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Anisotropic integral decomposition of depolarizing Mueller matrices

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Abstract: We propose a novel, computationally efficient integral decomposition of depolarizing Mueller matrices allowing for the obtainment of a nondepolarizing estimate, as well as for the determination of the elementary polarization properties, in terms of mean values and variances-covariances of their fluctuations, of a weakly anisotropic depolarizing medium. We illustrate the decomposition on experimental examples and compare its performance to those of alternative decompositions.

1. Introduction

The interpretation in physical terms of experimental Mueller matrices of complex media is one of the fundamental tasks of applied polarimetry. Algebraic decompositions of Mueller matrices constitute powerful tools for addressing this task [1] and are currently applied to various advanced areas of science and technology allowing for optical polarimetry characterization, such as microelectronics metrology [2] or biomedical diagnosis [3], to mention just two of them. The purpose of the present letter is to advance a novel, computationally efficient sum decomposition of depolarizing Mueller matrices. Belonging to the class of the so-called integral decompositions [4], it is based on the fluctuating medium phenomenological picture. Within this picture, depolarization, present in the experimental Mueller matrix and resulting from the partial or total loss of spatial or spectral coherence during the polarimetric experiment, is described as originating from the statistical fluctuations of the Jones matrix elements [5,6] or of the polarization properties of the medium [7,8].

2. Derivation and physical interpretation of the decomposition

Consider the general Jones matrix

$$\mathbf{J} = \begin{bmatrix} c_0 + c_1 & c_2 - ic_3 \\ c_2 + ic_3 & c_0 - c_1 \end{bmatrix} \tag{1}$$

parameterized by the components c_i, $i = 0, 1, 2, 3$, of its expansion on the Pauli basis [9] and assume that all components c_k, $k = 1, 2, 3$, but c_0 fluctuate, i.e. $c_k = c_{mk} + \Delta c_k$, where $c_{mk} = \langle c_k \rangle$ are the mean values of c_k and Δc_k, such that $\langle \Delta c_k \rangle = 0$, are their fluctuations or uncertainties (the brackets $\langle \ldots \rangle$ denote spatial or temporal averaging). Then $\mathbf{J}$ becomes the Jones generator [10] of a depolarizing Mueller matrix $\mathbf{M}$ whose coherency matrix $\mathbf{C}$ is given by [9]

$$\mathbf{C} = \langle \mathbf{c}\, \mathbf{c}^{+} \rangle = \mathbf{c}_m \mathbf{c}_m^{+} + \langle \boldsymbol{\Delta}\mathbf{c}\, \boldsymbol{\Delta}\mathbf{c}^{+} \rangle = \mathbf{C}_m + \boldsymbol{\Delta}\mathbf{C} \tag{2}$$

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where $\mathbf{C}_m$ and $\mathbf{\Delta C}$ are the mean and the fluctuating matrix components of $\mathbf{C}$, respectively (the superscript '+' denotes complex conjugate transpose). In Eq. (2)

$$\begin{aligned}\mathbf{c} &= \begin{bmatrix} c_0 & c_1 & c_2 & c_3 \end{bmatrix}^T = \mathbf{c}_m + \mathbf{\Delta c} \\ &= \begin{bmatrix} c_0 & c_{m1} & c_{m2} & c_{m3} \end{bmatrix}^T + \begin{bmatrix} 0 & \Delta c_1 & \Delta c_2 & \Delta c_3 \end{bmatrix}^T\end{aligned} \tag{3}$$

is the coherency vector [9,11] associated with the Jones generator $\mathbf{J}$, decomposed into its mean, $\mathbf{c}_m$, and fluctuating, $\mathbf{\Delta c}$, vector components (the superscript T stands for transpose). Substitution of Eq. (3) into Eq. (2) yields for the fluctuating part $\mathbf{\Delta C}$ of $\mathbf{C}$,

$$\mathbf{\Delta C} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \langle|\Delta c_1|^2\rangle & \langle\Delta c_1 \Delta c_2^*\rangle & \langle\Delta c_1 \Delta c_3^*\rangle \\ 0 & \langle\Delta c_2 \Delta c_1^*\rangle & \langle|\Delta c_2|^2\rangle & \langle\Delta c_2 \Delta c_3^*\rangle \\ 0 & \langle\Delta c_3 \Delta c_1^*\rangle & \langle\Delta c_3 \Delta c_2^*\rangle & \langle|\Delta c_3|^2\rangle \end{bmatrix} \tag{4}$$

