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Structure theorems for non-Noetherian modules aimed at non-Noetherian Iwasawa theory

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Abstract

We explore several structure theorems for modules over non-Noetherian rings, with a particular interest in finitely presented torsion modules. Our main objective is to prove structure theorems for admissible modules, a new class introduced by Burns et al. in [7]. This work intends to be mostly self-contained: a first chapter is devoted to fixing preliminaries, a second chapter is devoted to presenting some key homological results due to Endo [10] and to the proof of structure theorems for finitely presented torsion modules over EDR domains, valuation rings and Prüfer rings, the last two due to Warfield [28]. In the third and last chapter, we prove the classical structure theorem for Noetherian Krull domains, introduce admissible modules, and prove theorems describing their structure. Finally, we present an application to classical and non-Noetherian Iwasawa theory.

Introduction

The description of the structure of a module over a ring is an ever-present problem in mathematics. The earliest encounter with one such theorem is the classification of finitely generated (f.g.) k-vector spaces, which assures that any such vector space is isomorphic to the direct sum of a finite number of copies of k.

Another example of these structure theorems is the classification of f.g. abelian groups. This theorem can be reinterpreted as the classification of f.g. $\mathbb{Z}$-modules, which can be later extended to the description of f.g. R-modules over R a principal ideal domain (PID). In this instance, the theorem separates between a torsion part and a torsion-free part and describes the structure of both separately. This theorem motivates the proof of a similar statement for Dedekind domains.

The theorems described so far are well-known and widely used. However, they only concern Noetherian settings. In this work, we aim to explore the landscape of these types of theorems by focusing on non-Noetherian rings and modules. Our ultimate objective is to introduce admissible modules, a new class of modules first presented by Burns, Daoud, and Liang in [7], and a structure theorem for these modules. This theorem is relevant in the relatively new field of non-Noetherian Iwasawa theory, and we provide a hint of their connection with this new theory and its classical counterpart at the end of the work.

To achieve our goal, we organize this work into three chapters. The first chapter centers around fixing common definitions and recalling many results and characterizations of various classes of rings and modules. This is fundamental, for many classes of these structures have multiple characterizations and definitions in the literature.

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In addition, we introduce basic notions of homological dimensions and prove many generic results that will be useful in later chapters.

The second chapter focuses specifically on introducing non-Noetherian classes of rings and modules. The main objective of this chapter is to introduce the necessary tools to prove Warfield's Structure Theorem [28]. This theorem provides a description of finitely presented torsion modules over Prüfer domains. Furthermore, we present some results due to Endo [10] concerning a homological characterization of certain types of rings. All these results are to be used in Chapter 3 to prove the structure theorems for admissible modules.

Finally, in the last chapter we introduce the concept of pseudo-nullity of modules and its properties, the proof of a structure theorem for Noetherian Krull domains, and, finally, the introduction of admissibility and the relevant structure theorems of Burns, Daoud, and Liang [7]. The chapter finishes with a general picture of the applicability of this theorem to Iwasawa theory.

Chapter 1

Definitions and preliminary results

Ring theory and module theory are abundant in definitions and characterizations of different types of rings and modules. The interplay between the various classes of these structures is also especially intricate. This chapter aims to establish a common background for many of the ring and module theoretic concepts usually introduced in graduate-level courses. Many of the definitions and characterizations provided will be of use in later sections, so we use the chapter to fix notations and introduce common results relating the different structures introduced.

The first section involves mainly preliminaries and definitions of rings commonly found in the literature and in Noetherian settings. The second section focuses on introducing some module theoretic concepts in the commutative setting, with special attention to homological properties and dimensions.

1.1 Preliminaries on ring theory

In this section we present some definitions related to ring theory, mostly focusing on Noetherian types of rings. We also provide some basic results to be used in later sections. The complexity of the presented results varies, so we prove some and give a reference for the rest.

For the rest of the section, R always denotes a unital nonzero ring (not necessarily Noetherian). Additional hypotheses on R vary depending on the presented result. We start presenting two usual types of rings.

Definition 1.1.1 (Local ring)**.** A ring R is *local* if it satisfies one of the following equivalent conditions

(i) It has a unique maximal left ideal.

(ii) It has a unique maximal right ideal.

If R is commutative, we denote a local ring as a pair $(R, \mathfrak{m})$, where $\mathfrak{m}$ is the unique maximal ideal of R.

Definition 1.1.2 (Semi-local ring)**.** Let R be commutative, we say R is *semi-local* if it has a finite number of maximal ideals.

Remark 1.1.3. In our definitions, local and semi-local rings need not be Noetherian.

1.1.1 Regular and semisimple rings

We now briefly touch on the subject of regular rings and semisimple rings, their characterizations, and the relationship between the two.

Definition 1.1.4 (von Neumann regular ring)**.** A ring R is *von Neumann regular* (vNr) if for all $r \in R$ there exists $s \in R$ such that $rsr = r$

Proposition 1.1.5. *A ring R is von Neumann regular if and only if every principal right ideal is generated by an idempotent.*

Proof. $\boxed{\Rightarrow}$ Consider the ideal aR, if R is vNr then there exists $r \in R$ such that $ara = a$. Note that $arar = ar$ so $e = ar$ is an idempotent. Furthermore, $a = ara = ea \in eR$ and $e = ar \in aR$, so $aR = eR$.

$\boxed{\Leftarrow}$ Now assume that for all $a \in R$, there exists $e \in R$ idempotent such that $aR = eR$. Then $a = er$ for some $r \in R$ and so $ea = er = a$. Moreover, $e = as$ for some $s \in R$ so $a = ea = asa$. Thus, R is vNr. □

When the ring is commutative, we can provide an additional characterization.

Lemma 1.1.6. *Let R be a commutative vNr ring. Then $R_\mathfrak{p}$ is also vNr for any prime ideal $\mathfrak{p}$.*

Proof. Note that if $r/s \in R_\mathfrak{p}$ is invertible, then it satisfies the vNr condition. If r/s is not invertible, then $r \in \mathfrak{p}$ so there exists $t \in R$ with $rtr = r \implies (r/s)(st)(r/s) = r/s$, so $R_\mathfrak{p}$ is vNr. □

Proposition 1.1.7. *Let R be a commutative ring. R is vNr if and only if $R_\mathfrak{m}$ is a field for any $\mathfrak{m}$ maximal ideal.*

Proof. $\boxed{\Rightarrow}$ We show that any local vNr ring is a field. Let $(R, \mathfrak{m})$ be such a ring, let $r \in \mathfrak{m}$, since r is not a unit there exists $s \in R$ such that $rsr = sr^2 = r$, but then $(1 - sr)r = 0$ and since $sr \in \mathfrak{m}$, then $1 - sr$ is a unit, so $r = 0$. Thus it is a field.

$\boxed{\Leftarrow}$ Suppose $R_\mathfrak{m}$ is a field for any $\mathfrak{m}$ maximal ideal. Let $r \in R$ and consider the ideal $\mathrm{Ann}_R(r) = \{x \in R \mid rx = 0\}$. Since $R_\mathfrak{m}$ is a field, it has no zero divisors, so there cannot be any $\mathfrak{m}$ such that $a \in \mathfrak{m}$ and $\mathfrak{m} \cap \mathrm{Ann}_R(r) \neq \varnothing$. Thus, $aR + \mathrm{Ann}_R(r) = R$ since it is not contained in any maximal ideal. Furthermore, $aR \cdot \mathrm{Ann}_R(r) = 0$ so $R = aR \oplus \mathrm{Ann}_R(r)$, meaning aR is a direct summand of R so it is generated by an idempotent $e \in R$, $eR = aR$ and $\mathrm{Ann}_R(r) = (1 - e)R$. Consider $b = 1 - e + a$, note that b cannot belong to any maximal ideal, otherwise $eb = a$ and $(1 - e)b = 1 - e$ would also belong. Thus, it is a unit and $eb = a \Rightarrow e = b^{-1}a \Rightarrow ae = a = ab^{-1}a$. □

Remark 1.1.8. In particular, we have shown that commutative local vNr rings are fields.

Definition 1.1.9 (Semisimple ring)**.** A ring R is left (right) *semisimple* if it is the direct sum of minimal left (right) ideals.

Theorem 1.1.10 (Wedderburn–Artin)**.** *R is left (right) semisimple if and only if there exist division rings $D_1, \dots D_m$ and natural numbers $n_1, \dots n_m \geq 1$ such that*

$$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_m}(D_m) \tag{1.1}$$

The notation $M_n(D)$ denotes matrices of dimension $n \times n$ over D. Furthermore, the division rings $D_1, \dots D_m$ and natural numbers $n_1, \dots n_m$ are unique up to permutation.

Proof. See [23, Theorem 8.56]. □

Remark 1.1.11. By the previous theorem, a ring is left semisimple if and only if it is right semisimple. Thus, we simply call them semisimple from now on.

Corollary 1.1.12. *If R is semisimple then R is Noetherian and Artinian.*

Remark 1.1.13. Wedderburn-Artin theorem is not needed to show that semisimple rings are Noetherian and Artinian. In fact, some proofs of the theorem use the Artinian condition. However, we state the result like a corollary for convenience.

Corollary 1.1.14. *If R is commutative and semisimple, then R is the direct product of finitely many fields.*

Proof. Since R is commutative, by Wedderburn–Artin Theorem 1.1.10 we have that the n_i must be 1 for all i and the division rings must be commutative, so they are fields. □

Proposition 1.1.15. *All semisimple rings are vNr.*

Proof. Suppose R is right semisimple, by [23, Proposition 8.42] all right ideals of R are direct summands, so given $a \in R$, there exists J a right ideal of R such that $aR \oplus J = R$. Then $1 = e + f$ for some $e \in aR$ and $f \in J$. Then $e = e^2 + fe$ so $e - e^2 = fe \in J$, but $e - e^2 \in aR$ so $e - e^2 = 0$. Thus $e \in aR$ is an idempotent. Now to see $aR \subseteq eR$ note that for any $x \in aR$ we may write $x = ex + fx$ and $x - ex = fx \in J$ so $x - ex = 0$ meaning $x = ex \in eR$. Thus $aR = eR$ concludes the proof. □

1.1.2 Valuations and Dedekind domains

We now shift our focus towards some types of commutative domains. In particular, we introduce valuation rings, discrete valuation rings and Dedekind domains.

Definition 1.1.16 (Total quotient ring)**.** Let R be a commutative and consider the multiplicatively closed set $S = \{r \in R \mid r \text{ is not a zero divisor}\}$. We define $Q(R) = S^{-1}R$ the *total quotient ring* of R.

Remark 1.1.17. If R is commutative and a domain, then $Q(R)$ is the usual fraction field of R.

Definition 1.1.18 (Valuation ring)**.** Let R be commutative. We say R is a *valuation ring* if it is an integral domain and for every $x \in Q(R)$ if $x \notin R$ then $x^{-1} \in R$.

Lemma 1.1.19. *Let R be a valuation ring, then the following statements hold.*

(i) The ideals of R form a totally ordered set.

(ii) R is a local ring.

(iii) R is integrally closed.

(iv) All finitely generated ideals are principal.

Proof. (i) Take $I, J \subsetneq R$ ideals and suppose $I \nsubseteq J$, then there exists $x \in I$, $x \notin J$. Take any $y \in J$, since it is not the total, $y^{-1} \notin R$, so $xy^{-1} \notin R$. Since R is a valuation ring, $x^{-1}y \in R$ and so $y = xx^{-1}y \in I$. Thus, $J \subseteq I$.

(ii) Follows directly from the set of ideals being totally ordered, as there can only be a single maximal ideal. Call this maximal ideal $\mathfrak{m}$.

(iii) Consider a monic polynomial $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in R[x]$, suppose $y \in Q(R)$ is a root $p(y) = 0$. If $y \in R$ we are done. Otherwise, suppose $y \notin R$ then $y^{-1} \in R$. In particular, since y^{-1} is not invertible in R we have $y^{-1} \in \mathfrak{m}$. However,

$$y^n + a_{n-1}y^{n-1} + \cdots + a_0 = 0 \implies 1 + a_{n-1}y^{-1} + \cdots + a_0 y^{-n} = 0$$

so $1 \in \mathfrak{m}$, which is a contradiction.

(iv) By part (i), the ideals of R are totally ordered. It is enough to show that any two-generated ideal can be reduced. Let $I = xR + yR$, then either $xR \subseteq yR$ or $yR \subseteq xR$. In the first case $I = yR$ and in the second one $I = xR$. □

Remark 1.1.20. Note that the converse of (i) also holds if R is a domain. This motivates the following definition of a valuation ring that is not necessarily a domain.

Definition 1.1.21 (Generalized valuation ring)**.** A commutative ring R is called a *generalized valuation ring* if its ideals are totally ordered by inclusion.

The concept of generalized valuation ring will appear later on in contexts where R is not a domain. For now, we focus on valuation rings and introduce the concept of valuation, which is the essence of these types of rings.

Definition 1.1.22 (Ordered group)**.** Let $(G, +, \leq)$ be an abelian group with a total order relation $\leq$ such that if $x, y, z, t \in G$ and $x \leq y$, $z \leq t$ then $x + z \leq y + t$. We say that G is an *ordered group*.

Remark 1.1.23. Given an ordered group $(G, +, \leq)$ we can extend it by adding ∞ as follows. Consider $G_\infty = G \cup \{\infty\}$ such that $\infty \geq g$ for all $g \in G$. To make it compatible with the group operation we establish $\infty + g = \infty$ for all $g \in G$ and $\infty + \infty = \infty$. This is no longer a group (∞ does not have an inverse) but it preserves most of the structure and includes an element which is bigger (with respect to $\leq$) than any other element of G.

Lemma 1.1.24. *Let G be an ordered group and $g, h \in G$, then the following properties hold:*

(i) Let $g \geq h$ then $-h \geq -g$.

(ii) Let $g > 0$ and $h \geq 0$, then $g + h > 0$.

