are related through this automorphism.

This definition translates into surprising symmetries between the Dolbeault cohomology groups of X and X'. Mirror Symmetry is then the field that studies how to construct such spaces X' for a given X.

We will unfortunately not enter in these topics and we will skip most of the physical considerations on this work, as trying to study all of these mathematical objects while also giving the description of their applications to physics is too big of an endeavor for this text. Nevertheless, we will take the mathematical approach and try to understand all these types of mathematical structures, using the physical interpretations just as motivations.

Although we will try to briefly study all these types of manifolds in the text, our main goal will be to define Kähler manifolds in three different ways (using complex geometry, symplectic geometry and holonomy groups) and to convince ourselves that the three definitions are equivalent.

We assume the reader to have a good understanding of smooth manifolds and Lie groups as they are described in [Lee13] and of homology theory as presented in [Hat02], or on the level acquired by completing a master-level course in the University of Bonn (where I myself got instructed in differential geometry during an Erasmus program).

We will begin introducing complex manifolds and the structures that usually accompany them, without getting too much into complex algebraic geometry. We will continue with Hermitian manifolds and then we will arrive at our first definition of the Kähler manifold. We will follow with a brief discussion of Kähler manifolds and their properties to see why they are important, getting to the Hodge decomposition and eventually to the Hodge conjecture, as we have every tool we need in order to understand it.

Then we will drift into our second approach to Kähler manifolds, symplectic geometry, a field very related to physics and classical mechanics.

After a brief interlude for Riemannian geometry which we will use to prepare ourselves for what comes next we will introduce holonomy groups and our third approach to Kähler manifolds. Most of the results of this section are too ambitious for this work, so they are mostly referenced.

Introducing holonomy groups and their classification presents us with a great opportunity to study other, more restrictive kinds of manifolds; Hyperkähler and Calabi-Yau manifolds. Our final pages will include a brief study of them, presenting conjectures on their respective areas since they are a very active topic of research, especially from the perspective of string theory.

2 Kähler manifolds

2.1 Complex manifolds

Let us begin by extending the idea of smooth manifolds to complex manifolds. This is simply done by asking the chart functions to be holomorphic instead of just smooth.

Definition 2.1. Let $U \subseteq \mathbb{C}$ be an open subset. A function $f : U \to \mathbb{C}$ is called *holomorphic* if for every point $z_0 \in U$ there exists a ball $B_\varepsilon(z_0) \subseteq U$ of radius $\varepsilon > 0$ around z_0 such that f on $B_\varepsilon(z_0)$ can be written as a power series:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{for all } z \in B_\varepsilon(z_0)$$

Equivalently, let us consider $z = x + iy \in \mathbb{C}$ as an element $(x, y) \in \mathbb{R}^2$ and $f(x, y) = u(x, y) + iv(x, y) = (u, v) \in \mathbb{R}^2$ with $u(x, y)$ and $v(x, y)$ being the real and imaginary part of f, respectively. Then f is holomorphic if and only if f is continuously differentiable and the following equations (called the *Cauchy-Riemann equations*) are satisfied:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

which, by defining the operators $\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ and $\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$ we can summarize as $\frac{\partial f}{\partial \bar{z}} = 0$.

Now let us do the same but in multiple variables. First, let us define open sets in $\mathbb{C}^n$. We take as a basis of the topology the *polydiscs* $B_\varepsilon(w) = \{z : |z_i - w_i| < \varepsilon_i\}$ where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$. Now, a multivariable holomorphic function is just a holomorphic function in each of its variables. Equivalently:

Definition 2.2. Let $U \subseteq \mathbb{C}^n$ be an open subset and let $f : U \to \mathbb{C}$ be a continuously differentiable function. Then f is said to be *holomorphic* if the Cauchy-Riemann equations hold for all variables $z_i = x_i + iy_i$, that is:

$$\frac{\partial u}{\partial x_i} = \frac{\partial v}{\partial y_i}, \quad \frac{\partial u}{\partial y_i} = -\frac{\partial v}{\partial x_i} \quad i = 1, \ldots, n.$$

We also want our functions to take values in $\mathbb{C}^n$ instead of just $\mathbb{C}$.

Definition 2.3. Let $U \subseteq \mathbb{C}^m$ be an open subset. A function $f : U \to \mathbb{C}^n$ is called *holomorphic* if all its coordinate functions $f_1, \ldots, f_n$ are holomorphic functions $U \to \mathbb{C}$.

Definition 2.4. A map $f : U \to V$ between two open subsets $U, V \subseteq \mathbb{C}^n$ is called *biholomorphic* if f is bijective, holomorphic and its inverse $f^{-1} : V \to U$ is also holomorphic.

Definition 2.5. Let $U \subseteq \mathbb{C}^m$ be an open subset and let $f : U \to \mathbb{C}^n$ be a holomorphic map. The complex *Jacobian* of f at a point $z \in U$ is the matrix

$$J(f)(z) := \left(\frac{\partial f_i}{\partial z_j}(z)\right)_{\substack{1\leq i\leq n\\ 1\leq j\leq m}}$$

Multivariable holomorphic complex functions allow us to define complex manifolds, as well as morphisms between those:

Definition 2.6. A *holomorphic atlas* on a differentiable manifold M is an atlas $\{(U_i, \varphi_i)\}$ such that $U_i \simeq \varphi_i(U_i) \subseteq \mathbb{C}^n$ (in the sense that φ_i are homeomorphisms), and such that the transition functions $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ are holomorphic. As in the smooth case, the pair (U_i, φ_i) is called a *holomorphic chart.*

Two atlases (U_i, φ_i), (V_i, ϕ_i) are said to be equivalent if all their charts are compatible, that is, if the functions $\varphi_i \circ \phi_j^{-1} : \phi_j(U_i \cap V_j) \to \varphi_i(U_i \cap V_j)$ are holomorphic.

This gives an equivalence relation in the set of atlases of a differential manifold, which as in the smooth case, allows the following definitions:

Definition 2.7. A *complex manifold* X of dimension n is a real differentiable manifold M of dimension $2n$ with a choice of an equivalence class of holomorphic atlases.

Definition 2.8. A *holomorphic function* on a complex manifold X is a function $f : X \to \mathbb{C}$, such that $f \circ \varphi_i^{-1} : \varphi_i(U_i) \to \mathbb{C}$ is holomorphic for any chart (U_i, φ_i) of any holomorphic atlas in the equivalence class of atlases of X.

Note that the set of holomorphic functions on open sets U forms a sheaf.

Definition 2.9. Let X be a complex manifold. The *sheaf of holomorphic functions* is denoted by $\mathcal{O}_X$. Explicitly, for any open subset $U \subseteq X$:

$$\mathcal{O}_X(U) = \Gamma(U, \mathcal{O}_X) = \{f : U \to \mathbb{C} \mid f \text{ is holomorphic}\}$$

For a given point $x \in X$ the stalk $\mathcal{O}_{X,x}$ is a ring of germs of functions. It is constructed as a direct limit. By construction it is a local ring, that is, it has a unique maximal ideal $\mathfrak{m}_{\mathfrak{x}}$ corresponding to the germs of functions not vanishing at x.

Now, since the stalks of a sheaf are strictly local, by taking a small enough neighbourhood U of a point x so that it is included in a chart (U, φ) (with $\varphi(x) = 0$ by subtracting $\varphi(x)$ if necessary) we obtain the following ismorphism: $\mathcal{O}_{X,x} \simeq \mathcal{O}_{\mathbb{C}^n,0}$, where $n = \dim X$. Therefore all stalks are actually isomorphic.

Considering the stalk $\mathcal{O}_{X,x}$ as a ring of functions, we can consider its field of fractions which we denote $\mathcal{Q}(\mathcal{O}_{X,x})$.

One can immediately note the differences of a complex manifold and a smooth manifold. For instance, in a connected, compact complex manifold X the only global holomorphic functions are constants (put nicely, $\Gamma(X, \mathcal{O}_X) = \mathbb{C}$). This is due to the maximum's principle, which generalises to multivaraible holomorphic functions as shown by [Huy05, p. 5, Prop. 2.1.5].

Definition 2.10. Let X and Y be two complex manifolds. A continuous map $f : X \to Y$ is a *holomorphic* map, also called simply a *morphism*, if for any holomorphic charts (U, φ) of X and (V, ϕ) of Y the map $\phi \circ f \circ \varphi^{-1} : \varphi(f^{-1}(V) \cap U) \to \phi(V)$ is holomorphic. Two complex manifolds X and Y are called *isomorphic* (or *biholomorphic*) if there exists a holomorphic homeomorphism $f : X \to Y$.

Note that a bijective holomorphic function $f : U \to \mathbb{C}$ has always a holomorphic inverse, so this notion of biholomorphism agrees with the one we had before. To check this, one invokes the holomorphic version of the implicit function theorem and uses induction to prove that if the Jacobian is singular on a point, then it is identically zero at the point. By the inverse theorem function, one checks that if f is injective then $\det J(f)(z) \neq 0$ for all z in its domain (which implies that f is biholomorphic). This last step needs the fact that the function $\det J(f) : U \to \mathbb{C}$ has always a point where its Jacobian is surjective, and that comes from the *Weierstraß preparation theorem* (see [Sch12, Thm 3.2]) which gives a decomposition of a holomorpic function depending on their zeroes.

For a detailed and complete proof of the fact that a bijective holomorphic function is always a biholomorphism, check [Huy05, Prop. 1.1.13].

Example 2.11. As for examples, one could note that $\mathbb{C}^n$ is immediately a complex manifold with the identity being a global chart. Also any open subset of $\mathbb{C}^n$ is a complex manifold (in fact, any open set of a complex manifold is again a complex manifold, just as in the smooth case). Note that since the complex open ball of radius one is not biholomorphic to $\mathbb{C}^n$, one cannot describe complex manifolds as having a cover of open sets biholomorphic to $\mathbb{C}^n$.

Example 2.12. The most important example for us is the complex projective space $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{C}}$ defined as the set of lines in $\mathbb{C}^{n+1}$. Equivalently, $\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$ where $\mathbb{C}^*$ acts multiplicatively. As in the real case this allows us to express the points of $\mathbb{P}^n$ in coordinates $[z_0 : \ldots : z_n] = [\lambda z_0 : \ldots : \lambda z_n]$ for $\lambda \neq 0$, $z_0, \ldots, z_n, \lambda \in \mathbb{C}$. To check that $\mathbb{P}^n$ is a complex manifold one uses the covering $U_i = \{[z_0 : \ldots : z_n] \mid z_i \neq 0\} \subseteq \mathbb{P}^n$ and the charts $\varphi_i : U_i \to \mathbb{C}^n, [z_0 : \ldots : z_n] \to (\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i} \ldots, \frac{z_n}{z_i})$, for $i = 0, \ldots, n$.

Example 2.13. Another example would be that of complex tori. Let Λ be a lattice in $\mathbb{C}^n$ (that is, the image of $\mathbb{Z}^{2n}$ in $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ through an injective linear map). A complex tori of dimension n is then $\mathbb{C}^n/\Lambda$ where Λ acts on $\mathbb{C}^n$ by translation (we identify a point with all other points in its orbit as usual). In a small enough neighbourhood $U \subseteq \mathbb{C}^n$ of a point (for example a sphere with radius smaller than the complex norm of elements in the lattice) there are no points identified with one another and therefore the projection onto the torus defines a chart, making complex tori into complex manifolds. They are in fact quite interesting spaces: two different complex tori are not in general biholomorphic (despite the fact that complex tori are in particular real tori S^{2n} which are all difffeomorphic) and for $n > 1$ many of them are non-projective (we will shortly define what this means). For a deep study of complex tori see [BL99].

Now, one can define the notion of a complex submanifold using a concept similar to the k-slice condition (see [Lee97, Thm. 5.8]) that is used to define submanifolds on the smooth case:

Definition 2.14. Let X be a complex manifold of dimension n and $Y \subseteq X$ a differentiable manifold of dimension $2k$. We call Y a *complex submanifold* of X of dimension k if there exists a holomorphic atlas $\{(U_i, \varphi_i)\}$ of X such that $\varphi_i : U_i \cap Y \cong \varphi_i(U_i) \cap \mathbb{C}^k$ is a biholomorphism.
Here, as usual we identify $\mathbb{C}^k \subseteq \mathbb{C}^n$ with $(z_1, \dots, z_k, 0, \dots 0)$.

This atlas immediately provides Y with a complex structure and therefore, a submanifold $Y \subseteq X$ of a complex manifold X is also a complex manifold on its own, of complex dimension $dimY = k$. The *codimension* of Y in X is then $\operatorname{codim}(Y) = \dim(X) - \dim(Y) = n - k$.

Definition 2.15. Let X be a complex manifold. We say that X is *projective* if X is isomorphic to a closed complex submanifold of the projective space $\mathbb{P}^n$ for some n.

In particular since X is closed and the projective space is compact, any projective manifold is compact.

There is another concept of "subvariety" for complex manifolds, analytic subvarieties. In contrast to complex and smooth submanifolds, analytic subvarieties are allowed to have singularities.

Definition 2.16. Let X be a complex manifold. An *analytic subvariety* of X is a closed subset $Y \subseteq X$ such that for each point $x \in X$ there exist an open neighbourhood $U \in X$ of x such that $U \cap Y$ is of the form $U \cap Y = \{x \in X | f_1(x) = \dots = f_k(x) = 0\}$ for a finite amount of holomorphic functions $f_1, \dots, f_k \in \mathcal{O}(U)$.

Another indispensable tool for studying smooth manifolds are vector bundles and these also can be defined in the holomorphic case.

Definition 2.17. Let X be a complex manifold. A *holomorphic vector bundle* of rank r on X is a complex manifold E together with a holomorphic map

$$\pi : E \to X$$

Such that for $x \in X$ the fiber $E(x) := \pi^{-1}(x)$ over x is a r-dimensional complex vector space; and such that there is an open covering $X = \bigcup U_i$ and biholomorphic maps $\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^r$ that commute with the projections to U_i (that is, if we denote by π_1 the projection onto the fist coordinate of the cartesian product; $\pi_1 : U_i \times \mathbb{C}^r \to U_i$, we require $\pi = \pi_1 \circ \phi_i$) and such that for each $x \in U_i$ the map $\phi_i|_{E(x)} : E(x) \to \{x\} \times \mathbb{C}^r$ is $\mathbb{C}$-linear.

This is just the regular definition of bundle with the addition that we ask that the objects involved are complex and that the morphisms are holomorphic.

We define the transition functions as usual:

$$\phi_{ij}(x) := (\phi_i \circ \phi_j^{-1})(x, \cdot) : \mathbb{C}^r \to \mathbb{C}^r$$

Which are linear isomorphisms for each x, in other words, $\phi_{ij}(x) \in \mathrm{GL}(r, \mathbb{C})$.

The maps $\phi_{ij} : U_i \cap U_j \to \mathrm{GL}(r, \mathbb{C})$ are called cocycles. As we know, one can characterize a bundle by giving its cocycles or transition maps (see [Lee13, Thm. 10.6] for this characterization).

Note that any holomorphic vector bundle is in particular a smooth vector bundle over the underlying smooth manifold of our complex manifold.

Any construction that yields a vector space V' from a vector space V has its equivalent construction in vector bundles. In fact any linear algebra construction can be translated to vector bundles. For instance, given holomorphic vector bundles E, F over a complex manifold one can consider its direct sum $E \oplus F$, tensor product $E \otimes F$, exterior powers $\bigwedge^i E$, dual bundle E^*, and even projectivitzation. This last one is constructed in the following manner: Let $s : X \to E$ be the *zero section*, that maps every point of X to $0 \in E(x)$. Then in the complement $E \setminus s(X)$ the group $\mathbb{C}^*$ acts multiplicatively. We define the *projectivization* of E as $\mathbb{P}(E) := (E \setminus s(X))/\mathbb{C}^*$.

Also, the r-th exterior power of a holomorphic vector bundle of rank r is called the *determinant line bundle* and denoted $\det(E)$. It is indeed a line bundle, which means a bundle of rank 1.

As in the smooth case the tangent bundle plays an important role. Let $\{(U_i, \varphi_i)\}$ be a holomorphic atlas for a complex manifold X of dimension n. Let $\varphi_{ij} := \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \cong \varphi_i(U_i \cap U_j)$. The *Jacobian* of φ_{ij} at a point on $\varphi_j(U_i \cap U_j)$ is then the matrix

$$J(\varphi_{ij})(\varphi_j(z)) := \left(\frac{\partial \varphi_{ij}^k}{\partial z_l}(\varphi_j(z)) \right)_{1 \leq k,l \leq n}$$

Definition 2.18. Let X be a complex manifold of dimension n. The *holomorphic tangent bundle* of X is the holomorphic vector bundle $\mathcal{T}_X$ on X of rank n given by the transition functions $\phi_{ij} = J(\varphi_{ij})(\varphi_j(z))$. Note that since φ_{ij} are biholomorphisms, $J(\varphi_{ij})(\varphi_j(z)) \in \mathrm{GL}(n, \mathbb{C})$, and they are indeed transition functions. A section of $\mathcal{T}_X$ is called a holomorphic vector field.

The *holomorphic cotangent bundle* Ω_X is the dual of $\mathcal{T}_X$. The bundle of holomorphic p-forms is $\Omega_X^p := \bigwedge^p \Omega_X$ for $0 \leq p \leq n$ and the *canonical bundle* of X is $K_X := \det(\Omega_X) = \Omega_X^n$.

One ought to check that these definitions of $\mathcal{T}_X$ and Ω_X^p are independent of our choice of holomorphic atlas $\{(U_i, \varphi_i)\}$, and this turns out to be the case, as using other atlases yields canonically isomorphic vector bundles. Given a holomorphic vector bundle, one can easily associate a sheaf to it.

Definition 2.19. Let $\pi : E \to X$ be a holomorphic vector bundle. Its *sheaf of sections*, which we also call E, is defined by:

$$E(U) := \{s : U \to \pi^{-1}(U) \text{ holomorphic } \mid \pi \circ s = \mathrm{id}_U\}$$

So $E(U)$ are just the holomorphic sections of U, and the restriction maps are just restricting these sections. One can check that this is actually a sheaf and not just a pre-sheaf.

Note that as in the smooth case, sections form a module over the holomorphic functions (which, recall, we denote by $\mathcal{O}_X$). Therefore $E(U)$ is an $\mathcal{O}_X(U)$ module for every open subset U. We summarize this by saying that the sheaf of sections of a holomorphic vector bundle is in a natural way a sheaf of $\mathcal{O}_X$-modules. It turns out that such a sheaf of $\mathcal{O}_X$-modules also unequivocally determines the bundle.

Also note that if we consider the trivial line bundle $X \times \mathbb{C} \to X$, its sections are exactly the the holomorphic functions, so $\mathcal{O}_X$ can be seen as the sheaf of sections of $X \times \mathbb{C}$.

As in the smooth case we have an equivalent characterization of the tangent bundle, by defining it as the bundle given by the sheaf of derivations, that is, the sheaf that associates to an open subset $U \subseteq X$ the set of all $\mathbb{C}$-linear maps $D : \mathcal{O}_X(U) \to \mathcal{O}_X(U)$ satisfying the Leibniz rule $D(f \cdot g) = f \cdot D(g) + D(f) \cdot g$.

2.2 Almost complex structure

Let us now introduce almost complex structures, first on a real vector space and then on a smooth manifold. Let V be a finite-dimensional real vector space.

Definition 2.20. An endomorphism $I : V \to V$ with $I^2 = -\operatorname{id}$ is called and *almost complex structure* on V.

By taking determinants on the expression $I^2 = -\operatorname{id}$ one checks that $I \in \mathrm{GL}(V)$.

One can easily check that if $V_{\mathbb{C}}$ is a complex vector space, its underlying real vector space V has an almost complex structure (defined by complex multiplication; $I(v) = i \cdot v$) and that an almost complex structure can make V into a complex vector space (by defining $i \cdot v = I(v)$). Note this implies that if V has an almost complex structure then its real dimension must be even (since it can be considered as the underlying real vector space of a complex one).

In conclusion, in the case of vector spaces; almost complex structures and complex structures are essentially equivalent. The interesting behaviour appears once we consider the complexification of the vector space V together with an almost complex structure I and we look at how these interact.