By inverting the following expression [1]

$$\mathbf{C} = \frac{1}{4}\sum_{i,j} M_{ij}\mathbf{T}(\sigma_i \otimes \sigma_j)\,\mathbf{T}^{-1} \quad i,j = 0,\ 1,\ 2,\ 3 \tag{5a}$$

for the coherency matrix $\mathbf{C}$ in terms of the elements M_{ij} of its associated Mueller matrix $\mathbf{M}$ where the transformation matrix $\mathbf{T}$ is

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \end{bmatrix} \tag{5b}$$

and σ_i are the Pauli spin matrices ($\otimes$ is the Kronecker product), Eq. (2) transforms into its Mueller matrix counterpart [4,6],

$$\mathbf{M} = \mathbf{M}_m + \mathbf{\Delta M} \tag{6}$$

Notice that Eq. (6) could be also obtained directly by substituting the Jones generator $\mathbf{J}$ from Eq. (1) into the general expression $\mathbf{M} = \mathbf{T}\langle\mathbf{J}\otimes\mathbf{J}^*\rangle\mathbf{T}^{-1}$ for the depolarizing Mueller matrix $\mathbf{M}$ generated by the fluctuating $\mathbf{J}$ [5,10].

Equation (6) is nothing but a special integral decomposition of the depolarizing $\mathbf{M}$. [4] Thus, $\mathbf{M}_m$ is the nondepolarizing estimate of $\mathbf{M}$ while $\mathbf{\Delta M}$ is the residual. (If $\mathbf{M}$ is nondepolarizing, then $\mathbf{M} = \mathbf{M}_m$ and $\mathbf{\Delta M} = \mathbf{0}$.) From Eqs. (2), (3) and (5) one gets the explicit expression for the mean coherency vector $\mathbf{c}_m$,

$$\mathbf{c}_m = \frac{1}{2\sqrt{t}}\begin{bmatrix} M_{00} + M_{11} + M_{22} + M_{33} \\ M_{01} + M_{10} + i\,(M_{23} - M_{32}) \\ M_{02} + M_{20} - i\,(M_{13} - M_{31}) \\ M_{03} + M_{30} + i\,(M_{12} - M_{21}) \end{bmatrix} = \frac{1}{2\sqrt{t}}\begin{bmatrix} t \\ \alpha_A \\ \beta_A \\ \gamma_A \end{bmatrix} \tag{7}$$

parameterizing, in virtue of Eq. (1), the mean Jones matrix $\mathbf{J}_m$ associated with the nondepolarizing estimate $\mathbf{M}_m = \mathbf{T}\,(\mathbf{J}_m \otimes \mathbf{J}_m^*)\,\mathbf{T}^{-1}$ of $\mathbf{M}$. The quantity t is the trace of $\mathbf{M}$ whereas α_A, β_A and γ_A are the complex counterparts of the so-called anisotropy coefficients of $\mathbf{M}$ [11,12].

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Similarly, the expressions for the variances and covariances of the fluctuations Δc_k, obtained from Eqs. (2) , (4) and (5), are

$$\begin{aligned}
\langle|\Delta c_1|^2\rangle &= \tfrac{1}{4}(M_{00}+M_{11}-M_{22}-M_{33})-|c_{m1}|^2\\
\langle|\Delta c_2|^2\rangle &= \tfrac{1}{4}(M_{00}-M_{11}+M_{22}-M_{33})-|c_{m2}|^2\\
\langle|\Delta c_3|^2\rangle &= \tfrac{1}{4}(M_{00}-M_{11}-M_{22}+M_{33})-|c_{m3}|^2\\
\langle\Delta c_1\Delta c_2^*\rangle &= \tfrac{1}{4}[M_{12}+M_{21}+i\,(M_{03}-M_{30})]-c_{m1}c_{m2}^*\\
\langle\Delta c_1\Delta c_3^*\rangle &= \tfrac{1}{4}[M_{13}+M_{31}-i\,(M_{02}-M_{20})]-c_{m1}c_{m3}^*\\
\langle\Delta c_2\Delta c_3^*\rangle &= \tfrac{1}{4}[M_{23}+M_{32}+i\,(M_{01}-M_{10})]-c_{m2}c_{m3}^*
\end{aligned} \tag{8}$$