Proof. Suppose that $g \geq h$, then $-g \geq -g$ so $0 = g - g \geq h - g$ and $-h \geq -h$ so $-h \geq -g$. Now suppose that $g > 0$, then $g \neq 0$ and $g \geq 0$, so $g + h \geq 0$ and $g + h = 0$ if and only if $h = -g$ since $g \neq 0$. But $g \geq 0$ implies $-g \leq 0$ by the previous property, so $g + h \neq 0$ and $g + h > 0$. □

Definition 1.1.25 (Valuation of a field)**.** Let K be a field and G an ordered group. We define a *valuation of K* as a map $v : K \to G_\infty$ such that for all $x, y \in K$ it satisfies

(i) $v(xy) = v(x) + v(y)$.

(ii) $v(x + y) \geq \min\{v(x), v(y)\}$.

(iii) $v(x) = \infty \iff x = 0$.

Remark 1.1.26. From the first property it follows that $v(1) = 0$ since $v(1x) = v(1) + v(x)$ for all $x \in K$. Furthermore, $v(x^{-1}) = -v(x)$ is easily seen by taking $0 = v(1) = v(xx^{-1}) = v(x) + v(x^{-1})$.

Lemma 1.1.27. *Let K be a field and $K^* = K \backslash \{0\}$. Let G be an ordered group and $v : K \to G_\infty$ be a valuation of K. Then the following statements hold:*

(i) The restriction $v|_{K^*} : K^* \to G$ defines a group morphism.

(ii) The set $R_v = \{x \in K \mid v(x) \geq 0\}$ defines a valuation ring with maximal ideal $\mathfrak{m}_v = \{x \in K \mid v(x) > 0\}$.

Proof. Property (i) follows directly from axiom (i) of the definition of valuation of a field. The fact that R_v is a ring follows from axioms (i) and (ii), since if $x, y \in R_v$ then $v(x + y) \geq 0$ and $v(xy) \geq 0$. To see it is a valuation ring, take $x \in K$ such that $x \notin R$, this means $v(x) < 0$ and so $v(x^{-1}) = -v(x) \geq 0$, so $x^{-1} \in R$. The fact that $\mathfrak{m}_v$ is an ideal also follows from the basic properties of valuations and ordered groups. To see it is the unique maximal ideal of R_v, note that if $x \in R_v \backslash \mathfrak{m}_v$ then $v(x) = -v(x^{-1}) = 0$ and so $x^{-1} \in R_v$ so x is invertible in R_v. Thus, adding any elements to $\mathfrak{m}_v$ produces R_v, so it is the maximal ideal. □

Definition 1.1.28. Let K be a field and $K^* = K\backslash\{0\}$. Let G be an ordered group and $v : K \to G_\infty$ be a valuation of K. We call $v(K^*)$ the *value group* of v, R_v the *valuation ring* of v and $\mathfrak{m}_v$ the *valuation ideal* of v.

Proposition 1.1.29. *Let R be a commutative domain, then R is a valuation ring if and only if there exists a valuation v of $Q(R)$ such that $R_v = R$.*

Proof. The reverse implication is given by Lemma 1.1.27. Suppose R is a valuation ring and define the group $G = \{xR \mid x \in Q(R)\}$. Define the order for any $x, y \in Q(R)$ as $xR \geq yR$ if and only if $xR \subseteq yR$. The proof that this is a total order is similar as in the ideal only case. If $xy^{-1} \in R$ then $x = y(xy^{-1}) \in yR$ so $xR \subseteq yR$, otherwise, $x^{-1}y \in R$ and $y = x(x^{-1}y) \in xR$ so $yR \subseteq xR$. Define $v : Q(R) \to G_\infty$ where $v(x) = xR$, this is clearly a valuation as $xR \cdot yR = xyR$ and $xR, yR \subseteq (x+y)R$. Furthermore, $v(1) = R$ and $x \in R$ implies $xR \subseteq R$ so $v(x) \geq v(1)$ for all $x \in R$ meaning $R \subseteq R_v$. Conversely, if $v(x) \geq v(1)$ then $xR \subseteq R$ implies $x \in R$ so $R_v = R$. □

Definition 1.1.30 (Discrete valuation ring)**.** Let R be a valuation ring, we say R is a *discrete valuation ring* (DVR) if the value group of R is isomorphic to $\mathbb{Z}$.

There are multiple characterizations for DVRs which are useful in many contexts, we recall some of them.

Proposition 1.1.31. *Let $(R, \mathfrak{m})$ be a valuation ring, the following statements are equivalent.*

(i) R is a DVR.

(ii) R is a principal ideal domain (PID).

(iii) R is Noetherian.

Proof. (i) $\implies$ (ii) Suppose R is a DVR and consider its valuation $v : K \to \mathbb{Z}_\infty$ (Proposition 1.1.29). Since $v(\mathfrak{m}\backslash\{0\}) = \mathbb{Z}_{>0}$ there is some $r \in \mathfrak{m}$ with $v(r) = 1$. Consider an ideal $I \neq (0)$, the set $v(I\backslash\{0\}) \subseteq \mathbb{Z}_{>0}$ and so it has a minimum, call it n. If $n = 0$ then $I = R$ as it contains a unit, otherwise $n > 0$ and so there is an element $x \in I$ such that $v(x) = n$. Let $y \in I$ and $m = v(y)$, then $v(y/xr^{m-n}) = 0$ so $y = x(r^{m-n}u)$ for some unit $u \in R\backslash\mathfrak{m}$. Therefore, $I = xR$.

(ii) $\iff$ (iii) A PID is Noetherian. If R is Noetherian, then by Lemma 1.1.19(iv) it is a PID.

(ii) $\implies$ (i) Consider $r \in R$ such that $\mathfrak{m} = rR$ and let $I = \bigcap_{i=1}^{\infty} r^i R = xR$ for some $x \in R$. Since $I \subseteq \mathfrak{m}$ we have $x = rs$ for some $s \in R$. Then $x \in r^i R$ for all $i \geq 1$ implies $s \in r^{i-1}R$ for all $i \geq 1$ and so $s \in I$. Then we may write $x = rs = rxt$ for some $t \in R$ and so $x(1 - rt) = 0$ implies $x = 0$ as $r \in \mathfrak{m}$ is not invertible, so $I = (0)$. Therefore, to each $y \in R$ we may associate an $i \geq 0$ such that $y \in r^i R\backslash r^{i+1}R$. We may define $v^* : R \to \mathbb{Z}_\infty$ such that $v(y) = i$ and it is easy to see that the extension to $v : Q(R) \to \mathbb{Z}_\infty$ via $v(y/z) := v^*(y) - v^*(z)$ is well defined and a valuation giving $R_v = R$ so R is a DVR. □

Remark 1.1.32. In particular, we have seen that all ideals of a DVR are principal and generated by a power of an element of valuation 1.

Corollary 1.1.33. *Let R be a commutative domain with a unique non-zero prime ideal. The following are equivalent:*

(i) R is a DVR

(ii) R is a PID.

(iii) R is Noetherian and integrally closed.

Proof. (i) $\iff$ (ii) follows directly from the previous proposition. Note that in Proposition 1.1.31 (ii) $\implies$ (i) we only use the "valuation ring" hypothesis to assert it is local, and explicitly build the valuation. So we may exchange the hypothesis "valuation ring" for "local ring" and the corollary follows.
(ii) $\implies$ (iii) is direct.
(iii) $\implies$ (ii) is a bit longer, see [23, Lemma 11.89]. □

Definition 1.1.34 (Dedekind ring)**.** Let R be a commutative domain that is not a field, we say it is a *Dedekind ring* or *Dedekind domain* if it is Noetherian, integrally closed and all non-zero prime ideals are maximal.

The following characterizations of Dedekind rings will prove useful.

Proposition 1.1.35. *Let R be a commutative integral domain that is not a field. The following are equivalent:*

(i) R is a Dedekind ring.

(ii) All non-zero ideals of R uniquely factorize into a finite product of prime ideals.

(iii) For any prime ideal $\mathfrak{p}$ of R, the localization $R_\mathfrak{p}$ is a DVR.

Proof. (i) $\iff$ (ii) See [17, Theorem 11.6].
(i) $\implies$ (iii): Let R be Dedekind and $\mathfrak{p}$ be a prime ideal of R. Consider $R_\mathfrak{p}$ the localization at $\mathfrak{p}$. Since all prime ideals of R are maximal, $\mathfrak{p}R_\mathfrak{p}$ is the unique nonzero prime ideal of $R_\mathfrak{p}$. Moreover, since R is integrally closed and Noetherian so is $R_\mathfrak{p}$, and so by Corollary 1.1.33 it is a DVR.
(iii) $\implies$ (i): Suppose there exist prime ideals such that $\mathfrak{p} \subsetneq \mathfrak{q} \subseteq R$. Then $\mathfrak{p}R_\mathfrak{q} \subsetneq \mathfrak{q}R_\mathfrak{q}$ and $\mathfrak{p}R_\mathfrak{q}$ is a prime ideal of $R_\mathfrak{q}$. Since there is a unique prime ideal in $R_\mathfrak{q}$ then $\mathfrak{p}R_\mathfrak{q} = 0$ so $\mathfrak{p} = 0$ meaning $\mathfrak{q}$ is maximal. Now let $x \in Q(R)$ be integral over R, note that $Q(R) = Q(R_\mathfrak{p})$ for any prime ideal $\mathfrak{p}$ and since $R_\mathfrak{p}$ is a DVR, it is integrally closed so $x \in R_\mathfrak{p}$. If $x = a/b$ with $a, b \in R$, then we see that $b \in R \backslash \mathfrak{p}$ for any $\mathfrak{p}$ prime ideal, so it must me invertible in R and thus $x \in R$. □

An interesting property of Dedekind rings is that all its ideals can be generated by two elements or less.

Proposition 1.1.36. *Let R be Dedekind, let $I \subseteq R$ be an ideal. Then there exist $a, b \in I$ such that $I = aR + bR$.*

Proof. Write $I = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_n^{e_n}$ for different prime ideals $\mathfrak{p}_i$ and $e_i \geq 1$. Take any element $\alpha \in I$, then $\alpha R = \mathfrak{p}_1^{f_1} \cdots \mathfrak{p}_n^{f_n} \mathfrak{q}_1^{g_1} \cdots \mathfrak{q}_m^{g_m}$ for some $\mathfrak{q}_j$ different from the $\mathfrak{p}_i$ and $f_i \geq e_i$ since $\alpha R \subseteq I$. Then, by the Chinese Remainder Theorem, we may choose an element of $\beta \in I$ such that $v_{\mathfrak{p}_i}(\beta) = e_i$ and $v_{\mathfrak{q}_i}(\beta) = 0$, where $v_{\mathfrak{p}}$ denotes the valuation at $R_{\mathfrak{p}}$. Then $\gcd(\alpha, \beta) = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_n^{e_n} = I$ and so $\alpha R + \beta R = \gcd(\alpha, \beta)R = I$. □

The following property of Dedekind domains is very important in the field of number theory.

Theorem 1.1.37 (Krull-Akizuki)**.** *Let R be a Dedekind ring and $F/Q(R)$ be a finite algebraic extension. Let S be the integral closure of R in F, then B is a Dedekind ring.*

Proof. See [17, Theorem 11.7]. □

1.1.3 Properties related to prime ideals

We now shift our focus to definitions and properties mainly involving prime ideals. We first recall some miscellaneous results that will be of use later and then introduce the concepts of height and Krull dimension of a ring.

The following result is commonly known as the *prime avoidance lemma* and is in the proof of later results.

Lemma 1.1.38 (Prime avoidance)**.** *Let R be commutative, let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be prime ideals of R. Suppose we have an ideal $I \subseteq \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n$, then $I \subseteq \mathfrak{p}_i$ for some $i = 1, \ldots, n$.*

Proof. We proceed by induction on the number n of prime ideals $\mathfrak{p}_i$.

If $n = 1$ then the statement is obviously true. Suppose it is true for the union of n prime ideals and consider $I \subseteq \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n \cup \mathfrak{p}_{n+1}$. Define the sets

$$P_i = \bigcup_{j=1, j\neq i}^{n+1} \mathfrak{p}_j$$

If $I \subseteq P_i$ for some i, then the result follows by induction hypothesis. Suppose that $I \nsubseteq P_i$ for all $i = 1, \ldots, n+1$. Then there exist some elements $a_i \in I$ such that $a_i \notin P_i$ so $a_i \in \mathfrak{p}_i$. Consider now $a = a_1 + a_2 \cdots a_{n+1}$ and note that $a \in I$. Moreover, $a \notin \mathfrak{p}_1$, otherwise $a - a_1 \in \mathfrak{p}_1$ so some of the $a_i \in \mathfrak{p}_1$ for $i \geq 2$, which contradicts $a_i \notin P_i$ as $\mathfrak{p}_1 \subseteq P_i$ for $i \geq 2$. However, this implies that $a \in \mathfrak{p}_i$ for some $i \geq 2$, meaning $a_1 \in \mathfrak{p}_i$ again contradicting $a_1 \notin P_1$, so $I \nsubseteq P_i$ cannot happen. □

We subsequently provide a result describing the prime ideals of the direct product of commutative rings.

Proposition 1.1.39. *Let $R_1, \dots, R_n$ be commutative and consider $R = \prod_{i=1}^n R_i$. The prime ideals of R are of the form $\mathfrak{p} = R_1 \times \cdots \times R_{j-1} \times \mathfrak{p}_j \times R_{j+1} \times \cdots \times R_n$ for some $1 \geq j \geq n$ and $\mathfrak{p}_j$ a prime ideal of R_j.*

Proof. Note that we may rewrite $R = R' \times R_n$ where $R' = \prod_{i=1}^{n-1} R_i$. Thus, we may reduce to studying the case $n = 2$. Without loss of generality, consider $R = R_1 \times R_2$ alongside $\pi_i : R \to R_i$, $i = 1, 2$ the canonical projection morphisms. If $\mathfrak{p}_i \subseteq R_i$ are prime ideals of R_i then $\pi_i^{-1}(\mathfrak{p}_i)$ are also prime ideals. Thus, the ideals of the form $R_1 \times \mathfrak{p}_2$ and $\mathfrak{p}_1 \times R_2$ are prime. Now let $\mathfrak{p} \subseteq R$ be a prime ideal and note $(0,0) \in \mathfrak{p}$. Then $(0,0) = (1,0)(0,1)$ so either $(1,0) \in \mathfrak{p}$ or $(0,1) \in \mathfrak{p}$. Suppose $(1,0) \in \mathfrak{p}$, then $R_1 \times (0) \subseteq \mathfrak{p}$ and for any $(r_1, r_2) \in \mathfrak{p}$ we have $(r_1, 0) \in \mathfrak{p}$ so $(0, r_2) \in \mathfrak{p}$. Thus, we can write $\mathfrak{p} = R_1 \times \pi_2(\mathfrak{p})$. Since $\mathrm{Ker}(\pi_2) \subseteq \mathfrak{p}$ and π_2 is surjective, $\pi_2(\mathfrak{p}) = \mathfrak{q}$ must also be a prime ideal. The argument is symmetric for the other case. □

The following theorem, due to Cohen [9, Theorem 2], characterizes the Noetherianity of a commutative ring in terms of its prime ideals, rather than its ideals in general.