The complexification of a real vector space V is $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$. If V is endowed with an almost complex structure I, then we can consider the $\mathbb{C}$-linear extension of I to $V_{\mathbb{C}}$, which we also call I; that is, $I : V_{\mathbb{C}} \to V_{\mathbb{C}}$. Now since this new I is an endomorphism of a complex vector space with $I^2 + \operatorname{id} = 0$ we know it is diagonalizable (its minimal polynomial can only be $\lambda - i \cdot \operatorname{id}$, $\lambda + i \cdot \operatorname{id}$ or $(\lambda - i \cdot \operatorname{id})(\lambda + i \cdot \operatorname{id})$, in all cases having distinct roots).

From $I^2 + \operatorname{id} = 0$ one deduces that the eigenvalues of I are exactly i and $-i$. This gives us a rather natural decomposition for $V_{\mathbb{C}}$.

Definition 2.21. The spaces $V^{1,0}$ and $V^{0,1}$ are defined $V^{1,0} = \{v \in V_{\mathbb{C}} \mid I(v) = i \cdot v\}$; $V^{0,1} = \{v \in V_{\mathbb{C}} \mid I(v) = -i \cdot v\}$; so they are the $\pm i$ eigenspaces. We consider these complex vector subspaces of $V_{\mathbb{C}}$.

Due to the fact that I is diagonalizable we have a decomposition $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$. One can check that complex conjugation sends $V^{1,0}$ to $V^{0,1}$.

Now if $\dim(V) = d$ then its exterior algebra is $\bigwedge^* V = \oplus_{k=0}^{d} \bigwedge^k V$ which we can complexify $\bigwedge^* V_{\mathbb{C}} = \oplus_{k=0}^{d} \bigwedge^k V_{\mathbb{C}} = \bigwedge^* V \otimes_{\mathbb{R}} \mathbb{C}$.

Definition 2.22. We define:

$$\bigwedge^{p,q} V := \bigwedge^p V^{1,0} \otimes_{\mathbb{C}} \bigwedge^q V^{0,1}$$

where the exterior algebras of $V^{0,1}$, $V^{1,0}$ are taken as exterior algebras of complex vector spaces. For $\alpha \in \bigwedge^{p,q} V$ we say that α is of bidegree (p, q).

The interest we have in this construction is that it provides a decomposition of the complexified exterior algebra:

Proposition 2.23. *Let V be a finite-dimensional real vector space endowed with and almost complex structure I. Then* $\bigwedge^k V_{\mathbb{C}} = \oplus_{p+q=k} \bigwedge^{p,q} V$.

Proof. This is basically due to combinatorics. If $\dim(V) = d = 2n$ as a real vector space then $\dim_{\mathbb{C}}(V^{1,0}) = \dim_{\mathbb{C}}(V^{0,1}) = n$. So if $v_1, \ldots, v_n \in \bigwedge^{1,0} V = V^{1,0}$ and $w_1, \ldots, w_n \in \bigwedge^{0,1} V = V^{0,1}$ are $\mathbb{C}$-basis, then $v_{J_1} \otimes w_{J_2} \in \bigwedge^{p,q} V$ with $J_1 = \{i_1 < \cdots < i_p\}$ and $J_1 = \{j_1 < \cdots < j_q\}$ form a basis of $\bigwedge^{p,q} V$. □

An almost complex structure I on V induces an almost complex structure on its dual $V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ given by $I(f)(v) = f(I(v))$. This of course induces a similar decomposition as the previous one in $(V^*)_{\mathbb{C}} = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) = (V_{\mathbb{C}})^*$.

Definition 2.24. We define the natural projections:

$$\Pi^k : \bigwedge\nolimits^* V_{\mathbb{C}} \to \bigwedge\nolimits^k V_{\mathbb{C}} \text{ and } \Pi^{p,q} : \bigwedge\nolimits^k V_{\mathbb{C}} \to \bigwedge\nolimits^{p,q} V_{\mathbb{C}}$$

where we recall $\bigwedge^* V_{\mathbb{C}} = \oplus_{k=0}^d \bigwedge^k V_{\mathbb{C}}$ and $\bigwedge^k V_{\mathbb{C}} = \oplus_{p+q=k} \bigwedge^{p,q} V$.

As we mentioned before, every linear algebra construction has its counterpart on the tangent space of manifolds. So let us now explore these almost complex structures on smooth manifolds.

Definition 2.25. Let X be a differentiable manifold and let TX be the real tangent bundle of its underlying topological manifold. A vector bundle endomorphism I

$$I : TX \to TX, \text{ with } I^2 = -\,\mathrm{id}$$

Is called an *almost complex structure on X*.

A differentaible manifold X together with an almost complex structure I is called an *almost complex manifold.*

Every complex manifold admits such an almost complex structure in a natural way. Indeed, if we cover X by charts we only need to check that an open set $U \subseteq \mathbb{C}^n$ admits an almost complex structure. Writing the coordinates as $z_i = x_i + iy_i$ we consider $U \subseteq \mathbb{R}^{2n}$. Then define I as follows

$$I : TU_x \longrightarrow TU_x, \quad \frac{\partial}{\partial x_i} \longmapsto \frac{\partial}{\partial y_i}, \quad \frac{\partial}{\partial y_i} \longmapsto -\frac{\partial}{\partial x_i}$$

One then checks that this does not depend on the chart and that it is compatible with the trivializations (which is immediate since the partial derivatives are the trivializations). See [Huy05, Prop. 2.6.2] for the details.

Our next endeavor is to find when an almost complex manifold can be endowed with a complex atlas in order to turn it into a complex manifold. Let us begin with several definitions.

Let X be an almost complex manifold. The complexification of its real tangent space TX is denoted by $T_{\mathbb{C}}X = TX \otimes \mathbb{C}$.

Just as an almost complex structure on a vector space V provided a decomposition of its complexification $V_{\mathbb{C}}$, the same happens with tangent spaces.

Proposition 2.26. *Let (X, I) be an almost complex manifold. Then there exists a direct sum decomposition $T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X$ of complex vector bundles on X such that the $\mathbb{C}$-linear extension of I acts as multiplication by i on $T^{1,0}X$ an as multiplication by $-i$ on $T^{0,1}X$.*

Proof. Simply define $T^{1,0}X$ and $T^{0,1}X$ as the kernel of the vector bundle homomorphisms $I - i \cdot \mathrm{id}$ respectively $I + i \cdot \mathrm{id}$. The decomposition holds on every fiber so it holds for the whole bundle. □

Definition 2.27. Let X be an almost complex manifold. The vector bundles $T^{1,0}X$ and $T^{0,1}X$ are called the *holomorphic* and respectively *antiholomorphic tangent bundle* of X.

The name holomorphic tangent bundle is a reasonable one. As it suggests, it turns out that $T^{1,0}$ is indeed a holomorphic vector bundle over X when X is a complex manifold and the almost complex structure I is the natural one: it coincides with the holomorphic tangent bundle defined in Definition 2.18. In order to check this, we only need to proof that the transition functions of the holomorphic tangent bundle are the same as the ones given in our other definition of the holomorphic tangent bundle, which is not hard. See [Sch12, Example 8.1] for the precise calculation.

Definition 2.28. Let X be an almost complex manifold. We define the following vector bundles:

$$\bigwedge\nolimits_{\mathbb{C}}^k X := \bigwedge\nolimits^k (T_{\mathbb{C}}X)^* \text{ and } \bigwedge\nolimits^{p,q} X := \bigwedge\nolimits^p (T^{1,0}X)^* \otimes_{\mathbb{C}} \bigwedge\nolimits^q (T^{0,1}X)^*.$$

We denote the sheaf of sections of $\bigwedge_{\mathbb{C}}^k X$ by $\mathcal{A}_{\mathbb{C}}^k(X)$. Note that these are just the forms on the underlying real manifold that now admit multiplication by complex scalars. Let us denote the sheaf of real k-forms of the underlying real manifold by $\mathcal{A}^k(X)$.

We denote the sheaf of sections of $\bigwedge^{p,q} X$ by $\mathcal{A}^{p,q}(X)$. Global sections of $\mathcal{A}^{p,q}(X)$ are called forms of *type* or *bidegree* (p, q), or simply (p, q)-forms.

Thanks to the decomposition in Proposition 2.23 we have the following bidegree decompositions:

$$\bigwedge\nolimits_{\mathbb{C}}^k X = \bigoplus_{p+q=k} \bigwedge\nolimits^{p,q} X \text{ and } \mathcal{A}_{\mathbb{C}}^k(X) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(X)$$

As for vector spaces we define the natural projections $\Pi^k : \mathcal{A}_{\mathbb{C}}^*(X) \to \mathcal{A}_{\mathbb{C}}^k(X)$ and $\Pi^{p,q} : \mathcal{A}_{\mathbb{C}}^k(X) \to \mathcal{A}^{p,q}(X)$.

Thanks to this decomposition we can define a new type of differential operator. This will prove to be a very useful operator, as it will allow us to define a new type of cohomology for complex manifolds and characterize holomorphic forms, among many other uses.

Definition 2.29. Let X be an almost complex manifold. Let $d : \mathcal{A}_{\mathbb{C}}^k(X) \to \mathcal{A}_{\mathbb{C}}^{k+1}(X)$ be the $\mathbb{C}$-linear extension of the exterior differential. One defines

$$\partial := \Pi^{p+1,q} \circ d : \mathcal{A}^{p,q}(X) \longrightarrow \mathcal{A}^{p+1,q}(X) \text{ and } \bar{\partial} := \Pi^{p,q+1} \circ d : \mathcal{A}^{p,q}(X) \longrightarrow \mathcal{A}^{p,q+1}(X)$$

One can check without much trouble that the Leibniz rule for the exterior differential d (which of course still holds for its $\mathbb{C}$-linear extension) implies the following Leibniz rule for ∂ and $\bar{\partial}$:

$$\partial(\alpha \wedge \beta) = \partial(\alpha) \wedge \beta + (-1)^{p+q} \alpha \wedge \partial(\beta)$$

$$\bar{\partial}(\alpha \wedge \beta) = \bar{\partial}(\alpha) \wedge \beta + (-1)^{p+q} \alpha \wedge \bar{\partial}(\beta)$$

As we said one of the many uses of these ∂ and $\bar{\partial}$ is for instance a quick characterization of the holomorphic forms of a complex manifold X. Indeed, the holomorphic p-forms $\Omega_X^p(X) = H^0(X, \Omega_X^p)$ is the subspace $\{\alpha \in \mathcal{A}^{p,0}(X) \mid \bar{\partial}\alpha = 0\}$. For a proof see [Huy05, Prop. 2.6.11], but this amounts to finding local coordinate expressions for ∂ and $\bar{\partial}$.

Definition 2.30. Let (X, I) be an almost complex manifold. We say that I is integrable if $d\alpha = \partial(\alpha) + \bar{\partial}(\alpha)$ for all $\alpha \in \mathcal{A}^*(X)$.

There is an equivalent characterization:

Proposition 2.31. *Let (X, I) be an almost integrable manifold. Then I is integrable if and only if the Lie bracket of vector fields preserves $T_X^{0,1}$, that is, $[T_X^{0,1}, T_X^{0,1}] \subseteq T_X^{0,1}$.*

A proof for this can be found in [Huy05, Prop. 2.6.17].

Corollary 2.32. *If I is an integrable almost complex structure then $\partial^2 = 0$, $\bar{\partial}^2 = 0$ and $\partial\bar{\partial} = -\bar{\partial}\partial$.*

Proof. This follows directly from $d^2 = 0$ and $d = \partial + \bar{\partial}$. □

It turns out that the integrability of almost complex structures is the condition needed in order to ensure our almost complex manifold is indeed a complex one. In other words:

Theorem 2.33 (Newlander-Nierenberg)**.** *Any integrable almost complex structure is induced by a complex structure.*

Now proving this is far beyond the scope of this text. In [Voi02, Thm. 2.26] one can find a proof of a specific case, where (X, I) is assumed to be analytic.

One might note that the relation $\bar{\partial}^2 = 0$ invites us to construct a cohomology theory for this operator.

Definition 2.34. Let X be endowed with an integrable almost complex structure. The (p, q)-*Dolbeault cohomology* is the vector space

$$H^{p,q}(X) := H^q(\mathcal{A}^{p,\bullet}(X), \bar{\partial}) = \frac{\ker(\bar{\partial} : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q+1}(X))}{\operatorname{im}(\bar{\partial} : \mathcal{A}^{p,q-1}(X) \to \mathcal{A}^{p,q}(X))}$$

Also, it turns out that just as in the smooth case, all $\bar{\partial}$-closed forms are locally exact. This is known as the $\bar{\partial}$-Poincaré lemma (see [Voi02, Prop. 2.31] for more details). Using this, one proves the following:

Proposition 2.35. *The Dolbeault cohomology computes the cohomology of the sheaf of p-holomorphic forms Ω_X^p, that is, $H^{p,q}(X) \cong H^q(X, \Omega_X^p)$.*

A proof for this can be found in [Voi02, Cor. 4.38].

2.3 Hermitian structure

Here we shall introduce Hermitian manifolds. As in the previous section, we shall begin with a bit of linear algebra that we will then apply to the tangents spaces of our manifolds.

Having defined complex structures on smooth manifolds, we shall now approach Riemannian manifolds, and try to give them a complex structure that is compatible with its metric to obtain an object we will call Hermitian manifold. A metric of a Riemannian manifold is just an scalar product on each of its tangent spaces, so now our finite-dimensional real vector space V shall be endowed with an scalar product $\langle\,,\rangle$ (that is, $\langle\,,\rangle$ is a bilinear, symmetric and positive definite form).

Definition 2.36. We say an almost complex structure I on V is *compatible* with the scalar product $\langle\,,\rangle$ if $\langle v, w\rangle = \langle I(v), I(w)\rangle$ for all $v, w \in V$, that is, I is a orthogonal transformation with respect to $\langle\,,\rangle$; $I \in \mathrm{O}(V, \langle\,,\rangle)$.

Definition 2.37. Let $(V, \langle\,,\rangle)$ be a real finite dimensional euclidean vector space endowed with a compatible almost complex structure I. The *fundamental form* associated to $(V, \langle\,,\rangle, I)$ is the form

$$\omega(\_,\_) := \langle I(\_),\_\rangle = \langle I(I(\_)), I(\_)\rangle = -\langle \_, I(\_)\rangle$$

This form will play a very important role. For instance, it will be used to give the definition of a Kähler manifold. Let us take a closer look at it:

Proposition 2.38. *Let $(V, \langle\,,\rangle)$ be a real finite dimensional euclidean vector space endowed with a compatible almost complex structure I. Then its fundamental form ω is real and of type $(1,1)$, that is, $\omega \in \bigwedge^2 V^* \cap \bigwedge^{1,1} V_{\mathbb{C}}^*$.*

Proof. The form ω is real, since $\langle\,,\rangle$ is real and ω has no imaginary part in its definition.

It is also alternating, since $\omega(v, w) = \langle I(v), (w)\rangle = -\langle (v), I(w)\rangle = -\langle I(w), (v)\rangle = -\omega(w, v)$ for $v, w \in V$ so $\omega \in \bigwedge^2 V^*$.

Finally, if we define $\mathbf{I}$ as the $\mathbb{C}$-linear extension of I on $\bigwedge^* V_{\mathbb{C}}^*$ then $\mathbf{I}(\omega)(v, w) = \omega(\mathbf{I}(v), \mathbf{I}(w)) = \langle I(I(v)), I(w)\rangle = \omega(v, w)$. So therefore $\mathbf{I}(\omega) = \omega$.

One can easily check (by writing things out in coordinates) that for $\omega \in \bigwedge^{2,0} V$, one has $\mathbf{I}(\omega) = (i)^2\omega = -\omega$. Similarly, for $\omega \in \bigwedge^{0,2} V$, one has $\mathbf{I}(\omega) = (-i)^2\omega = -\omega$ and finally $\omega \in \bigwedge^{1,1} V$, one has $\mathbf{I}(\omega) = (-i)(i)\omega = \omega$. Since our ω has a unique decomposition as a sum of elements of these three alternating spaces, then $\omega \in \bigwedge^{1,1} V$. □

Now the reason they are called Hermitian manifolds is that we can obtain an Hermitian form from these structures:

Proposition 2.39. *Let $(V, \langle\,,\rangle)$ be a real finite dimensional euclidean vector space endowed with a compatible almost complex structure I. Let ω be its fundamental form. Then the form*

$$(\_,\_) := \langle \_,\_\rangle - i \cdot \omega(\_,\_)$$

is a positive Hermitian form on V; that is, $(v, v) > 0$ for $0 \neq v \in V$ and $(v, w) = \overline{(w, v)}$.

Proof. This is a simple calculation and it won't be done here. Simply use that ω is alternating and that $\langle\,,\rangle$ is symmetric. □

Note that one can recover the metric and the fundamental form from the Hermitian form through $\omega(\_,\_) = i\,\mathrm{Im}(\_,\_)$ and $\langle \_,\_\rangle = \mathrm{Re}(\_,\_)$.

If $(V, \langle\,,\rangle)$ is a Euclidian finite dimensional real vector space we can extend the scalar product $\langle\,,\rangle$ to $V_{\mathbb{C}} = V \otimes \mathbb{C}$ as follows: $\langle v \otimes \lambda, w \otimes \mu\rangle_{\mathbb{C}} := (\lambda\bar{\mu}) \cdot \langle v, w\rangle$ for $v, w \in V$ and $\lambda, \mu \in \mathbb{C}$. The extended form is no longer a scalar product, but a positive definite Hermitian form.

Proposition 2.40. *Let $(V, \langle\,,\rangle)$ be an euclidean vector space endowed a compatible almost complex structure I. Then $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ is an orthogonal decomposition with respect to the Hermitian product $\langle\,,\rangle_{\mathbb{C}}$.*

Proof. As we know form the definition of $V^{1,0}$ and $V^{0,1}$ and from linear algebra, the elements of $V^{1,0}$ are of the form $v - iI(v)$ and the elements of $V^{0,1}$ are of the form $w + iI(w)$ with $v, w \in V$. If we calculate their product we obtain

$$\begin{aligned}\langle v - iI(v), w + iI(w)\rangle_{\mathbb{C}} &= \langle v, w\rangle_{\mathbb{C}} + \langle -iI(v), w\rangle_{\mathbb{C}} + \langle v, iI(w)\rangle_{\mathbb{C}} + \langle -iI(v), iI(w)\rangle_{\mathbb{C}} \\ &= \langle v, w\rangle_{\mathbb{C}} - i\langle I(v), w\rangle_{\mathbb{C}} - i\langle v, I(w)\rangle_{\mathbb{C}} + (-i)(-i)\langle I(v), I(w)\rangle_{\mathbb{C}} \\ &= -i\langle I(v), w\rangle_{\mathbb{C}} + i\langle I(v), w\rangle_{\mathbb{C}} = 0\end{aligned}$$

So they are indeed orthogonal. □

Let us now introduce the definition of the Hodge $*$-operator, which can be defined on smooth manifolds and therefore without the need of an almost complex structure, but we need our space to be oriented. So let $(V, \langle\,,\rangle)$ be an oriented euclidean space of dimension d.

The inner inner product $\langle\,\rangle$ can be extended to elements in $\bigwedge^k V$ for $k = 1, \ldots, d$, by setting

$$\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k\rangle = \det(\langle v_i, w_j\rangle)_{1\le i,j\le k}$$

for $v_i, w_i \in V$ and extending it by bilinearity.

With this product on the exterior algebra and thanks to the fact that the decomposition $V = V^{1,0} \oplus V^{0,1}$ is orthogonal, it is immediate to check that the bidegree decomposition $\bigwedge^k V_{\mathbb{C}} = \bigoplus_{p+q=k} \bigwedge^{p,q} V$ is also orthogonal.

Given an oriented orthonormal basis $(e_1, \ldots, e_d)$ of V the orientation form is $vol = e_1 \wedge \cdots \wedge e_d \in \bigwedge^d V$, and it is one of the only two d-forms of length 1, that is, $\langle vol, vol\rangle = ||vol||^2 = 1$ (the other one being $-vol$).

Definition 2.41. The *Hodge $*$-operator* $* : \bigwedge^k V \to \bigwedge^{d-k} V$ is defined by the relation

$$\alpha \wedge *\beta = \langle \alpha, \beta\rangle \cdot vol$$

for $\alpha, \beta \in \bigwedge^k V$.

One then checks that this relation describes a unique operator $*$.

The product $\langle\,,\rangle$ also defines a scalar product on V^* and its exterior algebras. This is thanks to the isomorphism

$$\begin{aligned}\varphi : V &\simeq V^* \\ v &\to \langle v, \_\rangle.\end{aligned}$$

So we can define $\langle \omega_1, \omega_2\rangle = \langle \varphi^{-1}(\omega_1), \varphi^{-1}(\omega_2)\rangle$ for $\omega_1, \omega_2 \in V^*$ and then extent it to elements in $\bigwedge^k V^*$ in the same manner as before.