Eventually, the depolarizing residual $\mathbf{\Delta M}$ of $\mathbf{M}$ can be obtained from $\mathbf{\Delta M} = \mathbf{M} - \mathbf{M}_m$.

The special integral decomposition of $\mathbf{M}$ given by Eq. (6) is uniquely defined by Eqs. (7) and (8). Its physical meaning is clear: it assumes that the three anisotropy components c_k, $k = 1, 2, 3$ of the Jones generator $\mathbf{J}$ from Eq. (1) fluctuate while the isotropic one, c_0, remains constant. We can therefore name it the anisotropic integral decomposition of $\mathbf{M}$.

To gain a deeper physical insight into the anisotropic integral decomposition, consider the parameterization of the Jones matrix from Eq. (1)

$$\mathbf{J} = a\begin{bmatrix} \cos\frac{\mathrm{T}}{2} - \frac{i\mathrm{L}}{\mathrm{T}}\sin\frac{\mathrm{T}}{2} & -\frac{\mathrm{C}+i\mathrm{L}'}{\mathrm{T}}\sin\frac{\mathrm{T}}{2} \\ \frac{\mathrm{C}-i\mathrm{L}'}{\mathrm{T}}\sin\frac{\mathrm{T}}{2} & \cos\frac{\mathrm{T}}{2} + \frac{i\mathrm{L}}{\mathrm{T}}\sin\frac{\mathrm{T}}{2} \end{bmatrix} \tag{9}$$

in terms of its three elementary polarization properties, L (linear), L' (linear-45°) and C (circular). In Eq. (9) $\mathrm{T} = \sqrt{\mathrm{L}^2 + \mathrm{L}'^2 + \mathrm{C}^2}$ is the amplitude of the properties while a is the transmissivity or the reflectivity [1,4]. The three (complex) elementary polarization properties are defined through their (real) birefringence (B) and dichroism (D) components as $\mathrm{L} = \mathrm{LB} - i\,\mathrm{LD}$, $\mathrm{L}' = \mathrm{LB}' - i\,\mathrm{LD}'$ and $\mathrm{C} = -\mathrm{CB} + i\,\mathrm{CD}$ [1,13]. (Note that other sign conventions for the elementary properties likewise exist [1,13].)

Assume that the three elementary properties are fluctuating, i.e. $\mathrm{P}_k = \mathrm{P}_{mk} + \Delta\mathrm{P}_k$ where $\mathrm{P}_1 = \mathrm{L}$, $\mathrm{P}_2 = \mathrm{L}'$ and $\mathrm{P}_3 = \mathrm{C}$ [4,7]. If all mean values P_{mk} and fluctuations $\Delta\mathrm{P}_k$ are small, i.e. if $|\mathrm{P}_{mk}| \ll 1$ and $|\Delta\mathrm{P}_k| \ll 1$, then $|\mathrm{T}| \ll 1$ and $|\Delta\mathrm{T}| \ll 1$ and the expression for the fluctuating Jones generator from Eq. (9) simplifies considerably to

$$\mathbf{J}_{wad} = a\begin{bmatrix} 1 - i\frac{1}{2}\mathrm{L} & -\frac{1}{2}\mathrm{C} - i\frac{1}{2}\mathrm{L}' \\ \frac{1}{2}\mathrm{C} - i\frac{1}{2}\mathrm{L}' & 1 + i\frac{1}{2}\mathrm{L} \end{bmatrix} \tag{10}$$

Notice that the assumption $|\mathrm{P}_{mk}| \ll 1$ is physically equivalent to weak anisotropy whereas that of $|\Delta\mathrm{P}_k| \ll 1$ means weak depolarization. Therefore, $\mathbf{J}_{wad}$ from Eq. (10) is the Jones generator of both weakly anisotropic and depolarizing medium (or system).