Theorem 1.1.40 (Cohen)**.** *Let R be commutative. R is Noetherian if and only if all prime ideals of R are finitely generated.*

Proof. See [23, Theorem 6.53] for a modern proof or [9, Theorem 2] for the original. □

We now introduce the concepts of height of a prime ideal and Krull dimension of a ring. We also apply these concepts to some of the types of rings seen in the previous sections.

Definition 1.1.41 (Height)**.** Let R be commutative and $\mathfrak{p}_0, \dots, \mathfrak{p}_n$ be different prime ideals of R. A *prime chain* of *length* n is $\mathfrak{p}_n \subsetneq \cdots \subsetneq \mathfrak{p}_0$. The *height* of a prime ideal $\mathfrak{p}$ is the length of the longest prime chain with $\mathfrak{p}_0 = \mathfrak{p}$ and it is denoted $\mathrm{ht}(\mathfrak{p}) \leq \infty$.

Definition 1.1.42 (Krull dimension)**.** Let R be commutative, the *Krull dimension* of R is

$$\dim(R) = \sup\{\mathrm{ht}(\mathfrak{p}) \mid \mathfrak{p} \text{ prime}\}$$

the length of the longest prime chain in R.

Remark 1.1.43. It is not enough for a ring to be Noetherian to have finite Krull dimension, since the condition of all ascending chains being finite is not enough to assert we do not have a sequence of prime ideals whose height grows to infinity. See [19, Appendix, Example 1] for an example.

We present some examples of Krull dimension applied to rings presented in previous sections.

Proposition 1.1.44. *Let R be commutative, the following statements hold.*

(i) If $\mathfrak{p}$ *is a prime ideal,* $\operatorname{ht}(\mathfrak{p}) = \dim(R_{\mathfrak{p}})$.

(ii) If R *is Artinian then* $\dim(R) = 0$.

(iii) If R *is semisimple, then* $\dim(R) = 0$.

(iv) If R *is a DVR, then* $\dim(R) = 1$.

(v) If R *is a Dedekind domain, then* $\dim(R) = 1$.

Proof. (i) The ring $R_{\mathfrak{p}}$ is local with a unique maximal ideal $\mathfrak{m} = \mathfrak{p}R_{\mathfrak{p}}$, so $\dim(R_{\mathfrak{p}}) = \operatorname{ht}(\mathfrak{p}R_{\mathfrak{p}})$, as any prime ideal is contained in $\mathfrak{m}$. Furthermore, the prime ideals of $R_{\mathfrak{p}}$ are in correspondence with the prime ideals of R containing $\mathfrak{p}$, so any prime chain can be lifted and $\operatorname{ht}(\mathfrak{p}) = \operatorname{ht}(\mathfrak{p}R_{\mathfrak{p}})$.

(ii) Let $\mathfrak{p}$ be a prime ideal and consider $R/\mathfrak{p}$. Since R is Artinian, so is $R/\mathfrak{p}$. But $R/\mathfrak{p}$ is an Artinian domain meaning $R/\mathfrak{p}$ is a field. So, in fact, $\mathfrak{p}$ is maximal. If R is a field, then $\dim R = 0$ as the only prime ideal is (0). If it is not a domain, all prime ideals are maximal implies $\operatorname{ht}(\mathfrak{p}) = 0$ for all $\mathfrak{p}$ as (0) is not prime, so $\dim(R) = 0$.

(iii) By Corollary 1.1.12 we know semisimple rings are Artinian.

(iv) By Corollary 1.1.33 a DVR is a PID, so it is a domain where all prime ideals are maximal, so all maximal length prime chains are of the form $(0) \subseteq \mathfrak{p}$, so $\dim(R) = 1$.

(v) If R is Dedekind, then for any $\mathfrak{p}$ nonzero prime ideal, $R_{\mathfrak{p}}$ is a DVR, so $\operatorname{ht}(\mathfrak{p}) = \dim(R_{\mathfrak{p}}) = 1$. □

Even if the Noetherian condition is not enough to guarantee finite Krull dimension, it allows control over the height of ideals as shown in the next results due to Krull.

Definition 1.1.45. Let $I \subseteq R$ be an ideal. A *minimal* ideal over I is a prime ideal $\mathfrak{p}$ such that there is no other prime ideal $\mathfrak{q}$ with $I \subseteq \mathfrak{q} \subsetneq \mathfrak{p}$.

Remark 1.1.46. Minimal ideals over a proper ideal I always exist. This is an application of Zorn's Lemma. Consider $\mathcal{F}$ the family of all prime ideals containing I. Then $\mathcal{F}$ is non-empty since there exists a maximal ideal containing I. Now for any descending chain of prime ideals (given by inclusion), the intersection of the chain is also a prime ideal containing I, so it is bounded. Thus, there exists a minimal element.

Theorem 1.1.47 (Principal ideal theorem)**.** *Let* R *be a commutative Noetherian ring. Let* I *be a proper principal ideal of* R *and* $\mathfrak{p}$ *a prime ideal that is minimal over* I*, then* $\operatorname{ht}(\mathfrak{p}) \leq 1$.

Proof. See [23, Theorem 11.161]. □

Theorem 1.1.48 (Height theorem)**.** *Let* R *be a commutative Noetherian ring. Let* $I = (a_1, \ldots, a_n)$ *be a proper ideal generated by* n *elements of* R *and* $\mathfrak{p}$ *a prime ideal that is minimal over* I*, then* $\operatorname{ht}(\mathfrak{p}) \leq n$.

Proof. See [23, Theorem 11.162]. □

We can explore the notion of minimal ideal over I a bit further in the Noetherian setting.

Lemma 1.1.49. *Let R be Noetherian and $I \subseteq R$ an ideal. Then there are finitely many minimal ideals over I.*

Proof. Consider $\mathcal{F}$ to be the family of proper ideals of R such that they have infinitely many minimal ideals. Assume $\mathcal{F} \neq \varnothing$, then the family is non-empty, ordered by inclusion and any ascending chain terminates as R is Noetherian. Thus $\mathcal{F}$ has a maximal element by Zorn's Lemma. Let I be maximal in $\mathcal{F}$. We can assume it is not prime, otherwise I has only itself as a minimal ideal. Take $a, b \in R \backslash I$ such that $ab \in I$ and consider the ideals $I_a = I + aR$, $I_b = I + bR$. Since $I \subseteq I_a, I_b$, I_a, I_b must have finitely many minimal ideals by maximality of I. Now let $I \subseteq \mathfrak{p}$ be minimal, then $ab \in \mathfrak{p}$ so either $I_a \subseteq \mathfrak{p}$ or $I_b \subseteq \mathfrak{p}$. Thus, $\mathfrak{p}$ is minimal over I_a or I_b. This implies all minimal ideals of I are either minimal of I_a or I_b, but there are finitely many of these, so we have a contradiction. □

Proposition 1.1.50. *Let R be a Noetherian domain, let $I \subseteq R$ be a proper non-zero principal ideal. Then there are finitely many prime ideals of height 1 containing I.*

Proof. By Lemma 1.1.49 there are finitely many minimal ideals containing I. Furthermore, by Krull's principal ideal theorem 1.1.47 all minimal primes $\mathfrak{p}$ over I have $\operatorname{ht}(\mathfrak{p}) \leq 1$. Finally, as R is a domain, if $I \subseteq \mathfrak{p}$ with $\operatorname{ht}(\mathfrak{p}) = 1$ then $\mathfrak{p}$ is minimal over I. □

Armed with these notions we close the section on ring theory introducing a final class of rings. We have seen that Dedekind rings are 1-dimensional, we would like a generalization that allows to keep most of the properties of these rings but with higher dimensionality.

Definition 1.1.51 (Krull domain)**.** Let R be a commutative domain, let $\mathcal{P}$ denote the set of prime ideals of height 1, we say R is a *Krull domain* or *Krull ring* if it satisfies

(i) $R_{\mathfrak{p}}$ is a DVR for all $\mathfrak{p} \in \mathcal{P}$.

(ii) $R = \bigcap_{\mathfrak{p} \in \mathcal{P}} R_{\mathfrak{p}}$ where $R_{\mathfrak{p}}$ are viewed as subrings of $Q(R)$.

(iii) For any $r \in R$, the set $\{\mathfrak{p} \in \mathcal{P} \mid r \in \mathfrak{p}\}$ is finite.

Dedekind domains are obviously an example of Krull domains. Another example of Krull domains are UFD's. For a deeper exploration of Krull domains, we refer the reader to [17, §12] or [12, §IV.1]. We sum up some relevant results involving Krull domains.

Theorem 1.1.52. *Let R be a commutative domain.*

(i) *R is a Krull domain if and only if there exists a family of valuations $\{v_i\}_{i\in I}$ of $Q(R)$ such that for all $x \in Q(R)$, $v_i(x) = 0$ for almost all $i \in I$ and $R = \bigcap_{i\in I} R_{v_i}$.*

(ii) *If R is Noetherian, R is a Krull domain if and only if it is integrally closed.*

(iii) *R is a Krull domain with $\dim(R) = 1$ if and only if R is Dedekind.*

(iv) *Let R be a Krull domain and $K/Q(R)$ a finite algebraic extension of $Q(R)$, denote by B the integral closure of R in K, then B is a Krull domain.*

(v) *Let R be a Krull domain, then $R[x_1, \dots, x_n]$, $R[x_1, \dots]$, and $R[[x]]$ are also Krull domains.*

(vi) *Let R be a UFD, then R is a Krull domain. Furthermore, all height 1 prime ideals are principal.*

Proof. For (i) see [17, Theorem 12.3] and [12, Lemma IV.1.2], for (ii) see [12, Corollary IV.1.5], for (iii) see [17, Theorem 12.5], and for (iv) see [4, VII §1, Proposition 12]. For (v) see [25, Proposition 4.3, 4.4] and [11, Proposition 1.6]. Finally, for (vi) see [25, Theorem 5.3(g)] and [4, VII §3.2, Theorem 1]. □

Remark 1.1.53. In the context of (i), *almost all* means for all $i \in I\backslash I_0$ where I_0 is a finite subset of I.

Example 1.1.54. We provide an example of a non-Noetherian Krull domain and a Noetherian Krull domain that is not Dedekind.

(i) The ring $\mathbb{Z}[x]$ is a Krull domain, since $\mathbb{Z}$ is Dedekind then it is a Krull domain, so by Theorem 1.1.52(v) $\mathbb{Z}[x]$ is a Krull domain. However, $\mathbb{Z}[x]$ has prime ideals (p, x) where p is any prime of $\mathbb{Z}$ and $(0) \subsetneq (p) \subsetneq (p, x)$ is a chain of prime ideals, so the Krull dimension of $\mathbb{Z}[x]$ is at least 2.

(ii) For the non-Noetherian examples, take $\mathbb{Z}[x_1, \dots]$ which is Krull domain by Theorem 1.1.52(v). Another interesting example, due to Gilmer in [14], are the formal power series over a Krull domain in infinitely many variables. Gilmer proves this for the three usual topologies given to power series over infinitely many variables.

1.2 Preliminaries on module theory

This section is devoted to introducing and recalling some definitions and results pertaining to the theory of modules. In preparation for later discussions, we aim to introduce the concepts of global dimension and weakly global dimension and present a structure theorem for finitely generated torsion modules over Dedekind domains.

Throughout this section, we denote by R a nonzero unital commutative ring. Additional hypotheses on R may be added by each particular result. We start by providing some general definitions.

1.2.1 Types of modules and their characterizations

Definition 1.2.1 (Finitely generated)**.** An R-module M is *finitely generated* if there exist a finite number of elements $a_1, \dots, a_n \in M$ such that $M = \langle a_1, \dots, a_n \rangle$, meaning each element $a \in M$ is an R-linear combination of $a_1, \dots, a_n$.

Remark 1.2.2. We may rephrase this definition as follows: An R-module M is finitely generated if there exists an integer $n \geq 0$ and a surjective morphism $\pi : R^n \to M$. This morphism is $\pi(r_1, \dots, r_n) = \sum_{i=1}^{n} a_i r_i$.

Definition 1.2.3 (Finitely presented)**.** An R-module M is *finitely presented* if there exist integers $n, m \geq 0$ such that the sequence $R^m \to R^n \to M \to 0$ is exact.

Remark 1.2.4. By the previous observation, every finitely presented module is finitely generated.

Lemma 1.2.5. *Let R be Noetherian, then a finitely generated module is finitely presented.*

Proof. Let M be finitely generated and consider a surjective map $\pi : R^n \to M$ as in Remark 1.2.2. We proceed by induction on n. If $n = 1$ then $\mathrm{Ker}(\pi)$ is an ideal of R, so it is a finitely generated R-module and we have a surjective morphism $\tau : R^m \to \mathrm{Ker}\,\pi \hookrightarrow R$ for some $m \geq 0$. Then $R^m \xrightarrow{\tau} R \xrightarrow{\pi} M \to 0$ is exact and a presentation of M.

Now assume the statement is true for modules generated by $n - 1 \geq 0$ elements, consider a module generated by n elements $a_1, \dots, a_n$ and $R^n \xrightarrow{\pi} M \to 0$. Consider the R-submodule $N = \langle a_1, \dots, a_{n-1} \rangle$ of M, then N is finitely presented, in particular, it is finitely generated and take $p : R^{n-1} \to N$. Using the Snake lemma, we obtain the diagram in Equation 1.2.

By induction hypothesis, $\mathrm{Ker}\,p'$ and $\mathrm{Ker}\,p$ are finitely generated, so $\mathrm{Ker}\,\pi$ is finitely generated and so M finitely presented.