Thanks to this isomorphism φ we can obtain an orientation for V^* (an oriented basis of V^* will be the image through φ of an oriented basis of V) and therefore the Hodge $*$-operator can also be defined in $\bigwedge^k V^*$.

Let us now construct these objects on manifolds. As we stated, we'll need Riemannian manifolds and their metrics in order to have inner products on the tangent spaces of our manifolds.

Definition 2.42. Let X be a complex manifold with induced almost complex structure I and let g be a Riemannian metric on M, the underlying real manifold of X. We say that g is an *Hermitian structure* on X if for any point $x \in X$ the scalar product $g_x(\, , \,)$ on T_xM is compatible with the almost complex structure I_x. The induced real $(1,1)$-form $\omega := g(I(\_), \_)$ is called the *fundamental form*. The complex manifold X endowed with an Hermitian structure g is called an *Hermitian manifold*.

Of course, one does not need the manifold to be complex in the previous definition; it suffices that X is a Riemannian manifold with an almost complex structure I. But since we are expressedly interested in the property $d = \partial + \bar{\partial}$, exclusive to integrable almost complex structures (which, recall that automatically made X into a complex manifold) we will always assume that our Hermitian manifolds are also complex.

We would like to use the Hodge star operator, but we need our manifolds to be oriented in order to be able to introduce it. Luckily:

Lemma 2.43. *Every complex manifold X has a natural orientation.*

Proof. Let M be the real manifold underlying X. We will check that all real transition maps on M have positive determinant. These maps are just the transition maps for X but using the identification $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. Thanks to the holomorphic structure, one checks that if f is one of the transition maps, one has $\det_{\mathbb{R}} J(f) = |\det J(f)|^2$, which is non-negative. See [Huy05, Def. 1.1.9] for the details (and to see where this $J(f)$ comes from). □

Then, the Hodge star operator is defined naturally as follows.

Definition 2.44. Let (X, g) be an Hermitian manifold of complex dimension n (therefore real dimension $2n$) and let vol_g be the Riemann orientation form. As we've seen, X is automatically oriented. Define the *Hodge $*$-operator* $* : \bigwedge^k X \to \bigwedge^{2n-k} X$ through the relation

$$\alpha \wedge *\beta = g(\alpha, \beta) \cdot vol_g$$

for $\alpha, \beta \in \bigwedge^k X$, where we have extended the inner product $g(\_, \_)$ to the exterior algebras of the cotangent space in the manner we described before, for real vector spaces.

We then define $*$ on $\mathcal{A}^k(X)$ and we extend it $\mathbb{C}$-linearly to $\bigwedge^k_{\mathbb{C}} X$ and $\mathcal{A}^k_{\mathbb{C}}(X)$.

One can check (see [Sch12, Lem. 19.4]) that the Hodge $*$-operator maps $\mathcal{A}^{p,q}(X)$ to $\mathcal{A}^{n-q,n-p}(X)$, where n is still the complex dimension of X.

Let us now use this Hodge $*$-operator to construct several differential operators, some of them known from differential geometry, some of them belonging exclusively to complex geometry:

Definition 2.45. Let (M, g) be a Riemannian m-manifold. The *adjoint operator* is $d^* := (-1)^{m(k+1)+1} * \circ \, d \circ * : \mathcal{A}^k(M) \to \mathcal{A}^{k-1}(M)$.

One might ask why we have given such definition of d^*, and why we called it an adjoint. Recall from differential geometry that we were able to define integration of smooth real-valued functions on compact[2] oriented Riemannian manifolds (M, g). Now, thanks to the inner product $g_x(,)$ on each tangent space TM_x we can define an inner product $g_x(,)$ on $\bigwedge^k TM_x^*$ as we've seen before. This allows us to define a product of k-forms as follows:

$$\langle \alpha, \beta \rangle = \int_M g(\alpha, \beta) vol_g = \int_M \alpha \wedge *\beta$$

where $\alpha, \beta \in \mathcal{A}^k(M)$, $g(\alpha, \beta)$ is the smooth real-valued function $g(\alpha, \beta)(x) = g_x(\alpha_x, \beta_x)$ for $x \in M$, and vol_g is the Riemannian volume form (and we used the property that defined $*$). The fact that this is indeed an scalar product can be easily deduced from the fact that $g(,)$ is an scalar product (and using the linearity of the integral and the fact that integrating a positive function yields a positive value).

It turns out that d^* is the adjoint operator of d with respect to this inner product of forms (that is, $\langle d\alpha, \beta \rangle = \langle \alpha, d^*\beta \rangle$ for $\alpha \in \mathcal{A}^k(M), \beta \in \mathcal{A}^{k+1}(M)$). For a proof check [Voi02, Lem. 5.7].

Of course, if M is the real manifold underlying a Hermitian manifold (X, g), then its dimension is even and therefore $d^* = - * \circ d \circ *$. We will usually extend d^* to $\mathcal{A}^k_{\mathbb{C}}(X)$.

Definition 2.46. Let (X, g) be a Hermitian manifold. The *Laplace operator* or *Laplacian* is given by $\Delta = d^*d + dd^*$.

Definition 2.47. Let (X, g) be a Hermitian manifold of complex dimension n. We define the operators ∂^* and $\bar{\partial}^*$ as follows:

$$\partial^* := - * \circ \bar{\partial} \circ *$$

$$\bar{\partial}^* := - * \circ \partial \circ *$$

Since the Hodge $*$-operator maps $\mathcal{A}^{p,q}(X)$ to $\mathcal{A}^{n-q,n-p}(X)$, we have that $\partial^*(\mathcal{A}^{p,q}(X)) \subseteq \mathcal{A}^{p-1,q}(X)$ and similarly $\bar{\partial}^*(\mathcal{A}^{p,q}(X)) \subseteq \mathcal{A}^{p,q-1}(X)$.

Now these are also adjoint operators. We first need to expand our definition of scalar product of forms.

For a compact Hermitian manifold (X, g) we extend the product of forms $\langle , \rangle$ onto $\mathcal{A}^k_{\mathbb{C}}(X)$, the complexification of $\mathcal{A}^k(X)$ in the same way as we did for the complexification of Euclidean real vector spaces, that is, for $\lambda, \mu \in \mathbb{C}$ and $\alpha, \beta \in \mathcal{A}^k_{\mathbb{C}}(X)$ define $\langle \lambda\alpha, \mu\beta \rangle = (\lambda\overline{\mu}) \cdot \langle \alpha, \beta \rangle$. As before, we denote this extended Hermitian product in the same manner, $\langle , \rangle$. It is not an scalar product, but a positive definite Hermitian form, also called an Hermitian product.

We can also define the product on the whole $\mathcal{A}^*_{\mathbb{C}}$ by defining the product of two forms of different degree to be 0. Doing this makes $\mathcal{A}^*_{\mathbb{C},X} = \bigoplus_k \mathcal{A}^k_{\mathbb{C}}(X)$ a orthogonal decomposition. The norm of a form $\alpha \in \mathcal{A}^k_{\mathbb{C}}(X)$ is defined as usual, $||\alpha||^2 = \langle \alpha, \alpha \rangle$ and it is indeed a norm since $\langle , \rangle$ is a positive definite Hermitian form.

With respect to this extended product, ∂^* is the adjoint of ∂ and $\bar{\partial}^*$ is the adjoint of $\bar{\partial}$. This is proven in the exact same way as proving that d^* is the adjoint of d. Check [Sch12, Prop. 19.5] for an explicit computation.

[2]If M is not compact, one should restrict to compactly supported real-valued functions, but we will be interested in compact manifolds anyway.

In each tangent space we have that the bidegree decomposition is orthogonal (as we noted before) and so now with this extended product of forms (and thanks to the fact that the integral of the zero function is still zero) we have that the decomposition $\mathcal{A}^k_{\mathbb{C}}(X) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(X)$ is also an orthogonal decomposition.

Proposition 2.48. *Let (X, g) be an Hermitian manifold. One has $d^* = \partial^* + \bar{\partial}^*$.*

Proof. We had that for any complex manifold $d = \partial + \bar{\partial}$. By composing with $*$ on the left and on the right, using that $*$ is linear and multiplying by -1 we obtain $-*\circ d \circ * = -* \circ \partial \circ * - * \circ \bar{\partial} \circ *$ which is the desired result. □

Definition 2.49. Let (X, g) be an Hermitian manifold. We define the *Laplacians associated to ∂ and $\bar{\partial}$*, respectively, as

$$\Delta_{\partial} := \partial^* \partial + \partial \partial^*$$
$$\Delta_{\bar{\partial}} := \bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^*$$

They both respect the bidegree; $\Delta_{\partial}, \Delta_{\bar{\partial}} : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q}(X)$.

2.4 Kähler form

Now we have several differential operators, but they don't relate well to one another. For instance, a form that is $\bar{\partial}$-closed might not be d-closed. Luckily, with one extra condition, all these differential operators behave especially well. This is what us known as the Kähler condition.

Definition 2.50. Let (X, g) be an Hermitian manifold with induced fundamental form ω. We say that g is a *Kähler structure* (or *Kähler metric*) if the fundamental form ω is closed, $d\omega = 0$. In this case, the fundamental form ω is called the *Kähler form*. The complex manifold X endowed with the Kähler structure g is called a *Kähler manifold*.

Let us give a quick but important example:

Example 2.51. Recall our definition of the complex projective space $\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$ with the complex atlas (U_i, φ_i) where $U_i = \{[z_0 : \dots : z_n] \mid z_i \neq 0\} \subseteq \mathbb{P}^n$ and $\varphi_i : U_i \to \mathbb{C}^n, [z_0 : \dots : z_n] \to (\frac{z_0}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i} \dots, \frac{z_n}{z_i})$, for $i = 0, \dots, n$.

Using these coordinates we define the forms

$$\omega_i := \frac{i}{2\pi} \partial \bar{\partial} \log \left(\sum_{l=0}^{n} \left| \frac{z_l}{z_i} \right|^2 \right); i = 0, \dots, n.$$

These forms are in $\mathcal{A}^{1,1}(U_i)$ since we've taken $\partial\bar{\partial}$ of a smooth function, which is an element of $\mathcal{A}^{0,0}(U_i)$. Now one checks that $\omega_i|_{U_i \cap U_j} = \omega_j|_{U_i \cap U_j}$ and since $\mathbb{P}^n = \bigcup_{i=0}^n U_i$ we have that these forms glue together in order to define a global form $\omega_{FS} \in \mathcal{A}^{1,1}(\mathbb{P}^n)$.

One then checks that ω_{FS} is real, closed (which is easy thanks to the relations $d = \partial + \bar{\partial}$, $\bar{\partial}\partial = -\partial\bar{\partial}$ and $\partial^2 = \bar{\partial}^2 = 0$), and that it is positive definite (that is, that ω_{FS} is indeed the Kähler form associated to a metric). Check [Huy05, Example 3.1.9] for the explicit computations.

The metric associated to ω_{FS} is called the *Fubini-Study metric*, and it is a Kähler metric on $\mathbb{P}^n$.

We have the following result:

Proposition 2.52. *Let (X, g) be a Kähler manifold and let $Y \subseteq X$ be a complex sumbanifold. Then the restriction $g|_Y$ is again a Kähler metric on Y.*

Proof. We just note that all our definitions behave well with respect to restrictions. As we know $g|_Y$ is a metric on Y and our definition of the induced almost complex structure on a complex manifold implies that for $x \in Y$ the induced almost complex structure I_Y on $T_xY \subseteq T_xY$ is the restriction of the almost complex structure I_X on T_xX to T_xY and it leaves T_xY invariant, so the almost complex structure I_Y on Y is still compatible with the metric and thus the metric is Hermitian.

Now by definition the fundamental form on Y is $\omega_Y = g|_Y(I_Y(\_), (\_)) = g(I(\_), (\_))|_Y = \omega|_Y$ so the fundamental form on Y is just the restriction of the fundamental form on X, and therefore since it is closed on X it is also closed on Y, which means that $g|_Y$ is Kähler. □

Therefore, we conclude that any projective complex manifold X is Kähler since it can be considered as a submanifold of $\mathbb{P}^N$ for some N and we can endow X with the restriction of the Fubini-Study metric. Note that the Kähler structure on X will depend on the embedding.

Example 2.53. Complex tori $T = \mathbb{C}^n/\Lambda$ also turn out to be Kähler. To see this, one first endows $\mathbb{C}^n$ with a scalar product compatible with the natural almost complex structure of $\mathbb{C}^n$. This defines a constant metric g on $\mathbb{C}^n$ (which is, in particular, invariant by the action of Λ) that descends to T through the projection. As g is constant, the fundamental form will be closed.

We have promised some compatibility between d and $\bar{\partial}$ in Kähler manifolds. We say that two operators A, B acting on the same space commute if $A \circ B - B \circ A = 0$. We have the following compatibility condition:

Proposition 2.54. *Let (X, g) be a Kähler manifold. Then, $\frac{1}{2}\Delta = \Delta_\partial = \Delta_{\bar{\partial}}$ and Δ commutes with $*, \partial, \bar{\partial}, \partial^*$ and $\bar{\partial}^*$.*

The proof is long and full of computations and it is therefore omitted but one can find it in [Voi02, Thm. 6.7].

Many other interesting results stem form the Kähler condition, and we will see several of them in the next section.

2.5 Hodge decomposition

One of the central results regarding Kähler manifolds is the Hodge decomposition theorem, which we shall now introduce. Unfortunately, the proof of this theorem requires deep results in harmonic analysis, and so we won't be able to give the proof of the key result in this section, although that won't stop us from trying to understand it.

Definition 2.55. Let (X, g) be an Hermitian manifold and let $\alpha \in \mathcal{A}^k_{\mathbb{C}}(X)$. We say that α is *harmonic* if $\Delta(\alpha) = 0$. Similarly we will now say that α is $\bar{\partial}$*-harmonic* if $\Delta_{\bar{\partial}}(\alpha) = 0$ and that it is ∂*-harmonic* if $\Delta_\partial(\alpha) = 0$. We define

$$\mathcal{H}^k(X, g) := \{\alpha \in \mathcal{A}^k_{\mathbb{C}}(X) | \Delta(\alpha) = 0\} \quad \text{and} \quad \mathcal{H}^{p,q}(X) := \{\alpha \in \mathcal{A}^{p,q}(X) | \Delta(\alpha) = 0\}$$
$$\mathcal{H}^k_{\bar{\partial}}(X, g) := \{\alpha \in \mathcal{A}^k_{\mathbb{C}}(X) | \Delta_{\bar{\partial}}(\alpha) = 0\} \quad \text{and} \quad \mathcal{H}^{p,q}_{\bar{\partial}}(X) := \{\alpha \in \mathcal{A}^{p,q}(X) | \Delta_{\bar{\partial}}(\alpha) = 0\}$$
$$\mathcal{H}^k_{\partial}(X, g) := \{\alpha \in \mathcal{A}^k_{\mathbb{C}}(X) | \Delta_{\partial}(\alpha) = 0\} \quad \text{and} \quad \mathcal{H}^{p,q}_{\partial}(X) := \{\alpha \in \mathcal{A}^{p,q}(X) | \Delta_{\partial}(\alpha) = 0\}$$

Thanks to the fact that d^*, ∂^* and $\bar{\partial}^*$ are the adjoint operators of d, ∂ and $\bar{\partial}$ with respect to our Hermitian product of forms $\langle , \rangle$ we have the following characterization, which is not extremely useful but it is nice nonetheless:

Proposition 2.56. *Let (X, g) be a compact Hermitian manifold. A form α is harmonic (resp. $\bar{\partial}$-harmonic, ∂-harmonic) if and only if $d\alpha = d^*\alpha = 0$ (resp. $\bar{\partial}\alpha = \bar{\partial}^*\alpha = 0$, $\partial\alpha = \partial^*\alpha = 0$).*

Proof. As stated this is thanks to the adjointness of the operators. We shall give the proof for d, the other two are identical.

Suppose $\Delta\alpha = 0$. Then $0 = \langle \Delta\alpha, \alpha\rangle = \langle d^*d\alpha + dd^*\alpha, \alpha\rangle = \langle d^*d\alpha, \alpha\rangle + \langle dd^*\alpha, \alpha\rangle = \langle d\alpha, d\alpha\rangle + \langle d^*\alpha, d^*\alpha\rangle = ||d\alpha||^2 + ||d^*\alpha||^2$. Since both terms are non-negative and the only way their sum can equal zero is if both are zero. Since $||\_||$ is a norm, this implies $d\alpha = d^*\alpha = 0$.

The reverse implication is immediate. □

We will now see that the spaces of harmonic functions decompose in a special way.

Proposition 2.57. *Let (X, g) be an Hermitian manifold (not necessarily compact). We have the following decompositions:*

$$\mathcal{H}^k_{\bar{\partial}}(X, g) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\bar{\partial}}(X, g)$$

$$\mathcal{H}^k_{\partial}(X, g) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\partial}(X, g)$$

Proof. These two are proved in the exact same way so let us give the proof for the $\bar{\partial}$-harmonic forms.

Let $\alpha \in \mathcal{A}^k_{\mathbb{C}}(X)$ and let $\alpha = \sum \alpha^{p,q}$ be its bidegree decomposition. Consider $\bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\bar{\partial}}(X, g) \subseteq \mathcal{A}^k_{\mathbb{C}}(X)$. Each of the $\mathcal{H}^{p,q}_{\bar{\partial}}(X, g)$ is in $\mathcal{A}^{p,q}$ so they have zero intersection (since the $\mathcal{A}^{p,q}$ also have zero intersection) and therefore their sum is indeed a direct sum.

Now if $\alpha \in \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\bar{\partial}}(X, g)$, that is, each $\alpha^{p,q} \in \mathcal{H}^{p,q}_{\bar{\partial}}(X, g)$ and therefore $\Delta_{\bar{\partial}}\alpha^{p,q} = 0$ one has that $\Delta_{\bar{\partial}}\alpha = \sum \Delta_{\bar{\partial}}\alpha^{p,q} = 0$ and therefore $\alpha \in \mathcal{H}^k(X, g)$ and so we conclude $\bigoplus_{p+q=k} \mathcal{H}^{p,q}(X, g) \subseteq \mathcal{H}^k(X, g)$.

Now if $\alpha \in \mathcal{H}^k_{\bar{\partial}}(X, g)$ we have $0 = \Delta_{\bar{\partial}}\alpha = \sum \Delta_{\bar{\partial}}\alpha^{p,q}$. Now, since $\Delta_{\bar{\partial}}$ respects the bidegree, $\Delta_{\bar{\partial}}\alpha^{p,q} \in \mathcal{A}^{p,q}_X$ and since the bidegree decomposition is unique $\sum \Delta_{\bar{\partial}}\alpha^{p,q} = 0$ implies each $\Delta_{\bar{\partial}}\alpha^{p,q} = 0$ and so $\alpha \in \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\bar{\partial}}(X, g)$ which gives us the desired reverse inclusion $\mathcal{H}^k_{\bar{\partial}}(X, g) \subseteq \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\bar{\partial}}(X, g)$, and therefore the equality. □

Of course, if (X, g) is Kähler we have that $\frac{1}{2}\Delta = \Delta_{\bar{\partial}} = \Delta_{\partial}$ so for a Kähler manifold one has $\mathcal{H}^k(X, g) = \mathcal{H}^k_{\bar{\partial}}(X, g) = \mathcal{H}^k_{\partial}(X, g)$ and $\mathcal{H}^{p,q}(X, g) = \mathcal{H}^{p,q}_{\bar{\partial}}(X, g) = \mathcal{H}^{p,q}_{\partial}(X, g)$ and the two decompositions from the previous proposition coincide. Additionally, using these equalities we gain the decomposition $\mathcal{H}^k(X, g) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X, g)$.

Now our goal is to show that this decomposition carries over to cohomology groups. To do this we need the following two results, central in Hodge theory. As we stated before, their proofs require nontrivial harmonic analysis and thus we provide only their statements.

First, we have a decomposition of cohomology groups in Riemannian manifolds. Let us introduce them, as we haven't done so before:

Definition 2.58. Let M be a smooth manifold. Then the *de Rham cohomology* is defined as follows:

$$H^k(M,\mathbb{R}) := \frac{\ker(d : \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M))}{\operatorname{im}(d : \mathcal{A}^{k-1}(M) \to \mathcal{A}^k(M))}$$

As we know, this indeed turns out to be a cohomology and it is in fact isomorphic to singular cohomology, as shown by the de Rham theorem (check [Lee13, Thm. 18.14] for a explicit statement and a proof).