Comparison of the two Jones generators, $\mathbf{J}$ from Eq. (1) and $\mathbf{J}_{wad}$ from Eq. (10), shows that they are equivalent if one sets $c_0 = a$ and $c_k = -i\,a\,\frac{1}{2}\mathrm{P}_k$. Substitution of these relations in Eqs. (7) and (8) yields, respectively,

$$\mathbf{p}_m = \begin{bmatrix} \mathrm{L}_m & \mathrm{L}'_m & \mathrm{C}_m \end{bmatrix}^T = \frac{2i}{t}\begin{bmatrix} \alpha_A & \beta_A & \gamma_A \end{bmatrix}^T \tag{11}$$

and

$$\langle\Delta\mathrm{P}_k\Delta\mathrm{P}_l^*\rangle = \frac{16}{t}\langle\Delta c_k\Delta c_l^*\rangle \qquad k,\, l = 1,\, 2,\, 3 \tag{12}$$

for the vector $\mathbf{p}_m$ of the mean values P_{mk} of the elementary polarization properties, as well as for the variances-covariances $\langle\Delta\mathrm{P}_k\Delta\mathrm{P}_l^*\rangle$ of their fluctuations. Therefore, the anisotropic integral

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decomposition fully characterizes a weakly anisotropic and depolarizing medium in terms of the mean values and the variances-covariances of the fluctuations of its elementary polarization properties.

Most generally, if the condition for weak anisotropy and depolarization is relaxed, Eqs. (11) and (12) remain still valid (after rescaling) provided the elementary polarization properties P_k are replaced by their integral counterparts PI_k defined by [4]

$$\mathrm{PI}_k = \frac{2\mathrm{P}_k}{\mathrm{T}} \sin\frac{\mathrm{T}}{2} = \mathrm{P}_k \,\mathrm{sinc}\frac{\mathrm{T}}{2} \tag{13}$$

and the amplitude T of the elementary properties remains constant (i.e. does not fluctuate). Therefore, the anisotropic integral decomposition could be also termed a constant-amplitude one. Indeed, the identification of Eq. (9) with Eq. (1) yields $c_0 = a \cos(\mathrm{T}/2)$ and $c_k = -i\,a\,\frac{1}{2}\mathrm{PI}_k$ wherefrom $a = \sqrt{c_0^2 - c_{m1}^2 - c_{m2}^2 - c_{m3}^2}$. Equations (11) and (12) now hold for $\mathbf{pi}_m = \begin{bmatrix} \mathrm{LI}_m & \mathrm{LI}'_m & \mathrm{CI}_m \end{bmatrix}^T$ and $\langle \Delta\mathrm{PI}_k \Delta\mathrm{PI}_l^* \rangle$, respectively, provided the denominator t appearing in the first one is replaced by $t' = \sqrt{t^2 - \alpha_A^2 - \beta_A^2 - \gamma_A^2}$ whereas that in the second one by t'^2/t. Conversely, in the weak anisotropy and depolarization limit, $|\mathrm{T}| \ll 1$ and $|\Delta\mathrm{T}| \ll 1$; therefore, $\mathrm{PI}_k \to \mathrm{P}_k$, $t' \to t$, $t'^2/t \to t$, so that Eqs. (11) and (12) are recovered.

3. Application of the decomposition to experimental examples

In practice, the anisotropic integral decomposition can be used in two situations. First, if no assumptions on the anisotropy of the sample or on the depolarization present in the experiment can be made, the decomposition can be employed for obtaining a nondepolarizing estimate $\mathbf{M}_m$ of the experimental depolarizing $\mathbf{M}$ through the mean Jones matrix $\mathbf{J}_m$ associated with it; see Eq. (1), Eq. (7) and the discussion following it. Note that the relative simplicity of Eq. (7) makes the evaluation of this nondepolarizing estimate much more straightforward than that of any other [1,9,15]. Furthermore, $\mathbf{M}$ can be interpreted as resulting from fluctuating integral polarization properties PI_k and can be described in terms of their mean values and the variances-covariances of their fluctuations; see Eq. (13) and the related discussion.