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathrm{Ker}\,p' & \longrightarrow & R & \xrightarrow{p'} & M/N & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathrm{Ker}\,\pi & \longrightarrow & R & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathrm{Ker}\,p' & \longrightarrow & R & \xrightarrow{p} & N & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \\
0 & & 0 & & 0 & &
\end{array} \tag{1.2}$$

□

Definition 1.2.6 (Torsion)**.** Let M be an R-module and $a \in M$. We say m is a *torsion element* if there exists $r \in R$ non-zero divisor such that $rm = 0$. We define the set $M_{\text{tor}} = \{m \in M \mid m \text{ is torsion}\}$ as the *torsion submodule* of M. If $M = M_{\text{tor}}$ we say that M is a *torsion module* or *torsion*, if $M_{\text{tor}} = \{0\}$ then we say it is *torsion-free*.

Remark 1.2.7. We need to prove that M_{tor} is in fact a submodule of M.

Proposition 1.2.8. *Let M be an R-module, then M_{tor} is a submodule of M.*

Proof. Let $a, a' \in M_{\text{tor}}$ and $r \in R$. Then there exist $s, s' \in R$ non-zero divisors such that $sa = s'a' = 0$. Note that $ss' \neq 0$ is not a zero divisor and $ss'r \neq 0$, so $ss'(a + ra') = 0$ meaning $a + ra' \in M_{\text{tor}}$. □

Lemma 1.2.9. *An R-module M is torsion if and only if $M \otimes_R Q(R) = 0$, where $Q(R)$ is the total quotient ring.*

Proof. If M is torsion, then for any $m \in M$ there exists an $r \in R$ non-zero divisor with $rm = 0$. Then $0 = rm \otimes (1/r) = m \otimes 1$, so $M \otimes_R Q(R) = 0$. Conversely, if $M \otimes_R Q(R) = 0$ then $S^{-1}M = 0$, where S is the set of non-zero divisors of R, so $m/1 = 0$ implies that there exists some $r \in S$ such that $rm = 0$. □

Definition 1.2.10 (Free module)**.** Let M be an R-module. We say M is a *free module* if it is the direct sum of copies of R, meaning $M \cong \bigoplus_{i \in I} R_i$ where $R_i \cong R$ and I is an index set which need not be finite. We call I a *basis* of M.

Remark 1.2.11. Note that if R is a domain, then a free R-module is torsion-free.

Lemma 1.2.12. *Every R-module is a quotient of a free module.*

Proof. Given a module M choose a generating set S of M. Consider $F = \bigoplus_{s \in S} R$ and the R-linear morphism $p : F \to M$ defined by $p(e_s) = s$, where $e_s = (\delta_{st})_{t \in S}$ and δ_{st} is the Kronecker delta, and extended by R-linearity. Then p is surjective and $M \cong F/\text{Ker}(p)$. □

Definition 1.2.13 (Flat module)**.** Let M be an R-module. We say M is *flat* if the functor $M \otimes_R -$ is exact.

Remark 1.2.14. Note that free modules are flat since $(\bigoplus_{i \in I} R) \otimes_R M \cong \bigoplus_{i \in I} M$.

Lemma 1.2.15. *Let M be a flat R-module and R be a domain. Then M is torsion-free.*

Proof. Since M is flat, $0 \to R \to Q(R)$ exact implies $0 \to M \otimes_R R \to Q(R) \otimes_R M$ and $M \otimes_R R \cong R$ while $Q(R) \otimes_R M$ is the localization of M at $R \backslash \{0\}$ and is a $Q(R)$ vector space, so it is torsion-free and we can include M in it, so it is also torsion-free. □

Definition 1.2.16 (Projective module)**.** Let M, N, N' be R-modules. We say M is a *projective module* if for every surjective morphism $p : N \to N'$ and morphism $h : M \to N'$ there exists a lifting $g : M \to N$ such that $p \circ g = h$.

Lemma 1.2.17. *A free R-module M is projective.*

Proof. We use the notation of Definition 1.2.16. Take a basis $\{x_i\}_{i\in I}$ of M, then $h(x_i) = p(a_i)$ for some $a_i \in N$ by surjectivity of p. Now simply define $g : M \to N$ as the R-linear extension of $g(x_i) = a_i$. □

We provide some usual and useful characterizations and properties of projective modules.

Proposition 1.2.18. *Let M be an R-module. The following are equivalent:*

(i) *M is projective.*

(ii) *Every short exact sequence $0 \to N \to N' \to M \to 0$ splits.*

(iii) *The functor $\mathrm{Hom}_R(M, -)$ is exact.*

(iv) *M is a direct summand of a free R-module.*

(v) *There exist $a_i \in M$, $i \in I$ and R-linear maps $\phi_i : M \to R$, $i \in I$ such that for each $x \in M$, $\phi_i(x) = 0$ for almost all $i \in I$ and $x = \sum_{i\in I} \phi_i(x)a_i$. Moreover, $\{a_i \in M \,|\, i \in I\}$ generates M.*

Proof. See [24, Theorem 3.5, Propositions 3.2, 3.3, 3.10] □

Proposition 1.2.19. *Let M be a projective R module, the following statements hold:*

(i) *If M is finitely generated then it is finitely presented.*

(ii) *M is the direct sum of countably generated projective modules. Countably generated module refers to a module with a countable set of generators.*

(iii) *M is flat and if R is a domain, then it is torsion-free.*

Proof. See [24, Corollary 3.9, Proposition 3.11, 3.46] □

Theorem 1.2.20. *Let M be a projective R-module, R is free if any of the following conditions is satisfied:*

1. *R is a local ring.*
2. *R is a semi-local ring.*
3. *R is a PID.*

Proof. See [17, Theorem 2.5] for (i), [12, Theorem VI.1.11] for (ii) and [23, Theorem 9.8] for (iii). □

Definition 1.2.21 (Fractional ideal)**.** Let R be a commutative domain. Let $J \subseteq Q(R)$ be an R-submodule, we say J is a *fractional ideal* if there exists $r \in R$ such that $rJ \subseteq R$. We say an non-zero fractional ideal J is *invertible* if there exists another fractional ideal I such that $IJ = R$.

These are related to projective modules as follows.

Lemma 1.2.22. *A fractional ideal is projective if and only if it is invertible.*

Proof. See [17, Theorem 11.3]. □

Proposition 1.2.23. *Let R be a domain. The following are equivalent*

(i) R is Dedekind

(ii) All fractional ideals of R are invertible.

(iii) All fractional ideals of R are projective.

Proof. See [23, Proposition 11.93] for (i) $\iff$ (ii) and use the previous lemma for (ii) $\iff$ (iii). □

We now focus on the ideals of a ring as R-modules. We can wonder the effects of a ring having all its ideals be projective.

Definition 1.2.24 (Hereditary ring)**.** A ring R is *hereditary* if all of its ideals are projective R-modules.

Example 1.2.25. An easy example are PID's, where all ideals are isomorphic to the ring itself as modules, thus free. Another example of hereditary rings are semisimple rings. Recall that all ideals of a semisimple ring are direct summands of R, so they are projective as R is free as an R-module. Finally, we have also seen in Proposition 1.2.23(iii) that Dedekind domains are hereditary.

Proposition 1.2.26. *A commutative domain R is hereditary if and only if it is a Dedekind ring.*

Proof. We already know that Dedekind rings are hereditary. We only need to show that if all ideals are invertible, then so are all fractional ideals. Take J a fractional ideal, $rJ \subseteq R$ is an ideal so it has an inverse I. Then $rJI = R \implies J(rI) = R$ so J is invertible. □

The name "hereditary" comes from the following characterization.

Theorem 1.2.27 (Cartan-Eilenberg)**.** *A ring R is hereditary if and only if all submodules of projective modules are projective.*

Proof. See [24, Theorem 4.19]. □

1.2.2 Homological dimensions

We can now introduce the concepts of projective resolution and projective dimension.

Definition 1.2.28 (Projective resolution)**.** Let M be a non-zero R-module. Let P_i, $i \geq 0$ be projective R-modules such that $P_\bullet \to M \to 0$ is an exact sequence, where $P_\bullet$ denotes $\cdots \to P_i \to \cdots \to P_1 \to P_0$. We say $P_\bullet$ is a *projective resolution* of M. We say a resolution is finite if there exists some $n \geq 0$ such that $P_i = 0$ for all $i \geq n$.

Lemma 1.2.29. *Given a non-zero R-module M, there exists a projective resolution of M.*

Proof. This is a direct consequence Lemma 1.2.12, as there exists a free module F_0 and a module K_0 such that $0 \to K_0 \to F_0 \to M \to 0$. But now such a sequence also exists for K_0, so we can chain $0 \to K_1 \to F_1 \to K_0 \to 0$ and construct $F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$ inductively. $F_\bullet$ are free, in particular, projective. □

Remark 1.2.30. In the proof of Lemma 1.2.29, we have actually seen that any non-zero module has a free resolution.

This immediately begs the question of how small can a projective resolution be.

Definition 1.2.31 (Projective dimension)**.** Let M be a non-zero R-module, we define the projective dimension of M as $\mathrm{pd}(M) = n \geq 0$ as the length of the shortest finite projective resolution of M. If no finite resolution exists, $\mathrm{pd}(M) = \infty$.

Example 1.2.32. (a) If M is projective then $\mathrm{pd}(M) = 0$.

(b) If R is hereditary, then for every R-module M we have $\mathrm{pd}(M) \leq 1$. If we have the start of a projective resolution $P \to M \xrightarrow{\epsilon} 0$, note that $\operatorname{Ker}\epsilon$ is an R-submodule of P, so it is projective by Theorem 1.2.27, thus $0 \to \operatorname{Ker}\epsilon \to P \to M \to 0$ is exact.

(c) If R is semisimple, then all its R-modules are projective [24, Proposition 4.5]. Thus $\mathrm{pd}(M) = 0$ for all M.

These examples motivate the following definition.

Definition 1.2.33 (Global dimension)**.** The *global dimension* of a ring R is

$$\mathrm{gd}(R) = \sup\{\mathrm{pd}(M) \mid M \text{ an } R\text{-module}\}$$

Example 1.2.34. By the previous example, $\mathrm{gd}(R) \leq 1$ if R is hereditary and $\mathrm{gd}(R) = 0$ if R is semisimple. Furthermore, the converse also holds for semisimple rings [23, Example 11.126], so we may think of the global dimension as a measure of the semisimplicity of the ring.

In the same way we have created resolutions out of projective modules, we can create resolutions using flat modules.

Definition 1.2.35 (Flat resolution)**.** Let M be a non-zero R-module. Let N_i, $i \geq 0$ be flat R-modules such that $N_\bullet \to M \to 0$ is an exact sequence. We say $N_\bullet$ is a *flat resolution* of M. We say a resolution is finite if there exists some $n \geq 0$ such that $N_i = 0$ for all $i \geq n$.

Remark 1.2.36. By the same proof of Lemma 1.2.29 and Remark 1.2.14, flat resolutions exist for every non-zero module.

Definition 1.2.37 (Weak dimension)**.** Let M be a non-zero R-module, we define the *weak dimension* of M as $\mathrm{wd}(M) = n \geq 0$ the length of the shortest finite flat resolution of M. If no finite resolution exists, $\mathrm{wd}(M) = \infty$.

Definition 1.2.38 (Weak global dimension)**.** The *weak global dimension* of a ring R is

$$\mathrm{wgd}(R) = \sup\{\mathrm{wd}(M) \mid M \text{ an } R\text{-module}\}$$

Remark 1.2.39. Since all projective modules are flat, then $\mathrm{wd}(M) \leq \mathrm{pd}(M)$ for all R-modules M. Furthermore, $\mathrm{wgd}(R) \leq \mathrm{gd}(R)$ for any R. This motivates the "weak" denomination.

Example 1.2.40. Note that semisimple rings have weak global dimension 0. Flat and projective modules have weak dimension 0. Moreover, as in Example 1.2.32(b), $\mathrm{wgd}(R) \leq 1$ if and only if every submodule of a flat module is flat. We will provide examples of rings and modules having positive weak dimension in a later section.

We can characterize the weak dimension using localization as follows.

Lemma 1.2.41. *Let M be an R-module and $\mathfrak{m}$ a maximal ideal, then* $\mathrm{wd}_R(M) = \sup_{\mathfrak{m}} \mathrm{wd}_{R_\mathfrak{m}}(M_\mathfrak{m})$. *Furthermore,* $\mathrm{wgd}(R) = \sup_{\mathfrak{m}} \mathrm{wgd}(R_\mathfrak{m})$.

Proof. To check this we need an additional criterion for the weak dimension. We use that $\mathrm{wd}(M) = n$ if and only if $\mathrm{Tor}_k^R(M, N) = 0$ for all $k \geq n+1$ [24, Proposition 8.17]. Furthermore, $(\mathrm{Tor}_k^R(M, N))_\mathfrak{m} \cong \mathrm{Tor}_k^{R_\mathfrak{m}}(M_\mathfrak{m}, N_\mathfrak{m})$, so $\mathrm{wd}_R(M) \leq \mathrm{wd}_{R_\mathfrak{m}}(M_\mathfrak{m})$. In general, an R-module L satisfies $L_\mathfrak{m} = 0$ for all $\mathfrak{m}$ if and only if $L = 0$. Thus we taking the supremums we recover $\mathrm{wd}_R(M)$. □

Proposition 1.2.42. *A ring R is von Neumann regular if and only if* $\mathrm{wgd}(R) = 0$.

Proof. By Proposition 1.1.7, R is vNr if and only if $R_\mathfrak{m}$ is a field for any $\mathfrak{m}$ maximal ideal. Since $\mathrm{wgd}(R) \leq \mathrm{gd}(R)$ and the global dimension of a field is zero, so the previous condition is equivalent to $\mathrm{wgd}(R_\mathfrak{m}) = 0$ for all $\mathfrak{m}$ maximal ideal. By Lemma 1.2.41, this is equivalent to $\mathrm{wgd}(R) = 0$. □

Thus, vNr rings play the same role for weak global dimension as semisimple rings for global dimension. Finally, we present one last result regarding local valuation rings.