Theorem 2.59 (Hodge decomposition)**.** *Let (M,g) be a compact oriented Riemannian manifold. We have the following decomposition:*

$$\mathcal{A}^k(M) = d(\mathcal{A}^{k-1}(M)) \oplus \mathcal{H}^k_{\mathbb{R}}(M,g) \oplus d^*(\mathcal{A}^{k+1}(M)),$$

where we denoted $\mathcal{H}^k_{\mathbb{R}}(M,g) = \{\alpha \in \mathcal{A}^k(M) | \Delta\alpha = 0\}$. Moreover, this decomposition is orthogonal with respect to the scalar product of forms $\langle\,,\rangle$.

This gives us the following corollary:

Corollary 2.60. *Let (M,g) be a compact, oriented Riemannian manifold. Then the natural projection $\mathcal{H}^k_{\mathbb{R}}(M,g) \to H^k(M,\mathbb{R})$ sending $\alpha \to [\alpha]$ is an isomorphism.*

Proof. First of all we should check that the projection is indeed well-defined. For smooth manifolds, it is also true that a form is harmonic if and only if it is both d-closed and d^*- closed. In fact when we proved this for Hermitian manifolds we never used the complex structure. So any harmonic form is in particular d-closed so we can consider its cohomology class and the projection is well defined.

Injectivity can be shown without using the Hodge decomposition. If $[\omega] = [0]$ for $\omega \in \mathcal{H}^k_{\mathbb{R}}(M,g)$, this means ω is exact, $\omega = d\phi$ for some $\phi \in \mathcal{A}^{k-1}(M)$ but now we have $||\omega||^2 = \langle\omega,\omega\rangle = \langle\omega, d\phi\rangle = \langle d^*\omega,\phi\rangle = 0$ because $d^*\omega = 0$ since ω is harmonic. Therefore $||\omega|| = 0$ which implies $\omega = 0$ which proves injectivity.

For surjectivity, let $\alpha \in \mathcal{A}^k(M)$ be d-closed. Let us consider $[\alpha] \in H^k(M,\mathbb{R})$. We'll find an harmonic form whose projection into $H^k(M,\mathbb{R})$ is $[\alpha]$. By Hodge decomposition, we have that we can write $\alpha = d\beta + \omega + d^*\gamma$ where $\omega \in \mathcal{H}^k_{\mathbb{R}}(M,g)$, $\beta \in \mathcal{A}^{k-1}(M)$ and $\gamma \in \mathcal{A}^{k+1}(M)$. Now since α is d-closed and $dd = 0$ and ω is d-closed since it is harmonic we have $0 = d\alpha = dd\beta + d\omega + dd^*\gamma = 0 + 0 + dd^*\gamma$. Thanks to our scalar product of forms, we have that $dd^*\gamma = 0$ implies $0 = \langle\gamma, dd^*\gamma\rangle = \langle d^*\gamma, d^*\gamma\rangle = ||d^*\gamma||^2$ and since the scalar product is positive definite, this implies $d^*\gamma = 0$.

So for any α which is d-closed we can write $\alpha = d\beta + \omega$ with ω an harmonic form. Then, $[\alpha] = [\omega]$ and thus ω is an harmonic form whose projection is $[\alpha] = [\omega]$ so the projection map is surjective and therefore, an isomorphism. □

Equivalently, this says that every class in $H^k(M,\mathbb{R})$ has a unique harmonic representative. On a Hermitian manifold (X,g), we define $H^k(X,\mathbb{C}) = H^k(X,\mathbb{R}) \otimes \mathbb{C}$ and thus we obtain in the same manner (complexifying everything) an isomorphism $\mathcal{H}^k(X) \simeq H^k(X,\mathbb{C})$.

Theorem 2.59 has an analogous version for Hermitian manifolds. Its proof can't be reduced to the one of Theorem 2.59 but it uses similar techniques.

Theorem 2.61. *Let (X, g) be a compact Hermitian manifold. Then we have the following two orthogonal decompositions:*

$$\mathcal{A}^{p,q}(X) = \partial\mathcal{A}^{p-1,q}(X) \oplus \mathcal{H}^{p,q}_{\partial}(X, g) \oplus \partial^*\mathcal{A}^{p+1,q}(X)$$
$$\mathcal{A}^{p,q}(X) = \bar{\partial}\mathcal{A}^{p-1,q}(X) \oplus \mathcal{H}^{p,q}_{\bar{\partial}}(X, g) \oplus \bar{\partial}^*\mathcal{A}^{p+1,q}(X)$$

Moreover, the spaces $\mathcal{H}^{p,q}_{\partial}(X, g)$ and $\mathcal{H}^{p,q}_{\bar{\partial}}(X, g)$ are finite dimensional. If (X, g) is Kähler then $\mathcal{H}^{p,q}_{\partial}(X, g) = \mathcal{H}^{p,q}_{\bar{\partial}}(X, g)$.

As discussed, we won't prove these lasts two theorems, but [Sch12] goes into great detail in order to prove them in sections 16, 17 and 18, in particular, our desired results are an immediate consequence of [Sch12, Thm. 16.3]. Essentially, this decomposition comes from a general result that holds for any differential linear operator that is also elliptic, which we call elliptic regularity. For example, the operators Δ and $\Delta_{\bar{\partial}}$ turn out to be elliptic. This all comes from trying to solve equations of the form $\Delta\alpha = \beta$ and looking at the structure of the space of solutions, which is the reason we say these results belong to the field of harmonic analysis.

Similar to the smooth case, this decomposition gives us the following isomorphism:

Corollary 2.62. *Let (X, g) be a compact Hermitian manifold. Then the natural projection $\mathcal{H}^{p,q}_{\bar{\partial}}(X, g) \to H^{p,q}(X)$ is an isomorphism.*

Proof. The prof is essentially the same as in the real case and won't be repeated. Just use the previous decompositions and the fact that $\Delta_{\bar{\partial}}\alpha = 0$ if and only if $\bar{\partial}\alpha = \bar{\partial}^*\alpha = 0$. □

Finally, we arrive to our desired decomposition of cohomology groups:

Corollary 2.63. *Let (X, g) be a compact Kähler manifold. Then there exists a decomposition*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

This decomposition does not depend on the chosen Kähler structure.

Proof. The decomposition is given by the series of results that we've seen on this section:

$$H^k(X, \mathbb{C}) = \mathcal{H}^k(X, g) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\bar{\partial}}(X, g) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X, g) = \bigoplus_{p+q=k} H^{p,q}(X)$$

A priori, it might depend on the Kähler metric. To see that it does not, let us consider a second Kähler metric g'. This other metric will provide a different Riemannian volume form, and therefore a different Hodge $*$ operator, a different d^* and a different set of harmonic forms. However, we still have the identification $\mathcal{H}^{p,q}(X, g) \simeq H^{p,q}(X) \simeq \mathcal{H}^{p,q}(X, g')$. So for every harmonic form $\alpha \in \mathcal{H}^{p,q}(X, g)$ we have a corresponding $\alpha' \in \mathcal{H}^{p,q}(X, g')$. To prove that the decomposition does not depend on the metric we want to see that these α, α' correspond to the same element in $H^k(X,\mathbb{C})$, i.e., that the next diagram is commutative:

$$\begin{array}{ccccc} & \mathcal{H}^{p,q}(X,g) & \xhookrightarrow{i} & \mathcal{H}^k(X,g') & \\ \overset{\simeq}{\nearrow} & & & & \overset{\simeq}{\searrow} \\ H^{p,q}(X) & & & & H^k(X,\mathbb{C}) \\ \underset{\simeq}{\searrow} & & & & \underset{\simeq}{\nearrow} \\ & \mathcal{H}^{p,q}(X,g') & \xhookrightarrow{i'} & \mathcal{H}^k(X,g') & \end{array}$$

where $k = p + q$, i', i are natural inclusions and the isomorphisms are the one given by each metric.

So we have to check that $[\alpha] = [\alpha']$ in $H^k(X,\mathbb{C})$. Now, we have that $[\alpha] = [\alpha']$ in $H^{p,q}(X)$ so we can write $\alpha' - \alpha = \bar{\partial}\gamma$ for $\gamma \in \mathcal{A}^{p,q-1}(X)$. Now since α, α' were harmonic (with respect to their metrics) they are in particular d-closed and therefore since $d\bar{\partial}\gamma = d(\alpha' - \alpha) = d\alpha' - d\alpha = 0$, we have that $\bar{\partial}\gamma$ is d-closed. Let us consider the Hodge decomposition in the metric g with respect to the exterior derivative d for $\bar{\partial}\gamma$. We write $\bar{\partial}\gamma = d\beta + \omega + d^*\beta'$ with $\omega \in \mathcal{H}^k(X,g)$. In the proof of Corollary 2.60 we checked that if a form is d-closed its Hodge decomposition does not include a $d^*\beta'$ term, so we have $\bar{\partial}\gamma = d\beta + \omega$.

Using the Hodge decomposition with respect to $\bar{\partial}$ on $\bar{\partial}\gamma$ and the metric g we note that $\bar{\partial}\gamma$ is orthogonal to the whole space $\mathcal{H}^{p,q}(X,g)$ and since the bidegree decomposition was also orthogonal we have that $\bar{\partial}\gamma$ is orthogonal to the whole $\mathcal{H}^k(X,g)$. In particular it is orthogonal to ω and thus $0 = \langle \omega, \bar{\partial}\gamma \rangle = \langle \omega, \omega + d\beta \rangle = ||\omega||^2 + \langle \omega, d\beta \rangle$. Now by orthogonality of the Hodge decomposition with regard to d, we have $\langle \omega, d\beta \rangle = 0$, so $||\omega|| = 0$ and $\omega = 0$.

Therefore $\alpha' - \alpha = \bar{\partial}\gamma = d\beta$ and we have $[\alpha'] = [\alpha]$ and the two decompositions agree. □

2.6 Symplectic manifolds

Let us now turn our attention to a completely new kind of manifolds, symplectic manifolds. Well, they are technically a new kind of manifolds but as we'll see it turns out that any Kähler manifold is symplectic.

The notion of symplectic manifolds is motivated by physics, where they are used to describe the phase space of a system with several particles. This means that the coordinates of the manifold describe the position and the momentum of the particles, in what is known as *generalized coordinates*. Endowing these manifolds with this extra structure allows us to predict the trajectories of the particles through solving a set of differential equations that arise naturally. A change of generalized coordinates is then a diffeomorphism of symplectic manifolds that also preserves the symplectic structure.

But let us first define this symplectic structure. Just like in the previous cases, let us define it first for vector spaces and then we'll look at its correspondent notion on a manifold.

Definition 2.64. Let V be a real finite-dimensional vector space. An element $\omega \in \bigwedge^2 V^*$ is said to be a symplectic tensor if for each nonzero $v \in V$ there exists $u \in V$ such that $\omega(v,u) \neq 0$. We also say that ω is non-degenerate. The pair (V,ω) is called a symplectic vector space.

There is a useful equivalent definition of a symplectic tensor. Let us recall the notion of interior multiplication of forms. For a vector $v \in V$ and a form $\alpha \in \bigwedge^k V^*$ their interior multiplication is $v \lrcorner \alpha = i_v \alpha \in \bigwedge^{k-1} V^*$ and it is defined as $(v \lrcorner \alpha)(u_1, \ldots, u_{k-1}) := \alpha(v, u_1, \ldots, u_{k-1})$ for vectors $u_1, \ldots, u_{k-1} \in V$.

Now, $\omega \in \bigwedge^2 V^*$ is a symplectic tensor if and only if the linear map $\hat{\omega} : V \to V^*$ defined by $\hat{\omega}(v) = v \lrcorner \omega$ is an isomorphism.

The most simple example of a symplectic vector space is constructed as follows. Consider V to have dimension $2n$ and let $(A_1, B_1, \ldots, A_n, B_n)$ be a basis for V and let $(\alpha^1, \beta^1, \ldots, \alpha^n, \beta^n)$ be its dual basis. We define $\omega \in \bigwedge^2 V^*$ as:

$$\omega := \sum_{i=1}^{n} \alpha^i \wedge \beta^i$$

We can easily check that ω is indeed a symplectic tensor. Let us assume that it is not; then there exists a non-zero vector $v \in V$ such that $\omega(v, u) = 0$ for all $u \in V$. In particular, $\omega(v, A_i) = \omega(v, B_i) = 0$ for $i = 1, \ldots, n$. Now if we write v in terms of our basis, $v = \sum_{i=1}^{n}(a^i A_i + b^i B_i)$ where $a^i, b^i \in \mathbb{R}$ are its coordinates in this basis. But since we have an explicit description of ω and since α^i, β^i is the dual basis of A_i, B_i, we can explicitly calculate $\omega(v, A_i) = a^i, \omega(v, B_i) = b^i$ and since all its coordinates are zero the vector v must be zero. Therefore, ω is indeed a symplectic tensor.

Inner products on vector spaces induce a notion of orthogonality for linear subspaces. We can give an analogous definition in the symplectic case:

Definition 2.65. Let (V, ω) be a symplectic vector space and let $S \subseteq V$ be a linear subspace. The *symplectic complement of* S is the subspace

$$S^{\perp} := \{v \in V : \omega(v, w) = 0 \text{ for all } w \in S\}$$

The set $S^{\perp}$ is indeed a linear subspace thanks to the linearity of ω.

Just as for inner products, $\dim S + \dim S^{\perp} = \dim V$ ([Lee13, Lem. 22.3]). But when we had an inner product $\langle \, , \rangle$, a subspace and its orthogonal had zero intersection, thanks to the fact that, if for $v \in V$ one had $\langle v, v \rangle = 0$, then $v = 0$. This is not the case for symplectic vector spaces (V, ω). For example, if one assumes S to be 1-dimensional one has that $\omega(v, v) = 0$ for all $v \in S$ since ω is alternating. This fact allows the following definitions:

Definition 2.66. Let (V, ω) be a symplectic vector space, let $S \subseteq V$ be a linear subspace and let $S^{\perp}$ be the symplectic complement of S. S is said to be:

- *symplectic* if $S \cap S^{\perp} = 0$.
- *isotropic* if $S \subseteq S^{\perp}$.
- *coisotropic* if $S^{\perp} \subseteq S$.
- *Lagrangian* if it is both isotropic and coisotropic, that is, if $S = S^{\perp}$.

Since $\dim S + \dim S^{\perp} = \dim V$ for a Lagrangian subspace $\dim S = \dim S^{\perp} = \frac{1}{2}\dim V$. The condition for an isotropic subspace S implies that $\omega|_S = 0$ so therefore this will also be the case in Lagrangian subspaces. In fact, if a subspace $S \subseteq V$ satisfies both $\dim S = \frac{1}{2}\dim V$ and $\omega|_S = 0$ then it is immediately a Lagrangian subspace.

We have seen before a quick example of a symplectic vector space. As it turns out, this is the only example of a symplectic vector space, as any symplectic tensor can be written in that form. More explicitly:

Proposition 2.67. *Let (V, ω) be a symplectic vector space and let $m = \dim V$. Then V has even dimension $m = 2n$ and there exists a basis $(A_1, B_1, \ldots, A_n, B_n)$ of V with dual basis $(\alpha^1, \beta^1, \ldots, \alpha^n, \beta^n)$ in which ω has the form $\omega := \sum_{i=1}^n \alpha^i \wedge \beta^i$.*

The proof proceeds by induction and can be found in [Lee13, Prop. 22.7]. In particular, this shows that there can't exist symplectic vector spaces of odd dimension.

Before moving on to symplectic manifolds, let us show one more result regarding symplectic vector spaces. Given a form ω, we denote by ω^n the wedge product of ω by itself iterated n-times, that is:

$$\omega^n := \underbrace{\omega \wedge \omega \wedge \cdots \wedge \omega}_{n \text{ times}}$$

There is a simple way to characterize symplectic tensors using this:

Proposition 2.68. *Let V be a $2n$-dimensional real vector space and let $\omega \in \bigwedge^2 V^*$. Then ω is a symplectic tensor if and only if $\omega^n \neq 0$.*

Proof. Let us first suppose that ω is a symplectic tensor. By the previous proposition, we know that we can write $\omega = \sum_{i=1}^n \alpha^i \wedge \beta^i$ for some basis $(\alpha^1, \beta^1, \ldots, \alpha^n, \beta^n)$ of V^*. Now through a quick calculation (and using that 2-forms commute with respect to the wedge product) one obtains $\omega^n = n! \left(\alpha^1 \wedge \beta^1 \wedge \cdots \wedge \alpha^n \wedge \beta^n\right) \neq 0$.

Conversely, suppose that ω is not a symplectic tensor, and let us show that then $\omega^n = 0$. Since ω is a 2-form but not a symplectic tensor, we know that there exists a non-zero vector $v \in V$ such that $v \lrcorner \omega = \hat{\omega}(v) = 0$.

Now we know that for a p-form α and a q-from β the interior multiplication of forms satisfies the following Leibniz rule: $v \lrcorner (\alpha \wedge \beta) = (v \lrcorner \alpha) \wedge \beta + (-1)^p \alpha \wedge (v \lrcorner \beta)$. If we apply this inductively we can calculate $v \lrcorner \omega^n = v \lrcorner (\omega \wedge \omega^{n-1}) = n(v \lrcorner \omega) \wedge \omega^{n-1} = 0 \wedge \omega^{n-1} = 0$. Therefore $v \lrcorner \omega^n = 0$.

Now extend v to a basis $(E_1, \ldots, E_{2n})$ for V with $E_1 = v$. We have that $0 = (v \lrcorner \omega^n)(E_2, \ldots, E_{2n}) = \omega^n(E_1, \ldots, E_{2n})$. Now if $v_1, \ldots, v_{2n} \in V$ are arbitrary vectors using the multilinearity and alternativity of ω^n and writing each v_i in the basis $(E_1, \ldots, E_{2n})$ we obtain that $\omega^n(v_1, \ldots, v_{2n}) = K\omega^n(E_1, \ldots, E_{2n}) = 0$ for a suitable $K \in \mathbb{R}$. Since $v_1, \ldots, v_{2n}$ were arbitrary, we conclude that $\omega^n = 0$ which concludes the proof. □

Note that for $\omega \in \bigwedge^2 V^*$ we have that $\omega^n \in \bigwedge^{2n} V^*$, and if ω is symplectic then ω^{2n} is a non-zero $2n$-form, so it is an orientation form for V^*. In summary, any symplectic tensor ω on V endows V^* with a natural orientation.

Now let us apply these results and definitions to manifolds. For symplectic vector spaces, we made no distinction between a symplectic tensor and a non-degenerate form. In the case of manifolds, things are different:

Definition 2.69. Let M be a smooth manifold. We say that a 2-form ω is *non-degenerate* if ω_x is a symplectic tensor (that is, a non-degenerate 2-covector) for each $x \in M$. We say that a non-degenerate form ω is symplectic if it is closed, $d\omega = 0$. A pair (M, ω) where M is a smooth manifold and ω is a symplectic form is called a *symplectic manifold*. A choice of a symplectic form ω for a given smooth manifold M is called a *symplectic structure*.

An example would be $\mathbb{R}^{2n}$ with the form given before. If $(x^1, \dots, x^n, y^1, \dots, y^n)$ is a basis for $\mathbb{R}^{2n}$ the symplectic form would be $\omega = \sum_{i=1}^{n} dx^i \wedge dy^i$. Note that ω is closed thanks to the fact that $d \circ d = 0$.

Note that by Proposition 2.67 our manifold must be even dimensional, and therefore, there can't exist symplectic manifolds of odd dimension. This will not be a problem for us, as our intention is to have symplectic structures on complex manifolds, which are already even dimensional smooth manifolds. If we wanted something similar to symplectic structures on odd-dimensional smooth manifolds then we'd have to use contact structures ([Lee13, p. 581]), but we won't study them due to their lack of compatibility with complex manifolds.

Let (M, ω) be a symplectic manifold. Then, by Proposition 2.68 we know that ω^n must be a nowhere-vanishing top degree form, which as we know, is an orientation form. Therefore any symplectic manifold is oriented, which also implies that not every even-dimensional smooth manifold can be symplectic, as they must be oriented. For example, since the real projective space $\mathbb{P}^n$ is orientable if and only if n is odd, then no real projective space can ever be a symplectic manifold. Again, this will not be an issue when we deal with symplectic structures on complex manifolds, since complex manifolds are always oriented.