Second, if the weak anisotropy and depolarization condition is known to hold *a priori*, which is a relatively frequent case in practice, the decomposition can provide estimates of the mean values of the elementary polarization properties P_k of the medium, as well as of their variances-covariances; see Eqs. (11) and (12). Typically, one uses the Cloude decomposition [2,9], the Lu-Chipman decomposition [14], the virtual experiment method [15] or the recently proposed instrument-dependent method [16] for solving the first kind of problems. The second situation is commonly tackled with the differential decomposition [17–19] or with its equivalent, the matrix roots decomposition [20]. However, all these approaches require either matrix diagonalizations or numerical minimizations that make them much more intensive computationally than the anisotropic integral decomposition. The latter expresses the quantities of interest directly in terms of measured Mueller matrix elements.

To illustrate the anisotropic integral decomposition we shall consider two experimental examples. The following normalized Mueller matrix

$$\mathbf{M}^{(c)} = \begin{bmatrix} 1 & 0.000 & 0.006 & -0.012 \\ 0.002 & 0.911 & -0.050 & -0.015 \\ -0.004 & 0.047 & 0.890 & 0.166 \\ -0.018 & -0.002 & -0.182 & 0.877 \end{bmatrix} \tag{14}$$

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is that of a 0.065-mm-thick slab of c-cut nickel sulfate hexahydrate ($NiSO_4.6H_2O$) crystal measured in transmission at the wavelength of 380 nm on a UV-visible four-photoelastic-modulator-based polarimeter described in detail elsewhere [21]. The experimental matrix $\mathbf{M}^{(c)}$ is only weakly depolarizing since its Gil-Bernabeu depolarization index DI [22] equals 0.905 (recall that DI = 1 for a nondepolarizing Mueller matrix). This observation justifies the determination of a nondepolarizing estimate for $\mathbf{M}^{(c)}$. By applying Eqs. (7), (5), (2) and inverting the latter, one readily finds the two matrix terms of the anisotropic decomposition (6) to be

$$\mathbf{M}_m^{(c)} = \begin{bmatrix} 0.928 & 0.001 & 0.002 & -0.015 \\ 0.001 & 0.927 & -0.048 & -0.011 \\ 0.000 & 0.049 & 0.911 & 0.174 \\ -0.015 & 0.002 & -0.174 & 0.912 \end{bmatrix} \tag{15a}$$

and

$$\mathbf{\Delta M}^{(c)} = \begin{bmatrix} 0.072 & -0.001 & 0.004 & 0.003 \\ 0.001 & -0.016 & -0.002 & -0.004 \\ -0.004 & -0.002 & -0.021 & -0.008 \\ -0.003 & -0.004 & -0.008 & -0.035 \end{bmatrix} \tag{15b}$$

Clearly, $\mathbf{M}_m^{(c)}$ is a nondepolarizing estimate of the weakly depolarizing experimental $\mathbf{M}^{(c)}$ whereas $\mathbf{\Delta M}^{(c)}$ is the (depolarizing) matrix residual. The relative smallness (with respect to the unit) of the elements of $\mathbf{\Delta M}^{(c)}$ is in unison with the weakly depolarizing nature of $\mathbf{M}^{(c)}$ (recall that $\mathbf{\Delta M} = \mathbf{0}$ for a nondepolarizing $\mathbf{M}$). One likewise notices that the residual $\mathbf{\Delta M}^{(c)}$ is G-symmetric, i.e. it satisfies the relation $\mathbf{G}\,\mathbf{\Delta M}^{(c)T}\mathbf{G} = \mathbf{\Delta M}^{(c)}$ where $\mathbf{G} = diag\begin{pmatrix} 1 & -1 & -1 & -1 \end{pmatrix}$ is the Minkowski metric [1,17]. It readily follows from Eqs. (5) that G-symmetry is a general property of any Mueller matrix whose coherency matrix has the special form given by Eq. (4). Notice, however, that in general, the nondepolarizing estimate $\mathbf{M}_m$ does not obey any particular symmetry, i.e. Equation (6) is not simply the trivial decomposition of $\mathbf{M}$ into its G-symmetric and G-antisymmetric parts.