Theorem 1.2.43. *A local ring R is a valuation ring if and only if* $\mathrm{wgd}(R) \leq 1$.

Proof. See [10, §5, Theorem 4]. □

1.2.3 Miscellaneous results

We devote this last part of the section to presenting three unrelated results needed in later sections. The first one concerns the commutativity between the Hom functor and localization for finitely presented modules. The second is a specific version of Nakayama's lemma. Finally, we present a structure theorem for torsion modules over Dedekind domains.

Proposition 1.2.44. *Let R be a commutative ring and let M, N be R-modules. Let $S \subseteq R$ be a multiplicatively closed set. If M is finitely presented then*

$$S^{-1}\operatorname{Hom}_R(M,N) \cong \operatorname{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$$

as $S^{-1}R$-modules.

Proof. To ease notation, we denote S^{-1} as $(\bullet)'$, for example $R' \coloneqq S^{-1}R$. As M is finitely presented, there exist $m, n \geq 0$ such that $R^m \to R^n \to M \to 0$ is exact. Recall that localization is exact, or that localization can be seen as $R' \otimes_R -$ and so $(R^m)' \cong (R')^m$ for any m, then

$$R^m \to R^n \to M \to 0 \ \text{ exact } \implies (R')^m \to (R')^n \to M' \to 0 \ \text{ exact}$$

where we now have a sequence of R'-modules. Now we apply the contravariant $\operatorname{Hom}_{R'}(-, N')$ to the previous equation and use that the functor is left-exact.

$$0 \to \operatorname{Hom}_{R'}(M', N') \to \operatorname{Hom}_{R'}((R')^n, N') \to \operatorname{Hom}_{R'}((R')^m, N') \ \text{ exact}$$

Note that $\operatorname{Hom}_{R'}((R')^n, N') \cong (N')^n$, since any R'-linear map from $(R')^n$ to N' is determined by an element of $(N')^n$ associating each component to the image of $e_i = (\delta_{ij})$, $i = 1, \ldots, n$. Thus, we obtain the sequence

$$0 \to \operatorname{Hom}_{R'}(M', N') \to (N')^n \to (N')^m \ \text{ exact} \tag{1.3}$$

Let us go back to the exact sequence $R^m \to R^n \to M \to 0$. Now we start taking the contravariant functor $\operatorname{Hom}_R(-, N)$ and obtain

$$0 \to \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_R(R^n, N) \to \operatorname{Hom}_R(R^m, N)$$

Again, $\operatorname{Hom}_R(R^n, N) \cong N^n$ and we can take the localization at S obtaining

$$0 \to \operatorname{Hom}_R(M, N)' \to (N')^n \to (N')^m \tag{1.4}$$

Joining the sequences in equations (1.3) and (1.4) we obtain the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \operatorname{Hom}_R(M,N)' & \xrightarrow{\phi} & (N')^n & \longrightarrow & (N')^m \\ \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle Id} \\ 0 & \longrightarrow & 0 & \longrightarrow & \operatorname{Hom}_{R'}(M',N') & \xrightarrow{\psi} & (N')^n & \longrightarrow & (N')^m \end{array} \tag{1.5}$$

It can be checked that the maps $(N')^n \to (N')^m$ are the same in both rows. The map φ can be obtained using the universal property of the tensor product by taking $\bar{\varphi} : R' \times \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_{R'}(M',N')$ such that $\bar{\varphi}(a/s,\alpha)(b/t) = (a/st)\alpha(b)$. This is clearly bilinear and balanced, inducing φ. To see that the remaining square commutes, we use the tensor product interpretation of localization. Note that

$$\phi\left(\frac{a}{s}\otimes\alpha\right) = \left[(\mathrm{Id}\otimes g_*)\left(\frac{a}{s}\otimes\alpha\right)\right](e_i)_i = \frac{a}{s}\otimes\Big((\alpha\circ g)(e_i)_i\Big)$$

where $g : R^n \to M$ is the morphism given by the finite presentation of M and $g_* = \mathrm{Hom}_R(g,N)$. Furthermore, $\psi(\beta) = (\beta\circ(\mathrm{Id}\otimes g))(e_i)_i$, so all together

$$(\psi\circ\varphi)\left(\frac{a}{s}\otimes\alpha\right) = \left[\varphi\left(\frac{a}{s}\otimes\alpha\right)\circ(\mathrm{Id}\otimes g)\right](e_i)_i$$

Considering the elements of $\mathrm{Hom}_{R'}(M',N')$ as maps $R'\otimes_R M \to R'\otimes_R N$ then $\varphi(a/s,\alpha) = (a/s)\mathrm{Id}\otimes\alpha$ by definition. So

$$(\psi\circ\varphi)\left(\frac{a}{s}\otimes\alpha\right) = \frac{a}{s}\otimes\Big((\alpha\circ g)(e_i)_i\Big) = \phi\left(\frac{a}{s}\otimes\alpha\right)$$

So it commutes. By the Five Lemma, φ is an isomorphism. □

We proof the following equivalent statement of Nakayama's Lemma.

Lemma 1.2.45 (Nakayama)**.** *Let M be a finitely generated R module and $I \subseteq J(R)$ an ideal contained in the Jacobson radical of R.*

(i) If $IM = M$ then $M = 0$.

(ii) If $M = IM + a_1R + \cdots + a_nR$ for some $a_i \in M$ then $M = a_1R + \cdots + a_nR$.

Proof. (i) Consider $M \neq 0$ is finitely generated, let $a_1,\ldots,a_n$ be a minimal set of generators of M. Then $\sum_{i=1}^n a_ir_i = a_n$ for some $r_i \in I$ but then $a_n(1-r_n) = \sum_{i=1}^{n-1} a_ir_i$ and $1-r_n$ is invertible since $r_n \in J(R)$, so a_n is an R-linear combination of a_i for $i = 1,\ldots,n-1$ contradicting the minimality of the generator set. So $M = 0$.

(ii) Let $N = a_1R + \cdots + a_nR$, then $M = IM + N$. Consider the quotient $M/N = (IM+N)/N = IM/N$ and so by (ii) we have $M/N = 0$, thus $M = N$. □

We now present the well-known structure theorem for torsion modules over Dedekind domains.

Theorem 1.2.46 (Jordan decomposition)**.** *Let R be a Dedekind domain. Let M be a finitely generated R-module, then there exist prime ideals $\mathfrak{p}_i$ for $1 \leq i \leq n$ for some $n \geq 1$ and natural numbers $m_1,\ldots,m_n \geq 1$ such that*

$$M \cong \bigoplus_{i=1}^{n} R/\mathfrak{p}_i^{m_i} \tag{1.6}$$

Let $k \geq 1$, $\ell_j \geq 1$ for $j = 1, \ldots, k$ and consider prime ideals $\mathfrak{q}_j$ for $j = 1, \ldots, k$ such that

$$M \cong \bigoplus_{i=1}^{n} R/\mathfrak{p}_i^{m_i} \cong \bigoplus_{j=1}^{k} R/\mathfrak{q}_j^{\ell_j} \tag{1.7}$$

then $k = n$ and there exists a permutation $\sigma \in S_n$ such that $\mathfrak{p}_i = \mathfrak{q}_{\sigma(i)}$ and $m_i = \ell_{\sigma(i)}$ for all $i = 1, \ldots, n$.

We refer the interested reader to [6, Chapter 10], where a full description of the proof and necessary concepts is given. In particular, the presented theorem is [6, Theorem 10.3.9] and its proof is given in section [6, §10.3.4].

Chapter 2

Non-Noetherian rings and modules

This chapter aims to introduce definitions and properties describing rings and modules without the Noetherian property. Many of the rings and modules introduced in the previous section require properties that are too strong to be satisfied by most non-Noetherian structures.

The first section is devoted to introducing definitions and the interplay between the newly introduced structures. The philosophy of the section essentially reduces to proposing new definitions, imposing similar conditions to those of Chapter 1, but only requiring them to be satisfied for the finitely generated ideals or modules.

In the second section, we strive to present and prove characterizations of finitely presented modules over valuation rings and Prüfer domains. These results are mostly due to Warfield [28] and have been adapted by Fuchs and Salce in the book [12]. This section serves as a mostly self-contained approach to Warfield's structure theorems, providing the necessary preliminaries and proving the results with great detail.

Throughout this chapter, we assume R is a commutative ring with unit.

2.1 From Prüfer domains to semi-hereditary rings

2.1.1 Prüfer and Bézout domains

The first type of ring we introduce is a generalization of Dedekind domains.

Definition 2.1.1 (Prüfer domain)**.** A ring R is a *Prüfer ring* or *Prüfer domain* if it is a domain and for any maximal ideal $\mathfrak{m}$ the localization $R_{\mathfrak{m}}$ is a valuation ring.

Remark 2.1.2. Dedekind domains are obviously Prüfer domains. But not all Prüfer domains are Dedekind.

Lemma 2.1.3. *Let R be a Noetherian ring, R is a Prüfer domain if and only if R is Dedekind.*

Proof. We need only show that Noetherian Prüfer rings are Dedekind. If R is Noetherian and $\mathfrak{m}$ is a maximal ideal, $R_{\mathfrak{m}}$ is Noetherian. A Noetherian valuation ring is a DVR (Proposition 1.1.31), so R is Dedekind. □

Many properties from Dedekind rings generalize to Prüfer domains, for instance, the following lemma is analogous to Proposition 1.2.23(ii).

Lemma 2.1.4. *A ring R is a Prüfer domain if and only if all its non-zero finitely generated fractional ideals are invertible.*

Proof. See [12, Theorem III.1.1]. □

The following is a generalization of Krull-Akizuki's Theorem 1.1.37 and motivates an example of a non-noetherian Prüfer ring.

Theorem 2.1.5. *Let R be a Prüfer domain and $Q(R)$ its fraction field. Consider an algebraic extension $K/Q(R)$, then if S denotes the integral closure of R over K, S is Prüfer.*

Proof. See [12, Theorem III.1.2]. □

Example 2.1.6. We cite some concrete examples of Prüfer domains that are not Dedekind domains:

(i) The subring of $\mathbb{C}$ of algebraic integers, i.e. the ring of complex roots of all $f(x) \in \mathbb{Z}[x]$.

(ii) The ring of homolorphic functions $f : \mathbb{C} \to \mathbb{C}$ (entire functions).

(iii) The subring of $\mathbb{Q}[x]$ of integer valued polynomials, i.e. the ring defined as $R = \{p(x) \in \mathbb{Q}[x] \mid p(n) \in \mathbb{Z} \text{ for all } n \in \mathbb{Z}\}$.

These comprise Examples 1.3, 1.4 and 1.5 of [12, §III.1]. The third example is of special interest, Brizolis proved in [5, Theorem S] that it is Prüfer.

We now introduce a generalization of principal ideal domains.

Definition 2.1.7 (Bézout ring)**.** A ring R is a *Bézout ring* if any finitely generated ideal is principal. If R is a domain, we say it is a *Bézout domain*.

Example 2.1.8. PID's and valuation rings (Lemma 1.1.19) are Bézout. Any non-noetherian valuation ring is an example of a Bézout domain that is not a PID. By Proposition 1.1.31, it is enough to construct a non-discrete valuation. Let k be a field and consider $k(x, y) = Q(k[x, y])$. Consider $\mathbb{Z}^2$ with the lexicographic ordering and the valuation $v : k(x, y) \to \mathbb{Z}^2_\infty$ given by $v(c) = (0, 0)$ for all $c \in k$, $v(x^n) = (n, 0)$, $v(y^n) = (0, n)$ for any $n \in \mathbb{Z}$ and for any polynomial $f(x, y) \in k[x, y]$ with monomials f_i define $v(f) = \min\{v(f_i)\}$. Finally, take $v(f/g) = v(f) - v(g)$. The ring defined by R_v is a non-Noetherian valuation ring.

Lemma 2.1.9. *Let R be a Bézout domain, then R is Prüfer.*

Proof. Any finitely generated fractional ideal of R is principal, so it is invertible. Thus Lemma 2.1.4 implies R is a Prüfer domain. □

So Bézout domains are a particular type of Prüfer rings.

Example 2.1.10. We give non-Noetherian examples of a Bézout domain and a Prüfer ring that is not Bézout.

(i) The subring A of $\mathbb{C}$ of algebraic integers, introduced in Example 2.1.6(i), is Bézout by the Principal ideal theorem [20, Theorem VI.7.5], which states that in the Hilbert class field H of a number field k, any ideal of $\mathcal{O}_k = A \cap k$ becomes principal. Thus, given any finitely generated ideal $I \subseteq A$ with generators $a_1, \dots, a_n$, consider the number field $k = \mathbb{Q}(a_1, \dots, a_n)$, then $I \subseteq \mathcal{O}_k$ and $I\mathcal{O}_H \subseteq A$ is principal.

(ii) The ring of integer-valued rational polynomials (Example 2.1.6(iii)) is not Bézout, as (n, x) for some $n \in \mathbb{Z}$ is not principal. In fact, Gilmer and Smith showed in [15, Theorem 4.2] that any finitely generated ideal of this ring is 2-generated. This is a property that Dedekind domains have, see Proposition 1.1.36, but does not translate to all Prüfer domains, see for instance the example of Schülting in [26].

The following proposition allows us to identify more Bézout domains from Prüfer rings.

Proposition 2.1.11. *Let R be a semi-local Prüfer domain, then R is Bézout.*

Proof. Let $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ be the maximal ideals of R. It is enough to show that any 2-generated ideal can be reduced to a principal ideal. Let $a, b \in R$ and $I = aR + bR$. Since $R_{\mathfrak{m}_i}$ are valuation rings for $i = 1, \dots, n$, ideals are totally ordered, meaning $aR_{\mathfrak{m}_i} \leq bR_{\mathfrak{m}_i}$ or $bR_{\mathfrak{m}_i} \leq aR_{\mathfrak{m}_i}$. Up to a reordering the maximal ideals, we can choose an integer $k \geq 1$ such that $aR_{\mathfrak{m}_i} \leq bR_{\mathfrak{m}_i}$ for all $i = 1, \dots, k$ and $bR_{\mathfrak{m}_i} < aR_{\mathfrak{m}_i}$ for the rest. This implies $IR_{\mathfrak{m}_i} = bR_{\mathfrak{m}_i}$ for $i \leq k$ and $IR_{\mathfrak{m}_i} = aR_{\mathfrak{m}_i}$ for the rest.