In the introduction to this section we mentioned that physicists are interested in transformations of symplectic manifolds that preserve the symplectic structure, as these provide the changes of coordinates used in classical mechanics. Let us define these transformations:

Definition 2.70. Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds. A diffeomorphism $f : M_1 \to M_2$ is called a *symplectomorphism* if it satisfies $f^*\omega_2 = \omega_1$.

For a symplectic manifold (M, ω) and a submanifold $S \subseteq M$ we say that S is a *symplectic*, *isotropic*, *coisotropic* or a *Lagrangian submanifold* if T_xS (thought as a linear subspace of T_xM) has the corresponding propierty for every $x \in S$.

The most widely used example of symplectic manifold are cotangent spaces. As it turns out, given a smooth manifold M its cotangent bundle T^*M (which we know also has a smooth manifold structure) has a canonical symplectic form (see [Lee13, Prop. 22.11] for its definition). This is the reason symplectic manifolds are used in physics. A smooth manifold M is used as the space of coordinates of a particle system, its tangent bundle TM is then used as the space of velocities while the cotangent bundle T^*M is used as the space of momenta. Since we are interested in the relationships between momenta and coordinates (which is what ultimately gives raise to the motion equations), we study T^*M which just so-happens to naturally be a symplectic manifold.

Similar to the musical isomorphisms given by a Riemmannian metric on a smooth manifold M (see [Lee13, p. 341]) we also have a tangent-cotangent isomorphism on symplectic manifolds, given by the fact that for $x \in M$ the non-degenerate form ω_x gives us an isomorphism $\hat{\omega}_x : T_xM \to T_x^*M$, which ends up being a smooth vector bundle isomorphism.

The fact that a symplectic structure provides a tangent-cotangent isomorphism and that ω is a 2-form might lead one to think that Riemannian manifolds and symplectic manifolds are not very different from one another. However, some key differences exist. One notorious example of this is the fact that symplectic manifolds are locally equivalent to the standard flat model (which is $\mathbb{R}^{2n}$ with symplectic form $\omega = \sum_{i=1}^{n} dx^i \wedge dy^i$). This is in stark contrast with Riemannian geometry, whereas any cartographer would be able to tell you, only regions with curvature zero can be isometric to the standard flat example (which is $\mathbb{R}^n$ with the euclidean product). This result has a name, it is called the Darboux theorem.

Theorem 2.71 (Darboux)**.** *Let (M, ω) be a $2n$-dimensional symplectic manifold and let $x \in M$. Then there are smooth coordinates $(x^1, \dots, x^n, y^1, \dots, y^n)$ in which ω takes the form $\omega = \sum_{i=1}^{n} dx^i \wedge dy^i$.*

The proof is rather long and it involves time-dependant flows so we won't reproduce it here, though it can be found in its entirety in [Lee13, Thm. 22.13]. The proof explicitly uses that ω is closed; this is one of the reasons we ask for this condition. Also, ω being closed lets us consider its class in cohomology and it's what allows to have results such as Noether's symmetry theorem (which relates symmetries of a system with conserved quantities along vector fields), central in physics.

Let us now turn our attention back to complex manifolds. We want to know how does a symplectic structure relate to the others structures we have defined. For instance, let M be a smooth manifold endowed with an integrable almost complex structure I and a symplectic structure ω. Now we would like some compatibility between the two structures, but what should we ask? First we could try asking the same condition as for the inner product, that is, that for $u, v \in \mathfrak{X}(M)$ (where $\mathfrak{X}(X)$ denotes the vector fields of M) one has $\omega(u, v) = \omega(I(u), I(v))$. As it turns out we desire a bit more. Let us see why.

Let us consider the bilinear form α defined as $\alpha(u, v) := \omega(u, I(v))$. Using this property of compatibility between ω and I one has that $\alpha(u, v) = \omega(u, I(v)) = \omega(I(u), I(I(v))) = \omega(I(u), -v) = \omega(v, I(u)) = \alpha(v, u)$ since ω is alternating and $I^2 = -\operatorname{id}$. So we obtain that α is symmetric and it is bilinear since both ω and I are. Moreover, $\alpha(I(u), v) = \omega(I(u), I(v)) = \omega(u, v)$ and $\alpha(I(u), I(v)) = \omega(I(u), I(I(v)) = \omega(u, I(v)) = \alpha(u, v)$. This α looks suspiciously similar to a Riemannian metric on M that is compatible with its integrable almost complex structure I and that has ω as its fundamental form. There is one problem though: nothing assures us that this α is positive definite. So we impose it:

Definition 2.72. Let X be a complex manifold, let M be its underlying smooth manifold and let I be the integrable almost complex structure on M induced by X. Moreover, let ω be a symplectic form on M and let $u, v \in \mathfrak{X}(M)$. We say that ω is compatible with I if the bilinear form $g(u, v) := \omega(u, I(v))$ is symmetric and positive definite, and thus a Riemannian metric on M.

Note that imposing that g is symmetric is equivalent to our condition of compatibility $\omega(I(u), I(v)) = \omega(u, v)$, but asking the condition to be satisfied by g is more comfortable. Again we don't explicitly need that the almost complex structure is integrable to give this definition, but our intention is to study symplectic structures on complex manifolds, so we will only deal with the complex case. The following proposition is an immediate result of the preceding definition and calculations:

Proposition 2.73. *Let X be a complex manifold, M its underlying smooth manifold, I its* *endowed integrable almost complex structure and let ω be a symplectic form on M. Then (X, ω) is a Kähler manifold.*

Proof. That ω is compatible with I means that the bilinear form $g(\_,\_) := \omega(\_, I(\_))$ is a Riemannian metric on M, and by our previous calculations the Riemannian metric g is compatible with I, so (X, g) is a Hermitian manifold. We also previously checked that ω was the fundamental form associated to g, so for (X, ω) to be Kähler we only need ω to be closed. But ω is closed because it is symplectic. □

The relationships between symplectic and Kähler manifolds goes beyond this.

Proposition 2.74. *Let (X, ω) be a Kähler manifold. Then, ω is a symplectic form on M, the underlying smooth manifold of X.*

Proof. We already checked in Proposition 2.38 that ω was a real 2-form. Since it is a Kähler form, it is closed, so we only need to check that ω_x is non-degenerate for each $x \in M$.

Let us assume that ω_x is degenerate. Then there exists some non-zero $v \in T_xM$ such that $\omega_x(v, u) = 0$ for all $u \in T_xM$. Let us pick $u = -I(v)$ in particular. Then $0 = \omega_x(v, -I(v)) = \omega_x(I(v), v) = g(v, v)$ where g is the metric on M. Since g is a metric this implies $v = 0$, which is a contradiction, and so ω_x is non-degenerate and (X, ω) is symplectic. □

In summary, we have found an equivalent way to look at Kähler manifolds: instead of a complex manifold whose integrable almost complex structure is compatible with a metric (and has a closed fundamental form), we can consider it a complex manifold whose integrable almost complex structure is compatible with a symplectic form. We will see yet another way to introduce a Kähler manifold in the following sections.

3 Hodge conjecture

Before continuing with other types of manifolds, we might as well present the Hodge conjecture, a very important conjecture on complex manifolds, since all our work up until now leaves us in a position to understand its statement with just a little more effort.

This conjecture deals with the relationships between algebraic geometry and topology, and it is perhaps the most famous problem in the field of algebraic geometry, as it is one of the seven Millennium Prize Problems.

Nowadays, the Hodge conjecture is usually stated for projective algebraic varieties, that is, subsets of the projective space $\mathbb{P}^n$ that can be described as the set of solutions of a finite system of homogeneous polynomial equations ([Sch12, Def. 38.1]). However we will state it for complex projective complex manifolds. This turns out to be equivalent, since any projective complex manifold is a projective algebraic variety. This is known as Chow's theorem. See [Sch12, Thm. 38.2] for a proof of the specific case where we are dealing with analytic subvarieties instead of complex submanifolds.

Recall that any projective complex manifold was in fact a Kähler manifold, which is an indispensable fact for stating the conjecture.

We need a couple of additional definitions before being able to state the conjecture.

Definition 3.1. Let X be a complex manifold with complex dimension n and let $Z \subseteq X$ be a complex submanifold with codim $Z = p$. The *fundamental class of* Z is $[Z] \in H^{p,p}(X)$ and it is defined by the condition

$$\int_X \alpha \wedge [Z] = \int_Z \alpha|_Z$$

for all $\alpha \in H^{2n-2p}(X)$

One then checks that this indeed defines an unique class in cohomology. This is a simple task if one uses Poincaré duality for smooth manifolds.

Theorem 3.2 (Poincaré duality)**.** *Let* M *be a compact, oriented,* n*-dimensional smooth manifold. Then the bilinear pairing*

$$(\_,\_) : H^r(M, \mathbb{R}) \times H^{n-r}(M, \mathbb{R}) \to \mathbb{R}$$

that sends $([\alpha], [\beta]) \to \int_M \alpha \wedge \beta$ *is non-degenerate.*

In fact, since in the proof we'll make use of the Hodge $*$-operator, we need our manifold M to be Riemannian, but any smooth manifold can be endowed with a Riemannian metric, see [Lee13, Prop. 13.3]. The proof is short enough if we make use of Hodge decomposition theorem, so we might as well provide it.

Proof. It follows from its definition that the pairing is bilinear, so it is indeed a pairing. Let us see that it is non-degenerate. Using Stoke's theorem (see [Lee13, Thm. 16.11]) and the Leibniz rule for the exterior derivative one can proof that this pairing is well defined.

Recalling the definition of the Laplacian and with a small calculation one can check that the Laplacian commutes with the Hodge $*$-operator. Therefore the Hodge $*$-operator restricts to a map $* : \mathcal{H}^r(M, \mathbb{R}) \to \mathcal{H}^{n-r}(M, \mathbb{R})$, and since we have $H^r(M, \mathbb{R}) \simeq \mathcal{H}^r(M, \mathbb{R})$ (by Proposition 2.60) the Hodge $*$-operator defines a map $* : H^r(M, \mathbb{R}) \to H^{n-r}(M, \mathbb{R})$.

Let us fix a non-zero $[\alpha] \in H^r(M, \mathbb{R})$. We need to check that there is a $[\beta] \in H^{n-r}(M, \mathbb{R})$ so that $([\alpha], [\beta]) \neq 0$. Now as we have just seen $*[\alpha] \in H^{n-r}$ and $([\alpha], *[\alpha]) = \int_M \alpha \wedge *\alpha = \langle \alpha, \alpha \rangle \neq 0$ since the inner product of forms $\langle \, , \rangle$ was positive definite. Therefore by choosing $[\beta] = *[\alpha]$, the pairing is non-degenerate. □

As we know, a non-degenerate pairing gives an isomorphism with the dual, so we have $(H^{n-r}(M, \mathbb{R}))^* \simeq H^r(M, \mathbb{R})$. This is what allows us to proof that the fundamental form of a p-dimensional complex submanifold $Z \subseteq X$ defined in 3.1 is indeed an element of $H^{2p}(X, \mathbb{C})$. The previous isomorphism extends to an isomorphism $\left(H^{2n-2r}(M, \mathbb{C})\right)^* \simeq H^{2r}(M, \mathbb{C})$ if X is a complex manifold of dimension n and M is its underlying $2n$-smooth manifold. The fundamental class is defined through a map $\int_Z : H^{2n-2r}(M, \mathbb{C}) \to \mathbb{C}$ so it is an element of $\left(H^{2n-2r}(M, \mathbb{C})\right)^*$. By the isomorphism, this defines a unique element $[Z]$ in $H^{2p}(M)$, as we wanted. Moreover, the isomorphism is given by the non-degenerate pairing $\int_M \alpha \wedge [Z]$ which is what we used to define the fundamental class in Definition 3.1.

Note that Poincaré duality required our manifold M to be oriented, but that is not a problem when extending these results to complex manifolds since as we saw in Proposition 2.43, every complex manifold is oriented.

Checking that for complex submanifolds $Z \subseteq X$ of a complex manifold X with $\operatorname{codim} Z = p$ the fundamental class $[Z] \in H^{2p}(X, \mathbb{C})$ satisfies $[Z] \in H^{p,p}(X, \mathbb{C})$ is not hard to do from the integral condition that gives the definition of $[Z]$.

There is an alternative way to introduce a fundamental class, and it comes from the field of algebraic topology. Let us denote by $H_k(M, \mathbb{Z})$ the k-th singular homology group of a topological manifold M. It can be proved that for a closed oriented topological m-manifold one has that $H_m(M, \mathbb{Z}) \simeq \mathbb{Z}$ (see [Hat02, Thm. 3.26] for a detailed proof) and then one defines the fundamental class $[M]$ of a closed oriented topological manifold M as a generator of the group $H_m(M, \mathbb{Z}) \simeq \mathbb{Z}$.

For a topological submanifold $i : N \to M$, with $\dim N = n$ and $\dim M = m$, we can define a homology class $[N]_{top}$ of the submanifold N in $H_n(M, \mathbb{Z})$. If $H_n(i) : H_n(N, \mathbb{Z}) \to H_n(M, \mathbb{Z})$ denotes the group homeomorphism induced by i in homology, and $[N] \in H_n(N, \mathbb{Z})$ is the fundamental class of the manifold N, we define $[N]_{top} := H_n(i)[N]$.

Now let $H^k(M, \mathbb{Z})$ denote the k-th singular cohomology group of a topological manifold M. Poincaré duality for topological manifolds gives us an isomorphism $PD : H^k(M, \mathbb{Z}) \to H_{n-k}(M, \mathbb{Z})$ where PD is defined by $PD(\alpha) = [M] \frown \alpha$. Here $\frown$ denotes the cap product and $[M]$ is the fundamental class of the manifold M. For a proof of the fact that PD is indeed an isomorphism, see [Hat02, Thm. 3.30]. So given an n-submanifold $N \subseteq M$ with homology class $[N]_{top} \in H_n(M, \mathbb{Z})$ then we define the fundamental class of the submanifold N as $PD[N]_{top} \in H^{m-n}(M, \mathbb{Z})$. Let us denote it $[N]^{top}$.

If X is a complex n-manifold and $Z \subseteq X$ is a codimension p submanifold this fundamental class $[Z]^{top} \in H^{2p}(X, \mathbb{Z})$ defined in the algebraic topological way defines a class in $H^{2p}(X, \mathbb{C})$ through the morphism $H^{2p}(X, \mathbb{Z}) \to H^{2p}(X, \mathbb{Z}) \otimes \mathbb{C} \simeq H^{2p}(X, \mathbb{C})$ (where we used the de Rham's theorem. Note that this is not just an inclusion since it can kill the torsion of $H^{2p}(X, \mathbb{Z})$). We shall denote this induced class $[Z]^{top}$ too, and we still call it fundamental class of the submanifold Z.

Now for a complex submanifold $Z \subseteq X$ of a complex manifold X with $\operatorname{codim} Z = p$ we have two different definitions of fundamental classes, $[Z]$, $[Z]^{top} \in H^{2p}(X, \mathbb{C})$. It turns out that both definitions agree, $[Z] = [Z]^{top}$ (see, e.g., [CFGT14, Lem. 3.1.27]). The algebraic topology description of the fundamental class of a submanifold has an advantage, it shows that a codimension p complex submanifold $Z \subseteq X$ has indeed $[Z] \in H^{p,p}(X, \mathbb{C}) \cap H^{2p}(X, \mathbb{Z})$, where we have also denoted by $H^{2p}(X, \mathbb{Z})$ the image of $H^{2p}(X, \mathbb{Z})$ in $H^{2p}(X, \mathbb{C})$ through the morphism described above.

We can extend this construction of fundamental classes to analytic subvarieties, the process can be seen in [CFGT14, Lem. 3.1.23].

We define the p-th cohomology group with rational coefficients of a complex manifold X as $H^p(X, \mathbb{Z}) \otimes \mathbb{Q} := H^p(X, \mathbb{Q})$ and we identify it with its image in $H^p(X, \mathbb{C})$ through the obvious inclusions and the isomorphism given by the de Rham's theorem.

Definition 3.3. Let X be a complex manifold. An element of the group $H^{p,p}(X, \mathbb{Q}) := H^{p,p}(X, \mathbb{C}) \cap H^{2p}(X, \mathbb{Q})$ is called a *Hodge class*.

The group $H^{p,p}(X, \mathbb{Q})$ is also sometimes denoted $\operatorname{Hdg}^p(X)$. As we have seen, fundamental classes of complex manifolds (and in fact, of analytic subvarieties) and their linear combinations will be elements of $H^{p,p}(X, \mathbb{Q})$. In fact let us give a name to those:

Definition 3.4. Let X be a complex manifold. An element of $H^{p,p}(X, \mathbb{Q})$ is called an *analytic class* if it is a linear combination with rational coefficients of fundamental classes of manifolds, that is, it belongs to the $\mathbb{Q}$-vector space generated by all the fundamental classes $[Z]$ for an analytic subvariety $Z \subseteq X$.

So as we have seen, some elements of $H^{p,p}$ might be analytic. The Hodge Conjecture states that in fact all of them are:

Conjecture 3.5. Let X be a projective complex manifold. Then any class in $H^{p,p}(X, \mathbb{Q})$ is analytic.

The conjecture is known to be false for a general Kähler manifold X, or in the case where we are working only with the $\mathbb{Z}$-module generated by fundamental classes and $H^{p,p}(X, \mathbb{Z}) := H^{p,p}(X, \mathbb{C}) \cap H^{2p}(X, \mathbb{Z})$.

Some cases are known. For example, the Lefschetz Theorem ([Huy05, Prop.3.3.2]) gives the desired result for $p = 1$. This result stems from the well-understanding of line bundles (vector bundles of rank 1) and it is one of the results that motivates the Hodge conjecture.

The conjecture is known to hold in other cases but they mostly involve many other objects which we don't have time to define here, as this section is already lengthy.

Hodge conjecture provides a remarkable bridge between algebraic topology (which we use to define fundamental classes) and complex geometry (which we use to define the groups $H^{p,p}(X, \mathbb{Z})$. It would be no exaggeration to say that whoever solves it will carve himself a name in the story of mathematics.

4 Kähler manifolds and the holonomy group

Let us now move to other kinds of manifolds, Namely, we would like to define Hyperkähler and Calabi-Yau manifolds. We will introduce them as Riemannian manifolds with an extra condition, so first we need to introduce several concepts from differential geometry.

First of all let us recall several basic facts about connections and Riemmanian manifolds.

4.1 Holonomy groups

Let us begin with the notion of connection, which will lead us to the notion of parallel transport which will lead us to holonomy groups, which are our goal.

The notion of connection can be defined on any smooth vector bundle over a smooth manifold, but we will only be interested in the Levi-Civita connection of a Riemannian manifold so we only need the notion of connection on the tangent bundle. Nevertheless, we might as well define connections in arbitrary bundles:

Definition 4.1. Let $\pi : E \to M$ be a smooth vector bundle over a smooth manifold M. Recall that we denoted the space of vector fields on M (that is, the space of sections of the tangent bundle TM) as $\mathfrak{X}(M)$. Similarly, let us denote the space of sections of E as $E(M)$.

A *connection* in E is a map

$$\nabla : \mathfrak{X}(M) \times E(M) \to E(M)$$

that sends $(X, Y) \to \nabla_X Y$ and satisfies:

1. $\nabla_{fX_1+gX_2}Y = f\nabla_{X_1}Y + g\nabla_{X_2}Y$ for $f, g \in C^\infty(M)$;
2. $\nabla_X(aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2$ for $a, b \in \mathbb{R}$;
3. $\nabla_X(fY) = f\nabla_X Y + (Xf)Y$ for $f \in C^\infty(M)$.

where $C^\infty(M) := \mathcal{A}^0(M)$ denotes the space of real-valued global smooth functions of M. We say that $\nabla_X Y$ is the *covariant derivative of* Y *in the direction of* X.

A *linear connection* is a connection in the tangent bundle TM.

Connections turn out to be local operators: $\nabla_X Y$ evaluated on a point $p \in M$ depends only on the behaviour of X and Y at p ([Lee97, Ex. 4.7, Lem. 4.2]). An example of a linear connection would be the directional derivative in $\mathbb{R}^n$. Every smooth manifold admits a linear connection ([Lee97, Prop. 4.5]), and a linear connection can be uniquely extended to a connection on all tensor bundles $T_l^n M$, also denoted ∇ ([Lee97, Lem. 4.6]. In particular, for $f \in C^\infty(M)$, $\nabla_X f = Xf$).