To assess the performance of the decomposition with increasing the level of depolarization exhibited by the sample, consider a transmission Mueller polarimetry experiment through stacks of rectangular strips of scotch tape. The measurements were done at the wavelength of 490 nm. Due to its spatially inhomogeneous stretched-plastic nature, scotch tape (*3M Scotch Magic Tape* brand used) has been experimentally shown to be both depolarizing and linearly birefringent along the tape direction [23]. Two strips were stuck on a glass slide forming a cross and the Mueller matrices of up to four stacked crosses were measured consecutively. Figure 1 presents the configuration. The experimental normalized Mueller matrices $\mathbf{M}^{(k)}$ obtained respectively for $k = 1, 2, 3, 4$ crosses are reported below,

$$\mathbf{M}^{(1)} = \begin{bmatrix} 1 & 0.000 & 0.012 & 0.001 \\ 0.000 & 0.997 & -0.006 & 0.044 \\ 0.000 & 0.006 & 0.985 & 0.019 \\ 0.000 & -0.045 & -0.022 & 0.979 \end{bmatrix} \tag{16a}$$

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$$\mathbf{M}^{(2)} = \begin{bmatrix} 1 & 0.000 & 0.011 & 0.001 \\ 0.000 & 0.945 & -0.014 & 0.091 \\ 0.000 & 0.011 & 0.935 & 0.045 \\ 0.000 & -0.090 & -0.047 & 0.924 \end{bmatrix} \tag{16b}$$

$$\mathbf{M}^{(3)} = \begin{bmatrix} 1 & 0.002 & 0.010 & 0.001 \\ 0.002 & 0.776 & -0.019 & 0.109 \\ -0.001 & 0.011 & 0.771 & 0.056 \\ 0.000 & -0.108 & -0.052 & 0.741 \end{bmatrix} \tag{16c}$$

$$\mathbf{M}^{(4)} = \begin{bmatrix} 1 & 0.002 & 0.006 & 0.000 \\ 0.002 & 0.531 & -0.017 & 0.088 \\ -0.001 & 0.010 & 0.530 & 0.047 \\ -0.001 & -0.093 & -0.041 & 0.506 \end{bmatrix} \tag{16d}$$

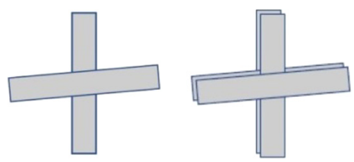

Fig. 1. A single cross (left) and two stacked crosses (right) made of scotch tape strips stuck on a glass slide.

An ideal, perpendicular-strips cross lacks all three birefringence (B) properties: LB, LB' vanish since the opposite birefringence values of the two uniaxially anisotropic strips effectively compensate one another while each individual strip does not exhibit any CB [23]. However, in our experiment the two strips of each cross were not strictly perpendicular − a 5 ± 2-deg angular misalignment was intentionally introduced − so that all three B-properties were present. (The weak CB property originates from the slightly helical structure formed by the two strips of the misaligned cross.)

Figure 2 shows the evolutions of the Gil-Bernabeu depolarization index DI, as well as of the mean values of the three elementary B-properties (LB, LB' and CB,) as a function of the number (denoted by "#") of stacked crosses. The mean values of the B-properties were obtained in three different ways.

First, they were evaluated from the G-antisymmetric part of the Mueller matrix logarithm, in accordance with the differential decomposition of depolarizing Mueller matrices [1,17–19]. Second, they were estimated from the non-depolarizing estimates obtained, respectively, with the Cloude [1,9] and anisotropic decompositions. (Given the coherency vector $\mathbf{c}_m$ of the nondepolarizing estimate $\mathbf{M}_m$ provided by the respective decomposition, the mean values P_{mk} of the elementary polarization properties are simply derived by identifying Eq. (1) with Eq. (9) for $\mathbf{c} = \mathbf{c}_m$ and $\mathrm{P}_k = \mathrm{P}_{mk}$.)