By the prime avoidance Lemma 1.1.38, there exist elements $c_j \in \mathfrak{m}_j \setminus \bigcup_{i \neq j} \mathfrak{m}_i$. Let $r = c_1 \cdots c_k$ and $s = c_{k+1} \cdots c_n$, note that $r \in \mathfrak{m}_i$ if and only if $i \leq k$ and similarly, $s \in \mathfrak{m}_i$ if and only if $i \geq k+1$. Consider $c = ra + sb$, then for all $1 \leq i \leq n$ we have $cR_{\mathfrak{m}_i} \leq IR_{\mathfrak{m}_i}$. If $i \leq k$ then $s \notin \mathfrak{m}_i$, $r \in \mathfrak{m}_i$, so s is invertible in $R_{\mathfrak{m}_i}$ and $r \in \mathfrak{m}_i R_{\mathfrak{m}_i}$. Now $a = b\alpha$ for some $\alpha \in R_{\mathfrak{m}_i}$ so $c = (r\alpha + s)b$ and $r\alpha + s$ must be a unit, so $cR_{\mathfrak{m}_i} = IR_{\mathfrak{m}_i}$. We can make the same argument for $i \geq k+1$ exchanging the roles of a and b, so $cR_{\mathfrak{m}_i} = IR_{\mathfrak{m}_i}$ for all maximal ideals, thus $I = cR$. □

2.1.2 Elementary divisor domains

We now introduce a generalization of unique factorization domains. This will allow us to talk about greatest common divisors in more general settings.

Definition 2.1.12 (GCD domain)**.** A ring R is a *GCD domain* if for every pair of elements $a, b \in R$ there exists $d \in R$ such that $d \mid a, b$ and, if $c \mid a, b$ then $c \mid d$. We denote $d = \gcd(a, b)$.

Remark 2.1.13. It is readily seen that usual properties of the gcd hold over GCD domains.

Example 2.1.14. All UFD's are obviously GCD domains. Furthermore, valuation rings are also GCD domains by the total ordering of the ideals.

Lemma 2.1.15. *Let R be a Bézout domain, then R is a GCD domain.*

Proof. Let $a, b \in R$, then let $d \in R$ such that $aR + bR = dR$, note that $d \mid a, b$. Then let $c \in R$ such that $c \mid a, b$, meaning there exist $r, s \in R$ with $a = cr$, $b = sc$, then $a, b \in cR$ so $dR \subseteq cR$, thus $c \mid d$. □

Example 2.1.16. Not all GCD domains are Bézout, in the same sense that not all UFD's are PID. For a non-Noetherian example, take the polynomial ring over countably infinite variables $k[x_1, \dots]$ over a field k. It is a GCD domain, since given two polynomials f, g we can reduce to the subring containing only the variables of f and g, which is a UFD.

In the same way that we study matrices over PID's, we may want to asses their behavior over their generalizations, so over Bézout domains. In particular, we will see that matrices are related to finite presentation of modules, and their "reduction" allows for a description of these types of modules.

Definition 2.1.17 (Diagonal matrix)**.** Let $\mathbf{A} = (a_{i,j})$ be an $n \times m$ matrix over a ring R. We say it is *diagonal* if $a_{ij} = 0$ for all $i \neq j$.

Definition 2.1.18 (Diagonal reduction)**.** Let $\mathbf{A}$ be an $n \times m$ matrix over a ring R. We say $\mathbf{A}$ admits a *diagonal reduction* over R if there exist invertible square matrices $\mathbf{P}, \mathbf{Q}$ of dimensions m and n respectively such that $\mathbf{D} = \mathbf{PAQ}$ is a diagonal matrix with diagonal entries d_i such that $d_i \mid d_{i+1}$ for all $i = 1, \dots, \min(n, m)$.

Armed with this concept, we can start by seeing how this applies to Bézout domains.

Proposition 2.1.19. *Let R be a domain, the following are equivalent:*

(i) R is a Bézout domain.

(ii) All diagonal matrices over R admit a diagonal reduction.

(iii) For every 1×2 matrix (a, b) there exists an invertible 2×2 matrix $\mathbf{Q}$ such that $(a, b)\mathbf{Q} = (d, 0)$ for some $d \in R$.

(iv) Every row matrix admits a diagonal reduction.

Proof. (i) $\implies$ (ii): Let R be a Bézout domain and let $\mathbf{A}$ be a diagonal $n \times m$ matrix. We proceed by induction on the length of the diagonal $\ell = \min(n, m)$.

If $\ell = 1$, then it trivially is in a reduced form. Suppose diagonal matrices admit a diagonal reduction length $\ell - 1$, and consider $\mathbf{A}$ diagonal of length ℓ, meaning either $n = \ell$ or $m = \ell$. We can separate the matrix by blocks in $\operatorname{diag}(a_1, \mathbf{A}')$ with $\mathbf{A}'$ an $(n-1) \times (m-1)$ matrix, so of diagonal length $\ell - 1$, by induction hypothesis $\mathbf{A}'$ admits a diagonal reduction $\mathbf{D}' = \operatorname{diag}(d_2', \dots, d_\ell')$, we can thus reduce $\operatorname{diag}(a_1, \mathbf{A}')$ to $\operatorname{diag}(a_1, \mathbf{D}')$. Since R is a GCD domain and all its finitely generated ideals are principal, we have $d = \gcd(a_1, d_2')$ and $d = a_1 r + d_2' s$ for some $r, s \in R$. Moreover, there exist $a_1', d_2 \in R$ such that $a_1 = d a_1'$. Then $\operatorname{diag}(a_1, d_2')$ can be transformed through elementary transformations into $\operatorname{diag}(d, -a_1' d_2')$, we briefly describe the steps:

$$\begin{pmatrix} a_1 & 0 \\ 0 & d_2' \end{pmatrix} \mapsto \begin{pmatrix} a_1 & 0 \\ d & d_2' \end{pmatrix} \mapsto \begin{pmatrix} 0 & -a_1' d_2' \\ d & d_2' \end{pmatrix} \mapsto \begin{pmatrix} d & d_2' \\ 0 & -a_1' d_2' \end{pmatrix} \mapsto \begin{pmatrix} d & 0 \\ 0 & -a_1' d_2' \end{pmatrix}$$

where in the first step we use $d = a_1 r + d_2' s$. Then we can transform $\operatorname{diag}(a_1, \mathbf{D}')$ into $\operatorname{diag}(d, -a_1' d_2', d_3', \dots, d_\ell')$ and since $d \mid d_2'$ and $d_2' \mid d_i'$ for all $i = 2, \dots, \ell$, we can apply the induction hypothesis on $\operatorname{diag}(-a_1' d_2', d_3', \dots, d_\ell')$ once again and reduce $\mathbf{A}$ to $\operatorname{diag}(d, d_2, d_3, \dots, d_\ell)$ where $d \mid d_2$ so the matrix is in reduced diagonal form.

(ii) $\implies$ (i): Suppose all diagonal matrices admit a diagonal reduction, then given any $a, b \in R$ the matrix $\operatorname{diag}(a, b)$ may be reduced to $\operatorname{diag}(c, d)$ where $c \mid d$ and $aR + bR = cR + dR = cR$, so all finitely generated ideals are principal.

(i) $\iff$ (iii): Let $a, b \in R$ be Bézout, then $aR + bR = dR$ with $d = \gcd(a, b)$ and we can write $a = da'$, $b = db'$. In turn, $\gcd(a', b') = 1$ so there exist $r, s \in R$ such that $a'r + b's = 1$. Then consider the matrix

$$\mathbf{Q} = \begin{pmatrix} r & -b' \\ s & a' \end{pmatrix}, \quad (a, b)\mathbf{Q} = (ar + bs, -ab' + a'b) = (d, 0) \tag{2.1}$$

The determinant of the matrix is 1, so it is invertible. Taking the same steps in the opposite direction gives the reverse implication.

(iii) $\iff$ (iv): That (iv) implies (iii) is obvious. Assume (iii), then proceed by induction on the number of columns. For $m = 1$ columns the matrix is already diagonal reduced. Suppose the statement holds for $m - 1$ columns, let $\mathbf{A} = (a_1, \dots, a_n)$ be the matrix to be reduced. By induction hypothesis, there exists an invertible matrix $\mathbf{Q}'$ such that $(a_2, \dots, a_n)\mathbf{Q}' = (b, 0, \dots, 0)$. Note we can omit a matrix $\mathbf{P}'$ in front since it is just multiplying by a scalar and we are in a domain. Consider $\mathbf{Q} = \operatorname{diag}(1, \mathbf{Q}')$, then $\mathbf{AQ} = (a, b, 0, \dots, 0)$. Now by assumption there is a 2×2 matrix $\mathbf{Q}''$ that reduces (a, b) to $(d, 0)$, so we are done. □

We can only guarantee reduction over a Bézout domain when the matrices are already in diagonal form. However, we can consider the class of rings in which all matrices admit reductions.

Definition 2.1.20 (Elementary divisor ring)**.** Let R be a ring, we say R is an *elementary divisor ring* or *EDR* if any matrix admits a diagonal reduction over R.

Remark 2.1.21. By the previous Proposition 2.1.19, EDR domains are Bézout domains. It is currently an open question whether there are Bézout domains that are not EDR domains. However, it is known that not all Bézout rings are EDR, see the examples in [13, §3, §4].

We can now characterize the subclass of Bézout domains that are EDRs.

Proposition 2.1.22. *Let R be a Bézout domain, the following are equivalent:*

(i) R is an elementary divisor domain.

(ii) All 2×2 matrices over R admit diagonal reduction.

Proof. (i) $\implies$ (ii): By definition of EDR.

(ii) $\implies$ (i): Let $\mathbf{A}$ be the $n \times m$ matrix to be reduced. Proceed by induction on the number of rows n. Note that we may assume that $n \leq m$, otherwise we can simply transpose the matrix and use the same procedures.

Case $n = 1$ is direct by Proposition 2.1.19. Suppose matrices with $n-1$ rows can be reduced. We want to reduce $\mathbf{A} = (a_{ij})$. Consider $\mathbf{A}'$ the lower right matrix of size $(n-1) \times (m-1)$ of $\mathbf{A}$. By induction hypothesis we may reduce $\mathbf{A}'$ to a diagonal reduced matrix $\mathbf{D}'$ and transform $\mathbf{A}$ itself. We can now exchange the first row and first column with the second row and second column and repeat the reduction on the same block. Finally, applying the assumption we can put the matrix in diagonal form. Since R is Bézout, any diagonal matrix reduces, so we are done. The steps are illustrated in the following equation.

$$\mathbf{A} = \begin{pmatrix} a_{11} & * \\ * & \mathbf{A}' \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & * \\ * & \mathbf{D}' \end{pmatrix} \mapsto \begin{pmatrix} d_1' & * & \mathbf{0} \\ * & a_{11} & * \\ \mathbf{0} & * & \mathbf{D}'' \end{pmatrix} \mapsto \begin{pmatrix} d_1' & * & \mathbf{0} \\ * & d_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{D}''' \end{pmatrix}$$

where $\mathbf{D}' = \operatorname{diag}(d_i')$, $i = 1, \ldots, n$ and $\mathbf{D}'' = \operatorname{diag}(d_i'')$ with $i = 2, \ldots, n-1$ and $\mathbf{D}'''$ some diagonal matrix. □

Before proceeding to analyze the structure of finitely presented modules over EDR domains, recall we have seen that when R is semilocal, Prüfer and Bézout are equivalent conditions, see Proposition 2.1.11. We wonder if the same might happen with Bézout and EDR. We need a preparatory lemma first.

Lemma 2.1.23. *Let R be Bézout and $\mathfrak{m}$ a maximal ideal of R. Let $\mathbf{A}, \mathbf{A}'$ be 2×2 matrices*

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathbf{A}' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

such that $\gcd(a, b, c, d) = 1$, $a \in \mathfrak{m}$ and $aR \subseteq a'R$, $a' \notin \mathfrak{m}$. Then $\mathbf{A}$ and $\mathbf{A}'$ are equivalent.

Proof. First notice that by Proposition 2.1.19(iii), $\mathbf{A}$ is equivalent to a matrix $\mathbf{A}''$ with $b'' = 0$ and $a'' = \gcd(a, b)$, thus $aR \subseteq a''R$. If $a'' \notin \mathfrak{m}$, we are done setting $A' = A''$. Otherwise, without loss of generality, we may assume we start with $b = 0$.

Assume $c \notin \mathfrak{m}$, then let $a' = gcd(a, c)$ and $a'R = aR + cR$, we can write $a' = ax + by$ and notice that $a'R \nsubseteq \mathfrak{m}$ as $c \in a'R$. Since $a = ra'$ and $b = sa'$, note $1 = sx + ry$, so the matrix

$$\mathbf{P} = \begin{pmatrix} x & y \\ r & s \end{pmatrix}$$

is invertible and $\mathbf{A}' = \mathbf{PA}$ has first entry $a' \notin \mathfrak{m}$ with $aR \subseteq a'R$.