Thanks to this and the characterization of tensors as multilinear morphisms ([Lee97, Lem. 2.4]), given a $\binom{k}{l}$-tensor field F we can define a $\binom{k+1}{l}$-tensor field ∇F through

$$\nabla F(\omega^1, \dots, \omega^k, Y_1, \dots, Y_k, X) = \nabla_X F(\omega^1, \dots, \omega^k, Y_1, \dots, Y_k)$$

for $\omega^1, \dots, \omega^k \in \mathcal{A}^1(M)$ and $Y_1, \dots, Y_k, X \in \mathfrak{X}(M)$. Since this is multilinear over $C^\infty(M)$ in each variable, this defines indeed a $\binom{k+1}{l}$-tensor field.

Definition 4.2. We say that a $\binom{k}{l}$-tensor field F is *parallel* if $\nabla F = 0$.

We have not yet defined curves on a manifold. For us, a *curve* in a smooth manifold M will mean a smooth map $\gamma : I \to M$ where $I \subseteq \mathbb{R}$ is an open interval (if I is closed, we can always extend the curve to an open interval and work with the extended map).

Given a curve $\gamma : I \to M$ we can define its velocity in the usual way. For each $t \in I$, define $\dot\gamma(t) \in T_{\gamma(t)}$ as the push-forward $\gamma_*\left(\frac{d}{dt}\right)$ where $\frac{d}{dt}\big|_t \in T_t\mathbb{R}$ is a generator of the one-dimensional tangent space. The velocity $\dot\gamma(t)$ acts on functions $f \in C^\infty(M)$ as follows: $\dot\gamma(t)f = \frac{d}{dt}(f \circ \gamma)(t)$.

Definition 4.3. Let M be a smooth manifold. A *vector field along a curve* $\gamma : I \to M$ is a smooth map $V : I \to TM$ such that $V(t) \in T_{\gamma(t)}M$ for every $t \in I$.

The most basic example is the velocity vector of a curve, $\dot\gamma \in T_{\gamma(t)}$. The space of vector fields along a given curve γ is denoted $\mathfrak{J}(\gamma)$.

Another simple example: Given a vector field $X \in \mathfrak{X}(M)$ and a curve $\gamma : I \to M$ we define $\tilde X(t) := X_{\gamma(t)}$ for each $t \in I$. Then through a quick calculation using coordinate representations one can see that $\tilde X(t)$ is a vector field along the curve γ. In fact, we say a vector field along a curve V is *extendible* if there exists a vector field $X \in \mathfrak{X}(U)$ (where U is a neighbourhood of the image $\gamma(I)$ of γ) such that $V(t) = X_{\gamma(t)}$ for each $t \in I$. We then say that X is an extension of V.

This next concept will allow us to define geodesics:

Proposition 4.4. *Let M be a smooth manifold, let ∇ be a linear connection on M, and let $\gamma : I \to M$ be a curve. Then we have a unique operator $D_t : \mathfrak{J}(\gamma) \to \mathfrak{J}(\gamma)$, determined by ∇ and γ, which for $V, W \in \mathfrak{J}(\gamma)$ satisfies:*

1. *D_t is linear over $\mathbb{R}$: $D_t(aV + bW) = aD_tV + bD_tW$ for $a, b \in \mathbb{R}$;*

2. *$D_t(fV) = \dot{f}V + fD_tV$ for $f \in C^\infty(I)$;*

3. *If V is extendible, then for any extension X of V one has $D_tV(t) = \nabla_{\dot{\gamma}(t)}X$.*

A proof of the existence and uniqueness of D_t can be found in [Lee97, Lem. 4.9]. For any $V \in \mathfrak{J}(\gamma)$, we call D_tV the *covariant derivative of V along γ*.

As stated, we can easily define a geodesic using this concept. Geodesics are extremely important in differential geometry, which means they are also essential to General Relativity, where free moving objects in space move following geodesics. Given a curve γ on a smooth manifold M with a linear connection ∇, we define the acceleration of γ as $D_t\dot{\gamma}$. We say a curve is a *geodesic with respect to* ∇ if its acceleration is identically zero, that is, $D_t\dot{\gamma} \equiv 0$.

Similarly, a vector field V along a curve γ is said to be *parallel along γ with respect to* ∇ if $D_tV \equiv 0$. A vector field $V \in \mathfrak{X}(M)$ is said to be *parallel* if its parallel along every curve γ on M, that is, for every curve γ we have $\nabla_{\dot{\gamma}}V \equiv 0$ (since V is global, it is extendible). This is actually equivalent to our definition of parallelism in Definition 4.2. Indeed, recall from differential geometry that we characterize the tangent space as the set of all curve velocities, so this last condition is equivalent to having $\nabla_XV \equiv 0$ for every $X \in \mathfrak{X}(M)$, and this is equivalent to $\nabla V = 0$, that is, that V is parallel in the sense of Definition 4.2.

This notion of parallelism allows to define the notion of *parallel transport*:

Proposition 4.5. *Let M be a smooth manifold with a linear connection ∇, and let $\gamma : I \to M$ be a curve. Given $t_0 \in I$ and a vector $V_0 \in T_{\gamma(t_0)}M$, there exists a unique parallel vector field V along γ such that $V(t_0) = V_0$. This vector field V is called the* parallel translate of V_0 along γ.

The proof heavily relies on the Theorem of Existence and Uniqueness of Solutions for Linear ODEs, and can be found in [Lee97, Thm. 4.11].

If $\gamma : I \to M$ is a curve and $t_0, t_1 \in I$, we can define an operator $P_{t_0t_1} : T_{\gamma(t_0)}M \to T_{\gamma(t_1)}M$ using parallel transport as follows: for V_0, set $P_{t_0t_1}V_0 = V(t_1)$ where $V(t)$ is the parallel translate of V_0 along γ, given by the previous proposition. It turns out that this $P_{t_0t_1}$ is a linear isomorphism between $T_{\gamma(t_0)}M$ and $T_{\gamma(t_1)}M$. This is due to the fact that the differential equations we solved to obtain the parallel translate of V_0 are linear. Since a linear connection induces connections on all tensor bundles, we can define the parallel transport of tensor field using those connections. The notion of a tensor F being parallel along every curve still agrees with our notion of parallelism defined in Definition 4.2, that is, $\nabla F = 0$, the proof is the same as before.

Imagine now that γ is a closed curve based at p, that is, a curve $\gamma : [0, 1] \to M$ with $\gamma(0) = \gamma(1) = p$. As we mentioned before, we can extend this curve γ so its domain is a bigger open interval I and do all previous constructions with this extended gamma. The various operators we defined will not depend on the extension. So let us consider this last isomorphism $P_{t_0t_1}$ for a closed curve γ based at p and $t_0 = 0$, $t_1 = 1$. In this case, we will denote $P_{01} := P_\gamma : T_pM \to T_pM$. So given a closed curve γ based at p we obtain a linear, invertible map of T_pM, which is an n-dimensional real vector space (where $n = \dim M$). Therefore we conclude that $P_\gamma \in \mathrm{GL}(T_pM) \simeq \mathrm{GL}(n, \mathbb{R})$.

Different closed curves based at p will yield different elements in $\mathrm{GL}(T_pM)$. One can check that for two different closed curves α, β based at p one has $P_\alpha \circ P_\beta = P_{\alpha\circ\beta}$ where the composition of curves is defined in the usual way. Additionally, one can also check that $P_{\alpha^{-1}} = (P_\alpha)^{-1}$ where $\alpha^{-1}(t) = \alpha(-t)$ for $t \in [0,1]$. Moreover, the constant curve $\gamma_p(t) = p$ defines $P_{\gamma_p} = \mathrm{id}$.

This previous results are just calculations (see [Bes08, 10.8]) and are used to justify the following definition:

Definition 4.6. Let M be a smooth manifold with a linear connection ∇. We define the *holonomy group of* ∇ *based at* p as

$$\mathrm{Hol}_p(\nabla) := \{P_\gamma | \gamma \text{ is a piecewise smooth closed curve based at } p\}$$

The previous results show that the holonomy group is indeed a group. By definition, it is a subgroup of $\mathrm{GL}(T_pM)$, although not necessarily proper. Since we can easily get a representation for $\mathrm{GL}(T_pM) \simeq \mathrm{GL}(\dim M, \mathbb{R})$, we can think of holonomy groups of a connection as subgroups of the linear group $\mathrm{GL}(\dim M, \mathbb{R})$. Moreover, it can be proven that $\mathrm{Hol}(\nabla)$ is actually a Lie subgroup of $\mathrm{GL}(T_pM)$ and that it is connected whenever M is simply connected ([GHJ03, Def. 2.2]).

Another interesting result: similar to the fundamental group, the holonomy group depends on the basepoint p only up to conjugation in $\mathrm{GL}(\dim M, \mathbb{R})$ if our manifold M is connected. In fact, for a connected smooth manifold M with a linear connection ∇ and given two points $p, q \in M$ and a curve γ from p to q one has $\mathrm{Hol}_q(\nabla) = P_\gamma \mathrm{Hol}_p(\nabla) P_\gamma^{-1}$ ([Bes08, 10.11]). So for a connected smooth manifold and a given linear connection, all holonomy groups are in fact isomorphic and we can drop the notation of the base point p when no confusion can arise. From now on we will only consider connected manifolds and therefore we will consider all manifolds with a linear connection ∇ to have a unique holonomy group $\mathrm{Hol}(\nabla)$.

Much more can be said of holonomy groups, and [Bes08, Section 10] is a great reference for precisely that, but we will not study them any further. Instead, let us now go back to Riemannian manifolds, and let us try to give them a linear connection which is somewhat compatible with their metrics.

Let (M, g) be a Riemannian manifold and let $X, Y, Z \in \mathfrak{X}(M)$. Denote its Lie Bracket $[X, Y]$ (for a definition of the Lie Bracket, see [Lee13, p. 186]). It turns out that if ∇ is a linear connection that satisfies

1. Compatibility with the metric: $\nabla_X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X, Z)$;

2. Symmetry: $\nabla_X Y - \nabla_Y X \equiv [X, Y]$

then the linear connection ∇ is uniquely determined by the metric. See [Lee97, Thm. 5.4] for a complete statement and proof of this fact, which is known as the Fundamental Lemma of Riemannian Geometry. This uniquely defined connection is called the *Riemannian connection* or the *Levi-Civita connection* of g.

The holonomy group of a Riemannian manifold (M, g) is then the holonomy group of the Levi-Civita connection and we denote it $\mathrm{Hol}(g)$. We also call it the holonomy group of the metric g. As it turns out, not every possible subgroup of $\mathrm{GL}(n, \mathbb{R})$ can be a holonomy group of a Riemannian manifold M. In fact the possibilities are quite limited, as we will explore in the next section.

4.2 Berger's classification

For a specific type of Riemannian manifolds, we can classify holonomy groups. We will focus on irreducible, nonsymmetric Riemannian manifolds, so let us define those concepts.

Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds of positive dimension n_1 and n_2 respectively. Their product $M_1 \times M_2$ is also a smooth manifold, of dimension $n_1 + n_2$ (charts of the product manifold are just products of charts of each manifold). At each $(p, q) \in M_1 \times M_2$ we can describe the tangent space as $T_{(p,q)}(M_1 \times M_2) \simeq T_pM_1 \oplus T_qM_2$. This allows us to define a metric on the product manifold:

Definition 4.7. Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds. We define the *product metric* $g_1 \times g_2$ on $M_1 \times M_2$ as $(g_1 \times g_2)_{(p,q)}(\_,\_) := (g_1)_p(\_,\_) + (g_2)_q(\_,\_)$ for all $p \in M_1$ and $q \in M_2$. We call a Riemannian manifold of the type $(M_1 \times M_2, g_1 \times g_2)$ a *Riemannian product.*

One can check through some calculations that this is indeed a Riemannian metric on $M_1 \times M_2$. We can make this concept a bit more general, if we ask that the metric looks like this only locally:

Definition 4.8. Let (M, g) be a Riemannian manifold. We say that (M, g) is *locally reducible* if every point has an open neighbourhood isometric to a Riemannian product $(M_1 \times M_2, g_1 \times g_2)$. We say that (M, g) is *irreducible* if it is not locally reducible.

When our Riemannian manifold is a Riemannian product, the holonomy group decomposes as follows:

Proposition 4.9. *Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds, and let $(M_1 \times M_2, g_1 \times g_2)$ be their Riemannian product. Then $M_1 \times M_2$ has holonomy group $\mathrm{Hol}(g_1 \times g_2) = \mathrm{Hol}(g_1) \times \mathrm{Hol}(g_2)$.*

Proof. We just need to check that all our definitions decompose in this manner. For example, a curve in $M_1 \times M_2$ will be a product of curves; a connection, a product of connections, etc. The parallel transport map between tangent spaces will also decompose, giving us the desired decomposition in holonomy. □

Thanks to this proposition, if we find a way to classify holonomy groups for irreducible Riemannian manifolds, the classification for reducible ones will be immediate, since their holonomy groups will just be products of the already classified holonomy groups of irreducible manifolds (for a more general statement of this and a proof, see [Bes08, Thm. 10.38]). This is the reason we will ask that our manifold is irreducible in our classification theorem.

Now, let us define the other concept necessary for our classification theorem, that is, symmetry:

Definition 4.10. Let (M, g) be a Riemannian manifold. We say that M is a *symmetric space* if for every point $p \in M$ there exists an isometry $s_p : M \to M$ that is an involution (that is, $s_s \circ s_p = \mathrm{id}_M$, the identity on M) and that p is an isolated fixed point of s_p (that is, $s_p(p) = p$ and there is a neighbourhood U of p such that the only fixed point of s_p in U is p).

Examples of these spaces are $\mathbb{R}^n$ (with the isometry s_p being the point symmetry with respect to p) or the sphere S^n (with the isometry being a rotation of π degrees along the axis defined by the point and its antipodal point). Symmetric spaces have been widely studied, and they can even be classified. We will not dabble much in these spaces, but we will give the following result (which, actually, it is an equivalent definition):

Proposition 4.11. *Let (M, g) be a connected, simply-connected, symmetric space. Let G be the group of isometries of (M, g) generated by elements of the form $s_q \circ s_p$ for $q, p \in M$. Then G is a connected Lie group acting transitively on M. If we fix $p \in M$, and we denote by H the subgroup of G fixing p, then H is a closed, connected Lie subgroup of G, and $M = G/H$ Moreover, H is the holonomy group of M.*

See [Bes08, Thm. 10.72, Prop. 10.79] for a proof. The classification of the holonomy groups then becomes a problem that can be solved using the theory of Lie groups, see [Hel01, Chapter 10] for the complete classification and a deep study of symmetric spaces in general, or [Bes08, Chapter 10 §K] for a summarized table of the classification. Basically what we do is first classify symmetric spaces: write $M = G/H$ as in the previous proposition and we have a finite amount of possibilities for these G, H. Since H is the holonomy group of M, the holonomy groups are automatically classified.

We say that a Riemmannian manifold (M, g) is *locally symmetric* if every point $p \in M$ has an open neighbourhood $U_p \subseteq M$ and an isometry $s_p : U_p \to U_p$ that is an involution and has a unique fixed point p. We say that (M, g) is *nonsymmetric* if it is not locally symmetric.

We are not interested in symmetric spaces, because if we considered them, then our classification theorem would have many more entries, and their classification belongs to the theory of Lie groups and not necessarily to the complex geometry that we are studying.

One last thing before our classification theorem: We will ask that our manifold M is simply connected, since then the holonomy group is connected. For non-connected holonomy groups, one checks that their connected components are holonomy groups of simply connected manifolds, which we will have already classified.

Finally we can introduce the classification theorem:

Theorem 4.12 (Berger's classification Theorem). *Let (M, g) be a simply connected Riemannian manifold of dimension n that is irreducible and nonsymmetric. Then exactly one of the following seven cases holds:*

1. $\mathrm{Hol}(g) = \mathrm{SO}(n)$,
2. $n = 2m$ *with* $m \geq 2$, *and* $\mathrm{Hol}(g) = \mathrm{U}(m)$ *in* $\mathrm{SO}(2m)$,
3. $n = 2m$ *with* $m \geq 2$, *and* $\mathrm{Hol}(g) = \mathrm{SU}(m)$ *in* $\mathrm{SO}(2m)$,
4. $n = 4m$ *with* $m \geq 2$, *and* $\mathrm{Hol}(g) = \mathrm{Sp}(m)$ *in* $\mathrm{SO}(4m)$,
5. $n = 4m$ *with* $m \geq 2$, *and* $\mathrm{Hol}(g) = \mathrm{Sp}(m)\,\mathrm{Sp}(1)$ *in* $\mathrm{SO}(4m)$,
6. $n = 7$ *and* $\mathrm{Hol}(g) = G_2$ *in* $\mathrm{SO}(7)$, *or*
7. $n = 8$ *and* $\mathrm{Hol}(g) = \mathrm{Spin}(7)$ *in* $\mathrm{SO}(8)$.

Here we asked in several occasions that $m \geq 2$ to avoid repetitions, since for example $\mathrm{SO}(2) = \mathrm{U}(1)$.

This classification theorem was first given in 1955 by Berger in [Ber55]. It is now known that there are examples of metrics in each of the categories. Let us take a deep look at each of the possibilities:

1. One can check that an orientable Riemannian n-manifolds will always have holonomy inside $\mathrm{SO}(n)$, and that a Riemannian manifold satisfying the theorem's hypothesis is orientable. Therefore $\mathrm{SO}(n)$ is the general class for manifold in this classification.

2. It can be proven that metrics with $\mathrm{Hol}(g) \subseteq \mathrm{U}(m)$ are automatically Kähler metrics, and we will do so later. Here $\mathrm{U}(m)$ refers to the unitary group, the group of unitary $n \times n$ matrices; a unitary matrix is a complex matrix whose conjugate transpose is also its inverse. These are also the isometries of $\mathbb{C}^m$ with its usual hermitian product. With the usual identification $\mathbb{C}^m \simeq \mathbb{R}^{2m}$ we obtain $\mathrm{U}(m) \subseteq \mathrm{SO}(2m)$, and in particular Kähler metrics are Riemannian metrics (as we already knew). Through holonomy groups, we have found another characterization of Kähler manifolds: a Kähler manifold is a Riemannian $2n$-manifold (M, g) with $\mathrm{Hol}(g) \subseteq \mathrm{U}(n)$.

3. Metric with $\mathrm{Hol}(g) \subseteq \mathrm{SU}(n)$ are called *Calabi-Yau* metrics. Here SU denotes the special unitary group, the group of $n \times n$ unitary matrices with determinant 1, equivalently, they are the orientation-preserving isometries of $\mathbb{C}^n$. Since special unitary matrices are unitary, we have $\mathrm{SU}(n) \subseteq \mathrm{U}(n)$ and in particular, every Calabi-Yau metric is a Kähler metric.

4. Metrics with $\mathrm{Hol}(g) \subseteq \mathrm{Sp}(m)$ are called *Hyperkähler metrics*. The group $\mathrm{Sp}(m)$ is the symplectic compact group and it is defined as $\mathrm{Sp}(m) = \mathrm{Sp}(2m, \mathbb{C}) \cap \mathrm{SU}(2m)$; where $\mathrm{Sp}(2m, \mathbb{C})$ is the symplectic group defined as the group of matrices that preserve the symplectic form of a symplectic vector space. Equivalently

$$\mathrm{Sp}(2m, \mathbb{C}) = \{M \in M_{2m \times 2m}(\mathbb{C}) | M^T \Omega M = \Omega\} \text{ where } \Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

   and I_n is the identity matrix. These can also be characterised as the isometries of the quaternions $\mathbb{H}$ with determinat 1. By definition, $\mathrm{Sp}(m) \subseteq \mathrm{SU}(2m) \subseteq \mathrm{U}(2m)$ so in particular Hyperkähler metrics will always be Calabi-Yau and Kähler metrics. We will later focus on Calabi-Yau and Hyperkähler metrics.

5. Metrics with $\mathrm{Hol}(g) \subseteq \mathrm{Sp}(m)\,\mathrm{Sp}(1)$ are called *quaternionic Kähler metrics*. We identify $\mathrm{Sp}(1)$ with the subset of $\mathbb{H}$ of quaternions of norm 1 and $\mathrm{Sp}(m)$ as with the isometries with determinant 1 of a quaternionic vector space of dimension m.

   Then $\mathrm{Sp}(m)\,\mathrm{Sp}(1)$ is just the group of elements $(\beta, B) \in \mathrm{Sp}(m) \times \mathrm{Sp}(1)$ acting on $\alpha \in \mathbb{H}^m$ by scalar multiplication by $\beta \in \mathrm{Sp}(1) \subseteq \mathbb{H}$ on the right and matrix multiplication by B on the left. These metrics are in fact not Kähler, so we will not focus much on their study. However, they are very related to twistor spaces, so they can still be studied within complex geometry.