As shown theoretically [8] and confirmed experimentally [23] the three B-properties obtained from the differential decomposition (denoted "Log" in Fig. 2) vary linearly (within the experimental accuracy) with the pathlength of light through the medium, i.e. with number of crosses. Those

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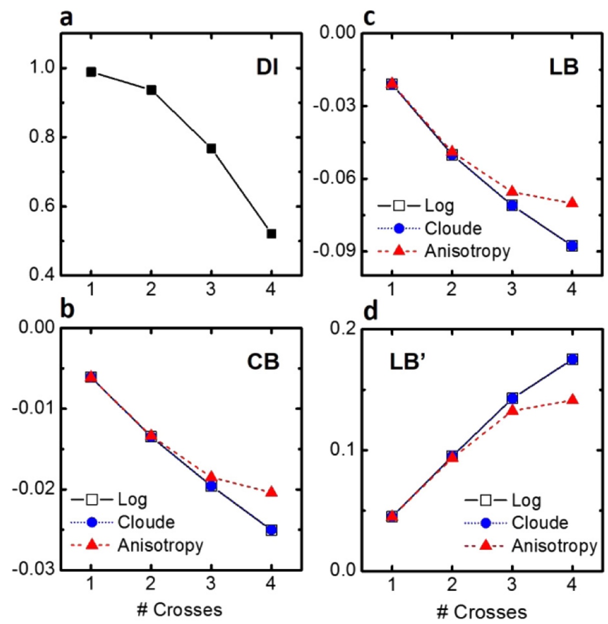

Fig. 2. Depolarization index DI (a) and the three elementary birefringence (B) properties, CB (b), LB (c) and LB' (d), obtained from three different decompositions as functions of the number of crosses.

derived from the Cloude decomposition (denoted "Cloude") exhibit the same trend; they are virtually indistinguishable from the first ones.

The B-properties derived from the anisotropic decomposition (denoted "Anisotropy") initially coincide with the previous two sets at weak depolarization (DI close to the unit) and low mean values of the properties before diverging from the linear trend at lower DI values and larger mean values of the properties. The experimentally observed behavior is fully consistent with the fact, arrived at in the theoretical part, that the nondepolarizing estimate of the anisotropic decomposition provides the elementary polarization properties of the medium in the weak depolarization and anisotropy limit. More quantitatively, we can conclude from Fig. 2 that if one is interested in determining the elementary polarization properties (in terms of mean values and variances-covariances of their fluctuations) then one should use the computationally advantageous anisotropy decomposition instead of the differential or Cloude ones as far as the depolarization index value does not drop below 0.8 and the mean values of the properties thus obtained do not exceed about 0.15. Notice that, even if the anisotropy of $\mathbf{M}$ is large, one can still apply the anisotropic decomposition to obtain the mean values of the elementary polarization properties from the nondepolarizing estimate $\mathbf{M}_m$ of $\mathbf{M}$, as explained in end of the previous paragraph. However, the variances-covariances of the properties cannot be derived straightforwardly from the depolarizing residual $\mathbf{\Delta M}$ since Eqs. (11) and (12) do not hold in this case.

4. Conclusion

To summarize, we have derived explicit expressions for a special integral decomposition of a depolarizing Mueller matrix $\mathbf{M}$, originating from the fluctuations of the anisotropic components of the Pauli expansion of the Jones generator $\mathbf{J}$ of $\mathbf{M}$. The decomposition can be used in practice either for obtaining a nondepolarizing estimate $\mathbf{M}_m$ of $\mathbf{M}$ or for evaluating the elementary polarization properties of $\mathbf{M}$ (in terms of their mean values and variances-covariances), if $\mathbf{M}$ is

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known to be weakly anisotropic and depolarizing. In the general case where no assumptions on $\mathbf{M}$ are made, the decomposition evaluates the integral polarization properties of $\mathbf{M}$. Authors believe these results to be of use to experimentalists who are willing to interpret phenomenologically measured depolarizing Mueller matrices.

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