Now assume $c \in \mathfrak{m}$, by hypothesis, $aR + cR + dR = R$ so $d' \notin \mathfrak{m}$, we transform $\mathbf{A}$ by adding the second column to the first.

$$\mathbf{A} = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 \\ c+d & d \end{pmatrix}$$

and now $c + d \notin \mathfrak{m}$, so we reduce to the previous case. □

Proposition 2.1.24. *Let R be a semilocal Bézout domain, then R is an elementary divisor domain.*

Proof. By Proposition 2.1.22, it is enough to check that any 2×2 matrix admits a diagonal reduction. The zero matrix is already reduced. Suppose

$$\mathbf{A} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a 2×2 non-zero matrix. By moving rows and columns we may assume $a \neq 0$. Furthermore, we may assume $\gcd(a, b, c, d) = 1$, as products by a scalar do not affect the block structure of the matrix. As R is semilocal, there are finitely many maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$. If a is not contained in any, then it is a unit, so we can easily transform $\mathbf{A}$ into a diagonal matrix. Up to reordering, we may assume $a \in \mathfrak{m}_1, \ldots, \mathfrak{m}_k$ for some $k \geq 1$. By Lemma 2.1.23, we can find an equivalent matrix $\mathbf{A}_1$ such that $aR \subseteq a_1R$ and $a_1 \notin \mathfrak{m}_1$. Note that a_1 may only be contained in some of the $\mathfrak{m}_2, \ldots, \mathfrak{m}_k$, if $a_1 \in \mathfrak{m}_2$ we can use the lemma again, otherwise we move on to $\mathfrak{m}_3$ and so on. Thus, we repeatedly use the lemma to build a sequence of inclusions $aR \subseteq a_1R \subseteq a_2R \subseteq \cdots \subseteq a_kR$ where the a_i are the first entry of some matrix $\mathbf{A}_i$ equivalent to $\mathbf{A}$ and $a_i \notin \mathfrak{m}_j$ for any $j \leq i$. Thus a_k is invertible and $\mathbf{A}_k$ can be put in diagonal form:

$$\begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix} \mapsto \begin{pmatrix} 1 & a_k^{-1}b_k \\ a_k^{-1}c_k & a_k^{-1}d_k \end{pmatrix} \mapsto \begin{pmatrix} 1 & a_k^{-1}b_k \\ 0 & a_k^{-1}d_k - a_k^{-2}c_kb_k \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & a_k^{-1}d_k - a_k^{-2}c_kb_k \end{pmatrix}$$

□

Corollary 2.1.25. *Let R be a semilocal domain. The following are equivalent:*

(i) R is a Prüfer domain.

(ii) R is a Bézout domain.

(iii) R is an EDR.

Proof. This is a collection of Proposition 2.1.11, Proposition 2.1.24, Lemma 2.1.9 and Remark 2.1.21. □

To close this discussion on elementary divisor domains, we present a characterization of finitely presented modules over EDR domains. To do so, we first need to relate finite presentations with matrices over R.

Definition 2.1.26. Let R be a domain, let M be a finitely presented R-module with a presentation $R^m \xrightarrow{\phi} R^n \to M \to 0$ with $\phi : R^m \to R^n$. Let $\{x_1, \dots x_n\}$ and $\{y_1, \dots, y_m\}$ be bases for R^n and R^m respectively. Then the $n \times m$ matrix $\mathbf{A} = (a_{ij})$ associated to the R-linear map ϕ is called a *presentation matrix* or *relation matrix* and it is described by

$$\phi(y_j) = \sum_{i=1}^{n} a_{ij} x_i \text{ for all } j = 1, \dots, \ a_{ij} \in R \tag{2.2}$$

Remark 2.1.27. Note that changing the basis choice for a presentation matrix corresponds precisely to taking a product $\mathbf{PAQ}$ for some $\mathbf{P}, \mathbf{Q}$ invertible matrices.

Theorem 2.1.28. *Let M be a finitely presented R-module where R is an EDR domain. Then $M \cong \bigoplus_{i=1}^{n} C_i$ for some $n \geq 1$ and $C_i = R/d_iR$ where $d_i \mid d_{i+1}$ for all $i = 1, \dots n-1$ and $d_nR \subseteq d_{n-1}R \subseteq \cdots \subseteq d_1R$.*

Proof. Let $R^m \xrightarrow{\phi} R^n \to M \to 0$ be a presentation of M and $\ell = \min(n, m)$. Let $\mathbf{A}$ be a presentation matrix of ϕ. Since R is an EDR, it admits a diagonal reduction, so we may choose basis $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_m\}$ of R^n and R^m respectively under which $\mathbf{D} = \operatorname{diag}(d_1, \dots, d_\ell)$ is a $n \times m$ presentation matrix of ϕ with $d_i \mid d_{i+1}$ for all $i = 1, \dots, \ell - 1$. Note that $\phi(y_j) = d_j x_j$ for any $j \leq \ell$ and $\phi(y_j) = 0$ for $j \geq \ell + 1$. From the presentation exact sequence we get $M \cong \operatorname{Coker}(\phi)$. If $\ell < n$, define $d_j = 0$ for $j = \ell + 1, \dots, n$ and then

$$\operatorname{Im}\phi = \bigoplus_{j=1}^{n} d_j R \implies M \cong \operatorname{Coker}\phi \cong \bigoplus_{j=1}^{n} R/d_jR \cong \bigoplus_{j=1}^{\ell} R/d_jR \oplus \bigoplus_{j=\ell+1}^{n} R \tag{2.3}$$

so we get the statement. The chain $d_nR \subseteq d_{n-1}R \subseteq \cdots \subseteq d_1R$ is direct from $d_i \mid d_{i+1}$ for $j = 1, \dots, \ell$ and $d_j = 0$ for $j > \ell$. □

2.1.3 Semi-hereditary rings

To end the section, we return to our process of generalizing rings. We introduce a generalization of hereditary rings alongside important characterizations.

Definition 2.1.29 (Semi-hereditary ring)**.** We say R is *semi-hereditary* if any finitely generated ideal of R is projective.

The following results were initially presented by Endo in [10], and relate semi-hereditary rings with certain homological properties.

Theorem 2.1.30 (Endo)**.** *Let R be a ring, the following are equivalent.*

(i) R *is semi-hereditary.*

(ii) $\operatorname{wgd}(R) \leq 1$ *and* $\operatorname{wgd}(Q(R)) = 0$.

(iii) $Q(R)$ *is regular and* $R_{\mathfrak{m}}$ *is a valuation ring for any maximal ideal* $\mathfrak{m}$.

(iv) Any torsion-free R*-module* M *satisfies* $\operatorname{wd}_R(M) = 0$.

Proof. (ii) $\iff$ (iii) Direct by Proposition 1.2.42, Theorem 1.2.43 and Lemma 1.2.41.

The proof is quite involved for the rest of equivalences, so we skip them. See [10, Theorem 2] for (i) $\iff$ (ii) and [10, §5, Theorem 5] for (ii) $\iff$ (iv). □

Corollary 2.1.31. *A domain R is semi-hereditary if and only if it is Prüfer.*

Proof. Follows directly from the definition and Theorem 2.1.30. □

Remark 2.1.32. This is analogous to how all hereditary domains are Dedekind domains.

Theorem 2.1.33 (Endo)**.** *Let R be a ring, the following are equivalent.*

(i) R is a direct product of Prüfer rings.

(ii) $\operatorname{wgd}(R) \leq 1$ *and* $\operatorname{gd}(Q(R)) = 0$.

(iii) $Q(R)$ *is semisimple and* $R_{\mathfrak{m}}$ *is a valuation ring for any maximal ideal* $\mathfrak{m}$.

(iv) Any finitely generated torsion-free R-module is projective.

Proof. We only need to prove (i) $\iff$ (ii) and (ii) $\iff$ (iv):

(i) $\implies$ (ii): Take $R = R_1 \times \cdots \times R_n$ for $n \geq 1$. Since Prüfer rings are domains, the total quotient ring of R is a product of fields, so it is semisimple. The structure of the maximal ideals is given in Proposition 1.1.39, consider $\mathfrak{m}_i$ a maximal ideal of R_i and denote its associated maximal ideal in R as $\mathfrak{m}$, then $R_{\mathfrak{m}} \cong (R_i)_{\mathfrak{m}_i}$. Since R_i is Prüfer, $(R_i)_{\mathfrak{m}_i}$ is a local valuation ring, so $\operatorname{wgd}(R_{\mathfrak{m}}) \leq 1$. By Lemma 1.2.41, $\operatorname{wgd}(R) \leq 1$.

(ii) $\implies$ (i) If $\operatorname{gd}(Q(R)) = 0$ then $Q(R)$ is semisimple, meaning it is a finite product of fields. Thus, R is a finite product of domains. $\operatorname{wgd}(R) \leq 1$ implies $R_{\mathfrak{m}}$ is a valuation ring for any maximal ideal $\mathfrak{m}$ so R is a finite product of Prüfer rings.

(ii) $\iff$ (iv): See [10, §5, Corollary]. □

Corollary 2.1.34. *Let R be a direct product of Prüfer rings, then R is semi-hereditary.*

Proof. By Theorem 2.1.33(ii), $\operatorname{wgd}(R) \leq 1$ and $\operatorname{gd}(Q(R)) = 0$, thus $\operatorname{wgd}(Q(R)) = 0$ and by Theorem 2.1.30, R is semi-hereditary. □

2.2 Warfield's structure theorems

In this section we aim to introduce two structure theorems for finitely presented modules over valuation rings and Prüfer rings respectively. We start introducing four new types of modules. The study of these modules is required to obtain the structure theorems we desire.

Definition 2.2.1 (Cyclic module)**.** Let M be an R-module, M is *cyclic* if it is generated by a single element.

Lemma 2.2.2. *Let M be an R-module, M is cyclic if and only if $M \cong R/I$ for some ideal I.*

Proof. Suppose M is cyclic, then $xR = M$ for some $x \in M$. Define $\varphi : R \to M$ as $\varphi(r) = xr$, notice $\operatorname{Ker}(\varphi) = \operatorname{Ann}_R(x)$ and φ is surjective, thus $M \cong R/\operatorname{Ann}_R(x)$. Now let I be an ideal and $M = R/I$ then $M = (1 + I)R$. □

Definition 2.2.3 (Cyclically presented module)**.** Let M be an R-module, we say it is *cyclically presented* if it is cyclic and $M \cong R/I$ with I principal.

Corollary 2.2.4. *Let R be an EDR domain. A finitely presented R-module M is the direct sum of cyclically presented R-modules.*

Proof. By definition and Proposition 2.1.28. □

We proceed to study two types of submodules that, as will be seen, control the structure of a module over R when R is a valuation ring, EDR domain or Prüfer ring.

Definition 2.2.5 (RD-module)**.** Let M be an R-module, $N \subseteq M$ an R-submodule is *relatively divisible* or an *RD-module* if $rN = N \cap rM$ for all $r \in R$.

Lemma 2.2.6. *Let M be an R-module, $N \subseteq M$ an R-submodule. $N \subseteq M$ is an RD-module if and only if for any $a \in N$, $r \in R$ the equation $rx = a$ has solutions in $x \in N$ if it is solvable in M.*

Remark 2.2.7. If N is a direct summand of M, then N is an RD-module. If $rx = a$ with $x \in M$, $a \in N$ and $r \in R$ then, since $M = N \oplus S$, we can write $x = x_N + x_S$ so $rx_N + rx_S = a \in N$ so $rx_S = 0$. Thus, x_N solves the equation and $x_N \in N$.

The concept of RD-module may be generalized through the previous characterization.

Definition 2.2.8 (Pure module)**.** Let M be R-modules, $N \subseteq M$ an R-submodule is *pure* if every finite system of equations

$$\begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1m} \\ r_{21} & r_{22} & \cdots & r_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nm} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \tag{2.4}$$

with $r_{ij} \in R$, unknowns x_j and $a_i \in N$, has a solution in N when it is solvable in M.

Remark 2.2.9. Pure modules are in particular RD-modules. Furthermore, direct summands of M are pure by a similar reasoning to the RD case. In the sequel, we will see that in our context, i.e. over Prüfer domains, the RD property is equivalent to the pure property.

The following characterization of purity is an analogy to the definition of RD.

Lemma 2.2.10. *A submodule N of M is pure if and only if for every pair $m, n \geq 1$ and every $m \times n$ matrix $\mathbf{T} = (t_{ij})$ with $t_{ij} \in R$*

$$\mathbf{T}M^m \cap N^n = \mathbf{T}N^m \tag{2.5}$$

where M^m is the direct sum of m copies of M interpreted as column vectors. Similarly for N^n and N^m.

Proof. The elements of $\mathbf{T}M^m$ represent the LHS of Equation 2.4 while N^n represents the RHS. Thus, $\mathbf{T}M^m \cap N^n$ are the elements of N^n such that Equation 2.4 has a solution in M^m. If these are exactly $\mathbf{T}N^m$, then the systems that have solutions in M^m also have solutions in N^m. Conversely, if N is pure, all systems with solutions in M^m have solutions in N^m, so $\mathbf{T}M^m \cap N^n \subseteq \mathbf{T}N^m$. □

Corollary 2.2.11. *Purity and relative divisibility are local properties.*

Proof. If an R-submodule N of M is pure, then $N_{\mathfrak{m}} \subseteq M_{\mathfrak{m}}$ is preserved by localization being exact, and so purity (RD) descends. If all $N_{\mathfrak{m}}$ are pure in $M_{\mathfrak{m}}$, then by the previous characterization (definition in the RD case) and the fact that localization commutes with finite intersection, it follows that N is pure (RD). □

Pure and RD submodules can be viewed as properties of an exact sequence in the following sense.

Definition 2.2.12. An exact sequence $0 \to A \xrightarrow{\iota} B \to C \to 0$ is *pure-exact* (*RD-exact*) if $\iota(A) \subseteq B$ is a pure module (RD-module).

This definition enables characterizations of pure and RD-modules in a homological setting.

Lemma 2.2.13. *An exact sequence $0 \to A \xrightarrow{\iota} B \to C \to 0$ is pure-exact if and only if for every finitely presented R-module M the functor $\mathrm{Hom}_R(M, -)$ preserves the exactness of the sequence, meaning*

$$0 \to \mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C) \to 0$$

is exact.

Proof. The proof is technical, so we omit it. See [12, Lemma I.8.5]. □

Remark 2.2.14. Note that in the lemma, the module M acts as if it were a projective module but for a specific exact sequence (recall Proposition 1.2.18iii). For convenience, we give a name to this kind of behavior.

Definition 2.2.15. Given an exact sequence $0 \to A \xrightarrow{\iota} B \to C \to 0$ of R-modules, we say an R-module M has the projective property with respect to it if

$$0 \to \mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C) \to 0$$

is exact. In particular, we say M is *RD-projective* (*pure-projective*) if it has the projective property with respect to any RD-exact (pure-exact) sequence.

Remark 2.2.16. We have seen that a pure module is RD. A pure-exact sequence is also RD-exact. However, a pure-projective module is not necessarily RD-projective. If a module has the projective property for RD-exact sequences, then it also does for pure-exact sequences, as they are RD-exact. However, having the projective property for pure-exact sequences is not enough to be RD-exact, as there might be non pure-exact but RD-exact sequences.