6. Let us do both last cases together. These two types of holonomy are called exceptional holonomies since they are not very related to the rest, and one cannot study them through

complex geometry. The notation G_2 refers to an exceptional Lie group (from the classification of simple Lie groups) and $\mathrm{Spin}(n)$ refers to the the spin group. To define it, one checks that $\mathrm{SO}(n)$ has, for $n \geq 3$, homology $H_1(\mathrm{SO}(n)) \simeq \mathbb{Z}_2$, so it admits a 2-covering, which we define to be $\mathrm{Spin}(n)$.

One might note the relation of this list to the four division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ (where $\mathbb{O}$ are the octionions). As it turns out, one can interpret $\mathrm{Sp}(m)\,\mathrm{Sp}(1)$ as isometries of $\mathbb{H}^m$, G_2 as the group of automorphisms of $\mathbb{O}$ with determinant 1 and $\mathrm{Spin}(7)$ as the group of automorphisms of $\mathbb{O}$. So in Berger's list we have various groups of automorphisms of the several division algebras that preserve their inner product ($\mathrm{U}(m)$, $\mathrm{Sp}(m)\,\mathrm{Sp}(1)$, $\mathrm{Spin}(7)$) and their subgroups of determinant 1 ($\mathrm{SO}(n)$, $\mathrm{Sp}(m)$, G_2).

We have arrived at our third and final equivalent definition of Kähler manifolds. We now provide a schematic proof, since checking all intermediate results would take too long and it is not excessively interesting.

First of all we need something called the *holonomy principle*:

Lemma 4.13 (Holonomy principle)**.** *Let (M, g) be a Riemannian manifold and let $x \in M$. Then $\mathrm{Hol}_x(g)$ acts naturally on the tensor bundle $(T_xM)^{\otimes k} \otimes (T_xM^*)^{\otimes l}$. If α_x is an $\mathrm{Hol}_x(M, g)$-invariant tensor (that is, if $P_\gamma(\alpha_x) = \alpha_x$ for each $P_\gamma \in \mathrm{Hol}_x(g)$) then α_x can be extended to a parallel tensor field α over M.*

A proof of this result can be found in [Joy07, Prop. 2.5.2] but essentially one defines α_y for $y \in M$ as $P_\gamma(\alpha_x)$ for a curve γ between x and y. This definition does not depend on the curve since if $\tilde{\gamma}$ is a different curve from x to y, then $\tilde{\gamma}^{-1} \circ \gamma$ is a piecewise smooth loop based on x so $P_{\tilde{\gamma}^{-1}\circ\gamma} = P_{\tilde{\gamma}}^{-1} \circ P_\gamma \in \mathrm{Hol}_x(g)$ and therefore α_x is invariant by it. So we have $P_{\tilde{\gamma}}(\alpha_x) = P_{\tilde{\gamma}} \circ P_{\tilde{\gamma}}^{-1} \circ P_\gamma(\alpha_x) = P_\gamma(\alpha_x)$. One then checks that it is indeed parallel, using that α is parallel if and only if $P_\gamma(\alpha_x) = \alpha_y$.

This holonomy principle goes both ways: it is actually a bijective correspondence between parallel tensor fields and tensors on T_pM invariant under $\mathrm{Hol}_x(g)$. Indeed if a tensor is parallel, for each curve γ between two points x, y we have by definition of parallel transport that $P_\gamma(\alpha_x) = \alpha_y$. Then if γ is a closed loop based on x, we will have $P_\gamma(\alpha_x) = \alpha_x$ so α_x is invariant under $\mathrm{Hol}_x(g)$.

Now we also need this standard result from differential geometry:

Lemma 4.14. *Let (M, g) be a Riemannian manifold and let ∇ be a connection on M. Then ∇ is compatible with the metric $g \iff g$ is parallel $\iff$ parallel transportation P_γ along a curve γ is an isometry for each curve γ on M.*

In particular for a closed loop γ based at x, the elements $P_\gamma \in \mathrm{O}(T_pM) \sim \mathrm{O}(n) \subseteq \mathrm{GL}(n, \mathbb{R})$ so the holonomy group of a Riemannian manifold is always in $\mathrm{O}(n)$. The proof is not hard, it can be found in [dC92, Prop. 3.2, Cor. 3.3].

Finally let us give one last important lemma. Let (M, g) be a Hermitian manifold. The integrable almost complex structure I of M is a section of the bundle of Endomorphisms $\mathrm{End}(TM)$. We if we consider an endomorphism to be a $\binom{1}{1}$-tensor, then we also have a connection on $\mathrm{End}(TM)$ since a linear connection induces a connection on all tensor bundles.

Lemma 4.15. *Let (M, g) be a Kähler manifold of dimension n. Then the induced integrable almost complex structure I and the the fundamental form ω are parallel.*

This result requires quite a bit of effort so we won't proof it here. See all of §4.A of [Huy05] for these results.

Now we already have everything we need in order to provide a sketch of the proof of the following proposition:

Proposition 4.16. *Let* (M, g) *be a Riemannian manifold of dimension* $2n$*. Then the metric* g *is Kähler if and only if* $\mathrm{Hol}(g) \in \mathrm{U}(n)$*.*

Proof. Let us first suppose that g is Kähler. Let I be the integrable almost complex structure of M and let $x \in M$.

Now we know thanks to Lemma 4.15 that I is parallel and thanks to Lemma 4.13 we know that I_x must be preserved by the action of elements $P_\gamma \in \mathrm{Hol}_x(g)$ and since $I_x \in \mathrm{GL}(n, \mathbb{C})$ the elements P_γ must also be in $\mathrm{GL}(n, \mathbb{C})$.

Therefore $\mathrm{Hol}(g) \subseteq \mathrm{GL}(n, \mathbb{C})$ and since by Lemma 4.14 we know $\mathrm{Hol}(g) \subseteq \mathrm{O}(2n)$, we conclude that $\mathrm{Hol}(g) \subseteq \mathrm{GL}(n, \mathbb{C}) \cap \mathrm{O}(2n) = \mathrm{U}(n)$.

Now we suppose that $\mathrm{Hol}(g) \subseteq \mathrm{U}(n)$. If we consider $\mathrm{U}(n)$ a group acting on $V = \mathbb{C}^n$ and by extension on its tensor products $V^{\otimes k} \otimes (V^*)^{\otimes l}$ (a linear map between vector spaces can always be extended into a linear map between their tensor powers) we note that $\mathrm{U}(n)$ preserves an element $J \in \mathrm{End}(\mathbb{C}^n)$ that satisfies $J^2 = \mathrm{id}$. This element is the diagonal matrix $J = i\,\mathrm{id}$ and it is the only one fixed by the whole $\mathrm{U}(n)$.

Recall that our representation of the holonomy group came from taking a representation of $\mathrm{End}(T_pM) \simeq \mathrm{GL}(2n, \mathbb{R})$. Choose $x \in M$ and then pick an element in $I_x \in \mathrm{End}(T_xM)$ represented by $J \in \mathrm{End}(\mathbb{C}^n)$, using the aforementioned representation. Since $J^2 = -\mathrm{id}$, we also have $I_x^2 = -\mathrm{id}$. Since elements $P_\gamma \in \mathrm{Hol}_x(g)$ are in particular in $\mathrm{U}(n)$, they preserve this element I_x when they act on $\mathrm{End}(T_xM)$.

Therefore thanks to Lemma 4.13 I_x extends to a parallel section I of $\mathrm{End}(TM)$. Since the identity is also parallel and the maps P_γ are linear and preserve I we have that globally $I^2 = -\mathrm{id}$.

Since I is parallel and by Lemma 4.14 the metric is parallel too, $\omega = g(I(\_), \_)$ will also be parallel. One can proof through a small calculation that any parallel form (with respect to a torsion free connection) is closed (see [Huy05, Prop. 4.A.4]) so in particular ω is closed.

What remains to be seen is that the almost complex structure I we defined is in fact integrable and that it is compatible with the metric. We won't do this here but the argument follows from the Newlander-Nirenberg Theorem, but with a different statement than the one we gave in Theorem 2.33 that involves the Nihenjuis tensor, which determines whether an almost complex structure is integrable (see [Bes08, 2.11, 2.28]). □

Note that if $\mathrm{Hol}(g) = \mathrm{U}(n)$ then the only complex structure $J \in \mathbb{C}$ fixed by the totality of the holonomy group is $J = i\,\mathrm{id}$ and so we get that the complex structure in our Kähler manifold is unique.

We would like to find an example of this, that is, a Kähler n-manifold with general holonomy $\mathrm{U}(n)$. Unfortunately both examples of compact Kähler manifolds that we have previously given can't be classified using this classification. Complex tori of dimension n are not simply connected (their underlying smooth manifold is a torus T^{2n} whose first Homology group known to be $\mathbb{Z}^{2n}$), and although the complex projective space $\mathbb{P}^n_{\mathbb{C}}$ is indeed simply-connected, it is a symmetric space. However, since they are Kähler we know that their holonomy is contained in $\mathrm{U}(n)$.

Example 4.17. We can nevertheless give a quick example of a Kähler manifold with general holonomy group. Consider in $\mathbb{P}^n_{\mathbb{C}}$ the zero loci of an homogeneous polynomial of degree $d \neq n+1$. This is called a hypersurface of degree d, and they are simply connected for $n \geq 3$. They are projective varieties so they are Kähler (with the metric g being the induces Fubini-Study metric) and in general they don't admit involutions (so they are not symmetric spaces) and therefore they can be classified with Berger's classification. We can calculate their canonical bundle using the theory of line bundles, and we check that it is not trivial (which means that the metric is not Calabi-Yau, see section 4.4) so they must have holonomy $\mathrm{U}(n)$.

The arguments used in the proof of Proposition 4.16 are standard in holonomy theory. Through the same arguments, for instance, one can prove the following results:

Corollary 4.18. *Let (M, g) be a Riemannian n-manifold. Then M is oriented if and only if* $\mathrm{Hol}(g) \subseteq \mathrm{SO}(n)$.

Proof. To proof this one notes that $\mathrm{SO}(n)$ is a group acting on $\mathbb{R}^n$ that preserves a $\binom{0}{2n}$ alternating tensor. This gets translated to a non-vanishing top degree form on M (if it vanishes at some point then the parallel transport map would not be an isomorphism between tangent spaces) which is also an orientation form. □

Corollary 4.19. *Let (X, g) be a Hermitian manifold of complex dimension n. Then* $\mathrm{Hol}(g) \subseteq \mathrm{U}(n)$ *if and only if there exist a nowhere-vanishing holomorphic n-form on X.*

Proof. The proof is very similar to the previous one. It can be checked that $\mathrm{SU}(n)$ acting on $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ preserves a $\binom{0}{n}$-tensor. This translates to a nowhere-vanishing n-form on the Riemannian $2n$-manifold M, which can be proven to automatically be holomorphic. □

This nowhere vanishing holomorphic n-form can be considered a volume form for the holomorphic tangent bundle. This will be the case of Calabi-Yau manifolds.

Corollary 4.20. *A Riemannian $4n$-manifold (M, g) has* $\mathrm{Hol}(g) \subseteq \mathrm{Sp}(n)$ *if and only if M has three different integrable almost complex structures I, J, K that satisfy $IJ = K$.*

Proof. In a similar manner to the previous results we note that there are three matrices $I, J, K \in \mathrm{End}(\mathbb{C}^{2n})$ that satisfy $I^2 = J^2 = K^2$ and $IJ = K$ and are left invariant by the action of $\mathrm{Sp}(n)$. □

One can proof that the three almost complex structures I, J, K are actually Kähler (meaning that they are integrable, compatible with the metric g and each induces a closed fundamental form). This is done by noting that I, J, K are left invariant by the holonomy group so they are parallel. Then one uses the same arguments used to prove that a Riemannian manifold with $\mathrm{Hol}(g) \subseteq \mathrm{U}(n)$ is in fact Kähler. This will be the case of Hyperkähler manifolds.

We have finally given the three different definitions of Kähler manifolds and we have seen that they are all equivalent. This last third definition through holonomy groups has opened a path to study more complex objects: Calabi-Yau and Hyperkähler manifolds. Ignoring them after having just encountered them would leave a feeling of incompleteness so we will briefly make their acquaintance.

4.3 Einstein and Ricci-Flat manifolds

We will first introduce a new type of manifolds, called Ricci-flat manifolds. We will then use them to define Calabi-Yau manifolds, but they are interesting on their own. First of all, we need the concept of Ricci curvature of a Riemannian manifold, and before that, the concept of curvature:

Definition 4.21. Let (M, g) be a Riemannian manifold. We define the *curvature endomorphism* $R' : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by $R'(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$.

Thanks to the characterization of tensors as multilinear morphisms ([Lee97, Lem. 2.4]) we can check that this curvature endomorphism defines a $\binom{3}{1}$-tensor field ([Lee97, Prop. 7.1]) which we denote by R. In local coordinates we write $R = R_{ijk}{}^l dx^i \otimes dx^j \otimes dx^k \otimes \partial_l$. Now we will lower the index l through the musical isomorphism (see [Lee13, p. 341]) as follows:

Definition 4.22. Let (M, g) be a Riemannian manifold. We define the *(Riemann) curvature tensor* as the covariant 4-tensor field $Rm = R^\flat$ where we lowered its last index. It acts on vector fields as follows: $Rm(X, Y, Z, W) = g(R(X, Y)Z, W)$. In local coordinates we write $Rm = Rm_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l$ with $R_{ijkl} = g_{lm} R_{ijk}{}^m$.

Here we have also written the metric in local coordinates, $g = g_{ij} dx^i dx^j$ and as usual all repeated indexes indicate a sum. Although we will not really need this definition, this section would certainly feel incomplete without it.

We can always contract an $\binom{n}{l}$-tensor to obtain an $\binom{n-1}{l-1}$-tensor ([Lee97, p. 13]) and we do precisely that to define the Ricci curvature tensor:

Definition 4.23. Let (M, g) be a Riemannian manifold with Riemannian curvature tensor Rm and curvature endomorphism R'. The *Ricci curvature tensor* Rc is the covariant 2-tensor defined as the trace of the curvature endomorphism on its first and last indices, that is, the contraction of their first and last index. Its components are $R_{ij} := R_{kij}{}^k = g^{km} R_{kijm}$.

The *curvature* S of M is then defined as the trace of the Ricci tensor, that is, $S := g^{ij} R_{ij}$.

It turns out that the Ricci curvature tensor is symmetric ([Lee97, Lem. 7.6]) so one might ask what is its relation with the metric. In general, they bear no relation, but when they do, we have a special kind of manifold:

Definition 4.24. Let (M, g) be a Riemannian manifold. We say that g is a *Einstein metric* if its Ricci tensor Rc is a scalar multiple of the metric at each point, that is, if we have a smooth function λ such that $Rc = \lambda g$ at each point of M.

A quick computation (taking traces on both sides of $Rc = g\lambda$) shows that if such λ exists, then $\lambda = \frac{S}{\dim M}$, where S is the curvature of M. Moreover, one can proof (see [Lee97, Prop. 7.8]) that when such λ exists, then S is in fact a constant. This allows the following definition:

Definition 4.25. Let (M, g) be a Riemannian manifold and let Rc be its Ricci curvature tensor. We say that (M, g) is an *Einstein manifold* if $Rc = kg$ for some constant k.

Thanks to the previous discussion, this is equivalent to impose that the Riemannian metric g is an Einstein metric. There is a special case, $k = 0$:

Definition 4.26. Let (M, g) be a Riemannian manifold and let Rc be its Ricci curvature tensor. We say that (M, g) is Ricci-flat if $Rc \equiv 0$.

In particular, Ricci-flat manifolds are Einstein manifolds. For a deep study of Einstein manifolds, see [Bes08].

Now we would like to extend our notion of Ricci curvature to Kähler manifolds.

Definition 4.27. Let (X, g) be a Kähler manifold, let ω be its fundamental form, let M be the underlying smooth manifold, let I be its induced integrable almost complex structure, and let Rc be the Ricci curvature of (M, g). The *Ricci form* $\mathrm{Ric}(X, g)$ of (X, g) is the real two-form defined as $\mathrm{Ric}(u, v) = Rc(I(u), v)$. The Kähler metric is called *Kähler-Einstein* if $\mathrm{Ric} = \lambda\omega$ for some constant λ (this is in fact equivalent to asking that our if Kähler metric g is also Einstein).

The Kähler metric is called Ricci-flat if $\mathrm{Ric}(X, g) = 0$. Again, this is equivalent to (M, g) being Ricci flat.

Now, why have we gone through all the trouble of defining curvature and these two new types of manifolds? It turns out that the Ricci-flat condition is tied to the holonomy group. Any metric g with holonomy group $\mathrm{Hol}(g) \subseteq \mathrm{SU}(m)$ is Ricci flat. In particular, Calabi-Yau metrics and Hyperkähler metrics are Ricci-Flat (see [Bes08, Prop. 10.29] or [Huy05, Prop 4.A.18] for a proof).

Let us now take a deeper look at Calabi-Yau and Hyperkähler metrics.

4.4 Calabi-Yau manifolds

Let us now look at a new type of manifold, the Calabi-Yau manifold. These are fairly recent and they play a very important roles in physics, specially in topics of string theory. Perhaps due to this, there are several inequivalent definitions of a Calabi-Yau manifold. We will present two of them, giving a couple of characterizations for each.

First of all we need to give a brief introduction to Chern classes. Chern classes are a particular type of characteristic classes. A characteristic class is just a functor that associates a class in cohomology of a manifold to every vector bundle over said manifold.

In order to define them we must first extend our notion of the curvature tensor of a Riemannian metric to arbitrary connections on a vector bundle E over our smooth manifold M. We define the sheaf $\mathcal{A}^p(E)$ as $\mathcal{A}^p(U, E) := \Gamma(U, \bigwedge^p(TM^*) \otimes E)$ for open sets $U \subseteq M$. In particular $\mathcal{A}^0(M, E) = \Gamma(E)$. There is an alternative characterization, we can consider an element $\alpha \otimes \varepsilon \in \mathcal{A}^p(M, E)$ as the $\mathcal{A}^0(M)$-multilinear map $\alpha \otimes \varepsilon : TM \times \cdots \times TM \to E$ defined as $\alpha \otimes \varepsilon(X_1, \ldots, X_p) = \alpha(X_1, \ldots, X_p) \cdot \varepsilon$.

As we have mentioned a connection ∇ on M defines a map $\nabla : \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M)$. This idea can be extended to connections on an arbitrary bundle, giving us a map $\nabla : \mathcal{A}^k(M, E) \to \mathcal{A}^{k+1}(M, E)$.

Then the *curvature endomorphism* is simply defined as $F_\nabla := \nabla \circ \nabla : \mathcal{A}^0(M, E) \to \mathcal{A}^2(M, E)$. Through a quick calculation (which can be found in [Huy05, p. 211]), we can show that this concept agrees with our previous definition of curvature for linear connections.

One can also check that $F_\nabla \in \mathcal{A}^2(M, \mathrm{End}(E))$. This is due to the fact that curvature will take a vector field ε of E and return an element $\alpha \otimes \varepsilon'$ where α is a two form (that depends on ε) and ε' is another vector field on E. Then we consider $F_\nabla : TM \times TM \to \mathrm{End}(E)$ as the

$\mathcal{A}^0(M)$ multilinear map that sends (X,Y) to the endomorphism φ_{XY} of E which we define as $\varphi_{XY}(\varepsilon)=\alpha(X,Y)\cdot\varepsilon'$.

We now consider homogeneous polynomials. Let V be a complex vector space. There is a correspondence between k-multilinear symmetric maps $P:V\times\cdots\times V\to\mathbb{C}$ and homogeneous polynomials $\tilde{P}$ of degree k given by $\tilde{P}(B)=P(B,\dots,B)$.

We take $V=\mathfrak{gl}(r,\mathbb{C})$ that is, the Lie algebra associated to the Lie group $\mathrm{GL}(r,\mathbb{C})$. One can interpret $\mathfrak{gl}(r,\mathbb{C})$ as the space of $r\times r$-matrices with complex coefficients, with their usual complex vector space structure (and with the commutator being the Lie Bracket). The group $\mathrm{GL}(r,\mathbb{C})$ acts on $\mathfrak{gl}(r,\mathbb{C})$ through usual matrix multiplication.

Definition 4.28. A symmetric multilinear map $P:\mathfrak{gl}(r,\mathbb{C})\times\cdots\times\mathfrak{gl}(r,\mathbb{C})\to\mathbb{C}$ is called *invariant* if

$$P(CB_1C^{-1},\dots,CB_kC^{-1})=P(B_1,\dots,B_k).$$

for all $B_1,\dots,B_k\in GL(r,\mathbb{C})$ and all $C\in\mathfrak{gl}(r,\mathbb{C})$.

We can express this condition on the corresponding homogeneous polynomial $\tilde{P}$ as follows: $\tilde{P}$ is invariant if and only if $\tilde{P}(CBC^{-1})=\tilde{P}(B)$ for all $B\in\mathrm{GL}(r,\mathbb{C})$ and all $C\in\mathfrak{gl}(r,\mathbb{C})$.

It turns out that for a symmetric multilinear invariant k-form on $\mathfrak{gl}(r,\mathbb{C})$ (that is, a map P as in the previous definition), a given a vector bundle E of rank r over a smooth manifold M, a natural number $m\geq k$ and a partition $m=i_1+\cdots+i_k$ we can define a map

$$P:\left(\bigwedge^{i_1}(TM^*)\otimes\mathrm{End}(E)\right)\times\cdots\times\left(\bigwedge^{i_k}(TM^*)\otimes\mathrm{End}(E)\right)\to\bigwedge\nolimits_{\mathbb{C}}^{m}TM^*$$

given by $P(\alpha_1\otimes\varphi_1,\dots,\alpha_k\otimes\varphi_k)=(\alpha_1\wedge\cdots\wedge\alpha_k)P(\varphi_1,\dots,\varphi_k)$. We can do this thanks to the fact that (choosing a trivialisation of E) the fibre $E_x\simeq\mathbb{C}^r$ so $\mathrm{End}(E)\simeq\mathfrak{gl}(r,\mathbb{C})$. Since P is invariant, this definition does not depend on the chosen trivialisation.

In particular we will focus on the case $m=2k=2+\overset{k}{\dots}+2$. The map on P defined in this previous way will induce a map on the level of global sections (since it will be defined on each fibre) so we will have a map

$$P:\mathcal{A}^2(M,\mathrm{End}(E))\times\overset{k}{\dots}\times\mathcal{A}^2(M,\mathrm{End}(E))\to\mathcal{A}^{2k}_{\mathbb{C}}(M)$$

For $\alpha\otimes\varphi\in\mathcal{A}^2(,\mathrm{End}(E))$ we define $\tilde{P}(\alpha\otimes\varphi):=P(\alpha\otimes\varphi,\dots,\alpha\otimes\varphi)$.

Now one then checks that for a vector bundle E of rank r over a smooth manifold M and for a connection ∇ on E and a homogeneous symmetrical invariant polynomial P of degree k, the form $\tilde{P}(F_\nabla)\in\mathcal{A}^{2k}_{\mathbb{C}}(M)$ induced by the curvature tensor $F_\nabla\in\mathcal{A}^2(M,\mathrm{End}(E))$ is d-closed (see [Huy05, Cor. 4.4.5]) so we can consider its class in cohomology. Moreover, if ∇' is another connection in E, $\tilde{P}(F_\nabla)$ and $\tilde{P}(F_{\nabla'})$ define the same class in cohomology ([Huy05, Lem. 4.4.6]), so the induced class in cohomology depends only on the vector bundle and not on the connection we choose on it (since every vector bundle admits a connection; see [Lee97, Prop. 4.5] for a proof for linear connections but the general case use the same argument).

So by giving an invariant, homogeneous polynomial of rank k we have a way to assign cohomology classes in $H^{2k}(M,\mathbb{C})$ to vector bundles. A choice of polynomial is what is known as a *characteristic class*.

For example, let $\tilde{P}_k$ be the polynomial with $\deg(\tilde{P}_k)=k$ be defined as $\det(\mathrm{id}+B)=1+\tilde{P}_1(B)+\cdots+\tilde{P}_r(B)$ where $B\in\mathfrak{gl}(r,\mathbb{C})$. These are invariant, since the determinant is invariant by changes of basis.

Definition 4.29. The *k-th Chern form* of a vector bundle $\pi : E \to M$ of rank r endowed with a connection ∇ are defined as $c_k(E, \nabla) := \tilde{P}_k\left(\frac{i}{2\pi} F_\nabla\right)$, for $k = 1, \dots, r$.

Then the k-th Chern class of E is just $c_k(E) := [c_k(E, \nabla)] \in H^{2k}(M, \mathbb{C})$. As we mentioned, they do not depend on the chosen connection.

The k-th Chern class of a manifold M is just the k-th Chern class of its holomorphic tangent bundle (which we consider a complex vector bundle). We denote it $c_k(M)$.

Now Chern classes, fundamental forms and Ricci curvature are deeply tied to each other in Kähler manifolds. This next result, kown as Calabi's conjecture, was first postulated by Calabi in 1954 and then proven by Yau in 1977. This will help us construct Calabi-Yau manifolds (hence, the name).

Theorem 4.30 (Yau)**.** *Let (X, g) be a compact Kähler manifold, let M be the underlying Riemannian manifold, let ω be the fundamental form of g and let $c_1(M)$ be the first Chern class of M. Suppose that ρ is a real, closed $(1,1)$-form with $[\rho] = 2\pi c_1(M)$. Then there exists a unique Riemannian metric g' on M with fundamental form ω' such that $[\omega] = [\omega'] \in H^2(X, \mathbb{R})$ and the Ricci form $\mathrm{Ric}(X, g')$ of g' is ρ.*

In particular if the first Chern class vanishes one chooses $\rho = 0$ and we obtain a metric g' with $\mathrm{Ric}(X, g') = 0$ which makes (X, g) into a Ricci-flat manifold.

A proof of the theorem can be found in [Joy07, Section 6].

Now we are finally ready to give the definition of a Calabi-Yau manifold. Unfortunately, many different, non-equivalent definitions appear on the literature. We will present the two most common ones.

Definition 4.31 (Calabi-Yau manifold, weak version)**.** Let (X, g) be a compact Kähler manifold of dimension n. We say that X a Calabi-Yau manifold if X is Ricci-flat.

Thanks to the Yau theorem it is easy to construct such manifolds. One only needs a Kähler manifold with vanishing first Chern class which, thanks to the theorem, will have a metric that makes it Ricci-flat and therefore Calabi-Yau. In fact, it can be proven that if X is Ricci-flat then its first Chern class vanishes, so asking that X has a vanishing first Chern class is an equivalent characterization of Calabi-Yau manifolds.

Now we present our second definition, which is more restrictive than the first.

Definition 4.32 (Calabi-Yau manifold, strong version)**.** Let (X, g) be a compact Kähler manifold of dimension n. We say that X is a Calabi-Yau manifold if $\mathrm{Hol}(g) \subseteq \mathrm{SU}(n)$.

So we only ask that our manifold has a Calabi-Yau metric. We saw in Corollary 4.19 that this was equivalent to having a nowhere-vanishing holomorphic n-form (where n is the dimension of the manifold). We also mentioned in Section 4.3 that this implied that X was Ricci-flat so this definition of Calabi-Yau is indeed stronger than the previous one.

There is another equivalent characterization which is often used: A Kähler manifold X is Calabi-Yau if and only if its canonical bundle is trivial. This is in fact easy to check, since a nowhere vanishing holomorphic n-form immediately gives a global trivialization of the canonical bundle where it lives. Other non-equivalent definitions include not asking for X to be compact or asking that $\mathrm{Hol}(g) = \mathrm{SU}(n)$, which excludes the manifolds with exceptional holonomy group and Hyperkäher manifolds.

Example 4.33. Before ending this section let us look at examples of Calabi-Yau n-manifolds X for small n, using our stronger version of the definition of the term. For $n = 1$ in particular X is a one dimensional complex manifold and therefore a Riemann surface. It is also a topological oriented 2-manifold, which we know are classified by their Euler characteristic. For a smooth surface M we have the Gauss-Bonnet theorem which relates the Euler characteristic $\chi(M)$ with the Gaussian curvature R as follows: $\int_M R = 2\pi\chi(M)$. A Calabi-Yau manifold will be Ricci-flat and therefore its curvature will also be 0. By the Gauss-Bonnet Theorem its Euler characteristic must also be 0, and by the classification theorem the underlying smooth manifold of X must be a torus, so X must be a complex torus of dimension 1, which are also usually called elliptic curves.

A reader familiar with elliptic curves will know that a complex torus of dimension 1 can be thought of as the set of solutions of a polynomial of degree three on two complex variables, and taking the projective completion of the space we can think of the complex torus as a cubic non-singular hypersurface inside $\mathbb{P}^2_{\mathbb{C}}$.

Example 4.34. Checking that a n-dimensional complex torus is a Calabi-Yau manifold is not hard, in the sense that its whole Riemann curvature tensor vanishes. However, they are not simply connected (as we mentioned before, their underlying smooth manifold is a torus T^{2n} which is not simply connected). Hence they are sometimes not considered as Calabi-Yau manifolds, as they don't belong to Berger's classification.

Example 4.35. For $n = 2$ we turn to Kodaira's classification of complex surfaces (see [Kod68]). The classification implies that the only complex surfaces that are Calabi-Yau manifolds are either K3 surfaces (if they are simply connected) or complex tori of dimension 2. K3 surfaces are defined as simply connected complex surfaces with trivial canonical bundle, so they are immediately Calabi-Yau. The most prominent examples of such surfaces are the quotient of a complex torus of dimension 2 by the (-1) involution (the so called Kummer surfaces whose generalizations to higher dimensions is one of the only examples of Hyperkähler manifold that we have been able to find) and non-singular quartic hypersurfaces inside $\mathbb{P}^3_{\mathbb{C}}$.

Since K3 surfaces can be considered also as Hyperkähler manifolds (note that $\mathrm{Sp}(1) \simeq \mathrm{SU}(2)$ and they have a holomorphic 2-form corresponding to the section of the trivial canonical bundle), they are again sometimes not considered as Calabi-Yau manifolds.

Example 4.36. This is why, in some areas of research, Calabi-Yau manifolds have at least dimension 3. Still, we notice a pattern in our last couple of examples: the zero-locus of a non-singular homogeneous polynomial of degree $d + 1$ inside $\mathbb{P}^d_{\mathbb{C}}$ (that is, with $d + 1$ variables) is a Calabi-Yau d-manifold for $d = 2, 3$. As it turns out, this is also true for general $d \geq 2$ (see [He18, Prop. 3] for a justification). Moreover, thanks to the Weak Lefschetz Theorem ([Huy05, Prop. 5.2.6]) starting from dimension 2 these hypersurfaces are simply connected and therefore they belong to Berger's classification. They are also compact, as they are closed subsets of the projective space (which is compact). Finally one could check that for $n \geq 3$ they are not in fact Hyperkähler.

Calabi-Yau manifolds play an important role in physics and in particular, in superstring theory where they are used to model the extra dimensions that are needed to get string theory to work, but that are not present in our macroscopical scale because they are "too curled up onto themselves". What this means is that string theory needs our universe to be 10-dimensional in order to make physical sense. These 10 dimensions are then broken up as the 4 in the usual Minkowsky space-time and 6 belonging to a Calabi-Yau 3-fold with a very small characteristic radius.

Calabi-Yau manifolds are a very interesting topic and appear often in very active research attached with many open questions such as Mirror Symmetry. We have only given their definition and we will not go any further, but more information can be found in sources such as [Bes08], [GHJ03] and [Voi96].

4.5 Hyperkähler manifolds

Now after this brief introduction to Calabi-Yau manifolds we will turn our attention to another type of manifold, the Hyperkähler manifold. We can immediately give their definition:

Definition 4.37. Let (M, g) be a compact Riemannian $4n$-manifold. We say that (M, g) is a *Hyperkähler manifold* of dimension $2n$ if the metric g is an Hyperkähler metric, that is, if $\mathrm{Hol}(g) \subseteq \mathrm{Sp}(n)$.

In particular since $\mathrm{Sp}(n) \subseteq \mathrm{U}(2n)$ an Hyperkähler manifold is Calabi-Yau and Ricci-flat. We have mentioned in Corollary 4.20 that $\mathrm{Hol}(g) \subset \mathrm{Sp}(n)$ implies that we have three integrable almost complex structures I, J, K on M satisfying $IJ = K$ (in fact, they are Kähler).

In fact one can easily check that if $I^2 = J^2 = K^2 = -1$ and $IJ = K$ then for $\alpha, \beta, \gamma \in \mathbb{R}$ with $\alpha^2 + \beta^2 + \gamma^2 = 1$ one obtains $(\alpha I + \beta J + \gamma K)^2 = -\alpha^2 - \beta^2 - \gamma^2 = -1$, so we not only have three isolated almost integrable structures, but a whole sphere S^2 of them. Moreover, an easy check shows that all these structures are Kähler. This sphere of Kähler structures is what gave Hyperkähler manifolds their name: they are Kähler in more ways than one.

There is another equivalent characterization of Hyperkähler manifolds. One notices that we can write $\mathrm{Sp}(n) = \mathrm{Sp}(2n, \mathbb{C}) \cap \mathrm{U}(2n)$ which can be interpreted to mean that $\mathrm{Sp}(n)$ preserves a symplectic complex 2-tensor when it acts on $\mathbb{H}^n \simeq \mathbb{C}^{2n}$. Through the holonomy principle (Lemma 4.13) this translates to an holomorphic non-degenerate 2-form on a Hyperkähler manifold. In fact one can check that if the fundamental forms associated to the integrable almost complex structures I, J, K are denoted by ω_I, ω_J and ω_K respectively, and the manifold is endowed with the complex structure given by I, then the holomorphic non-degenerate 2-form is $\omega := \omega_J + i\omega_K$. We give the following definition:

Definition 4.38. A pair (X, ω) where X is a complex manifold and ω is a holomorphic 2-form $\omega \in \mathcal{A}^{2,0}(X)$ that is also non-degenerate (in the sense that for every non-zero holomorphic vector field X there exists a holomorphic vector field Y such that $\omega(X, Y) \neq 0$) is called a *holomorphically symplectic manifold.*

This is in fact equivalent to our Kähler manifold X being Hyperkähler. Proving this relays on Yau's proof of the Calabi conjecture, so we will not do it here.

Examples of Hyperkähler manifolds of dimension 2 are not that hard to find, since $\mathrm{Sp}(1) \simeq \mathrm{SU}(2)$ Hyperkähler surfaces are just Calabi-Yau manifolds of dimension 2, which we have already discussed.

Examples of higher-dimensional Hyperkähler manifolds are notoriously hard to find. It can be proven that any given manifold only admits a finite number of Hyperkähler metrics (whereas Kähler manifolds admit infinitely many Kähler metrics) and that a submanifold of an

Hyperkähler manifold will not necessarily be Hyperkähler. In fact we only have two deformation families of examples in any even dimension and two sporadic deformation types in dimensions 6 and 10, which is not, by any means, a lot. See [GHJ03, Section 21.2] for a rundown of the known examples. Due to this lack of examples, one of the most active open topics of research regarding Hyperkählerian geometry is finding new families of examples which are non-equivalent by deformation to the known ones. It is not known whether such other families must exist.

Finally we would like to mention an open conjecture regarding irreducible holomorphically symplectic manifolds (that is, Hyperkähler manifolds that are irreducible in the sense of Definition 4.8). First we need to define Lagrangian fibrations.

Definition 4.39. Let X be a complex manifold and let B be a normal variety (this is a notion from algebraic geometry which allows B to have some "mild" singularities, but for all intents and purposes we can consider B to be a complex submanifold), such that $\dim B < \dim X$. An holomorphic map $\pi : X \to B$ is a *fibration* (or *connected map*) if all the fibers are connected.

Moreover, we say that the *general fiber* of $\pi : X \to B$ satisfies certain property if there exists an open set $U \subseteq B$ in the Zariski topology (again, this is a concept from algebraic geometry but we can think of it as the complement of the zero loci of an ideal of polynomials) such that for all $x \in U$ the fiber $\pi^{-1}(x)$ satisfies this property.

If X is a complex manifold (in particular, non-singular), then the general fiber of any fibration $\pi : X \to B$ will be non-singular. Since Zariski open sets are large (note that the Zariski topology is not Hausdorff), the set of singular fibers (understood ad the fibers that are not general) will be "small".

Definition 4.40. Let (X, ω) be an irreducible holomorphically symplectic manifold of dimension $2n$ and let $\pi : X \to B$ be a fibration onto a normal variety as above. The fibration π is called a *Lagrangian fibration* if the general fiber $\pi^{-1}(x) \subseteq X$ is a complex Lagrangian submanifold of X (meaning a complex submanifold such that ω restricted to $\pi^{-1}(x)$ vanishes for $x \in U$).

Now one can proof that actually fibrations on irreducible holomorphically symplectic manifolds are of this very special type:

Proposition 4.41. *Let (X, ω) be an irreducible holomorphic symplectic manifold of dimension $2n$, B a normal variety of positive dimension $k < 2n$ and $f : X \to B$ a fibration. Then f is a Lagrangian fibration. Moreover the general fiber is a complex torus of dimension n.*

This result was obtained by Matsushita in 1999 and can be found in [Mat01]. It characterises the non-singular fibers of a Lagrangian fibration. In contrast, the Lagrangian fibrations conjecture makes an statement about the base space B of our fibration $\pi : X \to B$.

Conjecture 4.42. Let X, B be as above and let $f : X \to B$ be a Lagrangian fibration. Then B is the projective space $\mathbb{P}^n$.

If it were true, this conjecture together with the previous proposition would give a very restrictive description of irreducible Hyperkähler $2n$-manifolds that admit Lagrangian fibrations. It has been proven that the Hodge numbers (that is, the dimensions of $H^{p,q}(B)$) of the base space B are the same as the ones for $\mathbb{P}^n$ (and this is enough to show that the conjecture holds for $n = 1$).

This conjecture has been shown to hold in some additional particular cases, such as in the case for when X and B are both assumed a priori to be projective and B is assumed to be non-singular (see [Hwa08]). It has also been proven in the general case (where we allow B to be singular) for $n = 2$ (see [HX19]). However the general case for $n > 2$ still remains open for now.

Not all irreducible holomorphically symplectic manifolds might admit a Lagrangian fibration, but there is a conjecture regarding which ones do (see the article [Ver10]), and others relating to how many manifolds that admit Lagrangian fibrations exist in a given deformation family. Many of these conjectures have been shown to hold on the known families but unfortunately we are not able to come up with a proof that doesn't rely on the way the specific families are constructed.

Hyperkähler manifolds are far from being well understood and they will surely remain an active topic of research for the time being.

5 Conclusions

Complex geometry is a hard but rewarding topic. It has surprised me, both in its seemingly infinite depth and its interplay with several other subjects of both physics and mathematics. I had never expected that classical mechanics systems could be described so simply by symplectic manifolds, or that symplectic forms are very related to complex geometry through Kähler manifolds. Or that Kähler and Hyperkähler manifolds had applications in supersymmetry theories. Or the deep relation of Hodge theory with functional analysis. My study of complex geometry also seemed a very natural extension of the courses on differential geometry I had previously taken which helped me build rather easily on my previous knowledge.

I would have very much liked to continue exploring these connections, especially in the topics of Mirror Symmetry and Hyperkähler and Calabi-Yau manifolds. This was in fact the original goal of this project, but as turns out, the background material necessary to understand such objects was both too interesting to not thoroughly study and too extensive to fit inside the limited space of this work. I would have also liked to take a look at supersymmetry and at how it relates to Kähler and Hyperkähler geometries (see [Huy05, Section 3.B]), and how all this fits into Quantum Field Theory.

As time went along, I started finding results that required a very broad understanding of the subject (mainly the ones regarding holonomy and Calabi-Yau manifolds, whose proofs required knowledge of algebraic geometry and Lie groups). I focused on understanding what I could, since often the arguments behind results were either nowhere to be found or too hard to even begin to tackle them (this mainly refers to results in the last two sections). I was finally starting to find myself in territory where I had to look for references in articles rather than in books, but I ran out of time and space before I could delve into the topics of Mirror symmetry, the tools used to study the Lagrangian fibrations conjecture and more results about Calabi-Yau manifolds. In particular, I would have liked to tackle texts such as [Voi96] or [Ver99], and maybe relating complex geometry to algebraic geometry more.

I will surely return to this topic to finish studying the things I couldn't get to this time around. Hopefully by then someone will have figured out some of these open questions, shining a bit more light onto this corner of mathematics.

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