Lemma 2.2.17. *An exact sequence $0 \to A \xrightarrow{\iota} B \xrightarrow{p} C \to 0$ is RD-exact if and only if every cyclically presented R-module has the projective property with respect to it.*

Proof. Recall there is a natural isomorphism

$$\mathrm{Hom}_R(R/rR, B) \cong \mathrm{Ann}_B(r) = \{b \in B \mid rb = 0\}$$

We focus on the map $\pi_r : \mathrm{Ann}_B(r) \to \mathrm{Ann}_{B/A}(r)$ as $C \cong B/A$, $\pi_r(b) = p(b) = b + A$. Suppose the sequence is RD-exact and take $b + A \in \mathrm{Ann}_{B/A}(r)$. Note that $r(b + A) = 0$ implies $rb = a$ for some $a \in A$. Then, by the RD property, there exists $a' \in A$ such that $ra' = a$ and so $r(b - a') = 0$, note that $\pi_r(b - a') = b + A$ so it is surjective.
Assume π_r is surjective for any $r \in R$. Consider the equation $rx = a$ for some $a \in A$ such that it has solution for some $x \in B$. This equation reduces to $r(x + A) = 0$ modulo A, since π_r is surjective, there is some $b \in B$ such that $rb = 0$ and $x - b = a' \in A$. Then $a = rx = r(a' + b) = ra'$ and so it has a solution in A. □

Remark 2.2.18. Lemma 2.2.17 shows that cyclically presented R-modules are RD-projective. Lemma 2.2.13 shows finitely presented R-modules are pure-projective.

Our aim is to eventually describe the structure of modules over Prüfer domains, so we start analyzing the structure of modules over valuation rings. Before proceeding to the result, we require a small lemma.

Lemma 2.2.19. *Let $L \subseteq N \subseteq M$ be R-modules such that L is an RD-submodule of M and N/L is an RD-submodule of M/L, then N is an RD-submodule of M.*

Proof. Note that $rL = rM \cap L$ and $rN/L = rM/L \cap N/L$ for any $r \in R$. We need to prove $rM \cap N \subseteq rN$, take $ra \in N$ such that $a \in M$. By the second property, we can choose $b \in N$ and $\ell \in L$ such that $ra = rb + \ell$. Now $\ell = r(a - b) \in rM \cap L$ so $\ell = r\ell'$ for some $\ell' \in L$, and finally $ra = r(b + \ell') \in rN$. □

Proposition 2.2.20 (Fuchs-Salce)**.** *Let R be a valuation ring and M a finitely generated R-module. There exists a finite chain of modules*

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M \tag{2.6}$$

such that each M_i is an RD-module of M and M_i/M_{i-1} is cyclic for all $i = 1, \ldots, n$.

Proof. Let $\mathfrak{m}$ be the unique maximal ideal of R (Lemma 1.1.19), since M is finitely generated, the $R/\mathfrak{m}$-vector space $M/\mathfrak{m}M$ is finitely generated.

Consider $a_1+\mathfrak{m}M, \ldots, a_n+\mathfrak{m}M$ a basis of $M/\mathfrak{m}M$. Then $M = \mathfrak{m}M+a_1R+\cdots+a_nR$ and $J(R) = \mathfrak{m}$, so $M = a_1R + \cdots + a_nR$ by Nakayama's Lemma 1.2.45. Using the generators we can write $\operatorname{Ann}_R(M) = \bigcap_{i=1}^n \operatorname{Ann}_R(a_i)$, and $\operatorname{Ann}_R(M) = \operatorname{Ann}_R(a_j)$ for at least one j since the ideals of R are totally ordered (Lemma 1.1.19). We can choose a_1 to be $\operatorname{Ann}_R(a_1) = \operatorname{Ann}_R(M)$, furthermore, we can also choose it such that all other elements of $a_1 + \mathfrak{m}M$ have the same annihilator. Otherwise, we could simply change all the representatives of the basis and obtain a larger $\operatorname{Ann}_R(M)$.

We proceed to show that a_1R is an RD-submodule of M. Suppose there exist $r, s \in R$ and $x \in M$ such that $sR \subsetneq rR$, r and $0 \neq ra_1 = sx$ meaning $r \notin \operatorname{Ann}_R(M)$. Let $s = rp, p \in \mathfrak{m}$, note that we can choose $p \in \mathfrak{m}$ as otherwise it would be a unit and $sR \subsetneq rR$. Then $a_1 - px \in a_1 + \mathfrak{m}M$ is annihilated by r as $ra_1 - rpx = ra_1 - sx = 0$. Then $r \in \operatorname{Ann}(a_1 - px) = \operatorname{Ann}(a_1)$ but this contradicts $ra_1 \neq 0$. Hence a_1R is RD.

Now set $M_1 = a_1R$. By the same process, M/M_1 has an RD-submodule of the form $(a_2 + M_1)R$ for some $a_2 + M_1 \in M/M_1$. By Lemma 2.2.19 the preimage $M_2 = a_2M_1+M_1$ is also an RD-submodule of M and M_2/M_1 is cyclic. This inductive process may be carried until the generators have been exhausted. □

We are now almost ready to present Warfield's structure theorem for finitely presented modules over valuation rings. We must prove first a technical lemma.

Lemma 2.2.21. *Let R be a valuation ring, then a finitely presented cyclic R-module is cyclically presented.*

Proof. If M is a finitely presented cyclic R-module then $M = mR$ for some $m \in M$. Thus, $\phi : R \to M$ with $\phi(r) = ar$ is a surjection with $\operatorname{Ker}(\phi) = \operatorname{Ann}_R(a)$. We have the short exact sequence $0 \to \operatorname{Ann}_R(a) \to R \to M \to 0$ and so $\operatorname{Ann}_R(a)$ is finitely generated since R is finitely generated and M is finitely presented. As R is a valuation ring, $\operatorname{Ann}_R(a)$ is principal and so M is cyclically presented. □

Theorem 2.2.22 (Warfield [28, Theorem 1])**.** *A finitely presented module over valuation ring is a direct sum of cyclically presented modules.*

Proof. Recall that the quotient of a finitely presented module M modulo a finitely generated submodule is finitely presented. By Proposition 2.2.20, there is a sequence of RD-submodules

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$$

such that M_i/M_{i-1} are cyclic for all $i = 1, \dots, n$. In particular, M/M_{i-1} is finitely presented and cyclic, so it is cyclically presented by Lemma 2.2.21. By Lemma 2.2.17, M/M_{n-1} has the projective property relative to RD-exact sequences, so the RD-exact sequence $0 \to M_{n-1} \to M \to M/M_{n-1} \to 0$ splits and $M \cong M_{n-1} \oplus M/M_{n-1}$. We may now repeat the process inductively for M_{n-1} and obtain the full decomposition. □

Using the description of finitely presented modules over valuation rings, we can now proceed to analyze its consequences for Prüfer rings.

Theorem 2.2.23 (Warfield [29])**.** *Let R be a Prüfer domain, let $N \subseteq M$ be R-modules. N is an RD-submodule if and only if N is a pure submodule.*

Proof. It is clear that any pure module is RD. By Corollary 2.2.11, we can reduce to studying the local case. Since R is Prüfer, we may reduce to R being a valuation ring. We proceed to show that any RD-exact sequence $0 \to A \to B \to C \to 0$ is pure-exact. By Lemma 2.2.13, it is enough to show that any finitely presented module M has the projective property with respect to it. Using Warfield's Theorem 2.2.22 for valuation rings, we know that $M = \bigoplus_{i=1}^{n} M_i$ is a direct sum of cyclically presented modules. By Lemma 2.2.17, all M_i have the projective property with respect to the sequence, so M also satisfies it. Thus, by Lemma 2.2.13, the sequence is pure-exact. □

Remark 2.2.24. The previous statement essentially states that relative divisibility and purity are equivalent over Prüfer domains.

Using the previous theorem and Lemma 2.2.13, we deduce that finitely presented modules over Prüfer rings are RD-projective. Thus, we can reduce to describing the structure of RD-projective modules.

Lemma 2.2.25. *Let M be an R-module, then there exists an RD-exact sequence $0 \to N \to C \to M \to 0$ where C is a direct sum of cyclically presented R-modules.*

Proof. Consider the set $\{R/r_jR \mid j \in J\}$ of representatives of all cyclically presented R-modules up to isomorphism. Denote them by $C_j := R/r_jR$. Consider $\Delta(M) := \bigoplus_{j\in J} \bigoplus_{\phi_j} C_j$ where ϕ_j is indexed by the elements $\phi_j \in \mathrm{Hom}_R(C_j, M)$. Consider the morphism $\phi : \Delta(M) \to M$ defined by

$$\phi((c_{j,\phi_j})_{j\in J,\phi_j}) = \sum_{j\in J,\phi_j} \phi_j(c_{j,\phi_j})$$

Note that the morphism ϕ is surjective. Indeed, for any $m \in M$ we can consider $r \in \mathrm{Ann}_R(m)$ and the cyclic module R/rR. There exists some $i \in J$ such that $C_i \cong R/rR$ and we have an R-linear morphism $\psi' : R/rR \to M$ such that $\psi'(1) = m$. Thus, there is a map $\psi \in \mathrm{Hom}_R(C_i, M)$ such that $\psi(1) = m$. Take $c \in \Delta(M)$ such that it has all components zero except for $j = i, \phi_i = \psi$, where it has a 1. Then $\phi(c) = m$.

Since ϕ is surjective the sequence $0 \to \nabla(M) \to \Delta(M) \to M \to 0$ is exact where $\nabla(M) = \mathrm{Ker}(\phi)$. By Lemma 2.2.17, to prove the sequence is RD-exact it is enough to check that $\phi_* : \mathrm{Hom}_R(C_j, \Delta(M)) \to \mathrm{Hom}_R(C_j, M)$ is surjective for any $j \in J$. Let $\varphi \in \mathrm{Hom}_R(C_j, M)$ and consider the inclusion $\iota_\varphi : C_j \to \Delta(M)$ that sends C_j to the copy in position φ. Then clearly $\phi \circ \iota_\varphi = \varphi$. □

Theorem 2.2.26. *A module is RD-projective if and only if it is a direct summand of a direct sum of cyclically presented R-modules.*

Proof. By Lemma 2.2.17, we know cyclically presented modules are RD-projective. It follows that the direct sum of cyclically presented modules is RD-projective and so are its direct summands. Suppose now that M is RD-projective, by Lemma 2.2.25 there is an exact sequence $0 \to N \to C \to M \to 0$ such that C is a direct sum of cyclically presented modules. Since M is projective with respect to the sequence, it splits, so M is a summand of C. □

Remark 2.2.27. The statements and proofs of Lemma 2.2.25 and Theorem 2.2.26 also hold exchanging RD by pure and cyclically presented with finitely presented. See [12, Lemma VI.12.1, Theorem VI.12.2].

We are finally ready to prove Warfield's structure theorem [28] characterizing the finitely presented modules of Prüfer rings.

Theorem 2.2.28 (Warfield [28, Theorem 3])**.** *Let R be a Prüfer domain, then every finitely presented R-module is a direct summand of a direct sum of cyclic modules.*

Proof. Suppose R is Prüfer and M is a finitely presented R-module. Lemma 2.2.13 ensures M is pure-projective and, since R is Prüfer, Theorem 2.2.23 guarantees that purity and RD are equivalent, so M is RD-projective. The implication follows from Theorem 2.2.26. □

Remark 2.2.29. Warfield also proved the converse statement, providing a characterization of Prüfer rings. The interested reader may consult [12, Theorem III.3.3] for its proof.

Remark 2.2.30. At no point throughout the proof of Warfield's structure theorem and its components have we used the "domain" property. In fact, Warfield originally proved [28, Theorem 3] the previous result already omitting this hypothesis.

Warfield called rings that satisfy the same properties as valuation rings, except for not being domains, as *generalized valuation rings* (recall Definition 1.1.21).

Definition 2.2.31. A ring R such that for all $\mathfrak{m} \subseteq R$ maximal ideal the localization $R_\mathfrak{m}$ is a generalized valuation ring is called a *generalized Prüfer ring*.

Using the previous definitions Theorem 2.2.22 and Theorem 2.2.28 can be restated as follows.

Theorem 2.2.32 (Warfield [28, Theorem 1]). *A finitely presented module over a generalized valuation ring is a direct sum of cyclically presented modules.*

Theorem 2.2.33 (Warfield [28, Theorem 3]). *Let R be a generalized Prüfer domain, then every finitely presented R-module is a direct summand of a direct sum of cyclic modules.*

Chapter 3

Structure of admissible modules

This chapter is centered around the study of a new class of modules called *admissible*. Burns, Daoud, and Liang introduced these modules in [7, §2] to study $\mathbb{Z}_p[[\mathbb{Z}_p^{\mathbb{N}}]]$-torsion modules, relevant to the budding field of non-Noetherian Iwasawa theory. At the time of writing this chapter, the paper [7] is still a preprint undergoing peer-review, so special care is taken in the description and analysis of the proofs.

The first section of this chapter is devoted to introducing pseudo-null modules and the proof of a structure theorem for finitely generated Noetherian Krull domains. This serves as a stepping stone towards the study of admissible modules, for the proofs we provide for their respective structure theorems employ very similar techniques. For a deeper discussion of these results, we refer the reader to [4, VII, §4] and [21, V, §1].

The third section is devoted to the introduction of admissible modules and the proof of a structure theorem for admissible modules due to Burns, Daoud, and Liang [7, §2, Theorem 2.3]. The last section focuses on the connection between these results and the field of Iwasawa theory.

Throughout this chapter, we assume R is a commutative ring with unit. Given an ideal $I \subseteq R$ and S a multiplicatively closed set, we will denote the localization of the ideal as $IS^{-1}R$. At some points, we may abuse notation and, given $S^{-1}I$, call I its preimage under localization.

3.1 Pseudo-null modules and Noetherian Krull domains

We start this section exploring pseudo-nullity and pseudo-isomorphy. We later use this notion to describe finitely generated modules over Noetherian Krull domains.

3.1.1 Pseudo-nullity of modules

We start introducing some notation to be used throughout this chapter.

Definition 3.1.1. Let R be a ring and M and R-module, we define the following sets of prime ideals of R:

(i) $\mathrm{Spec}(R) = \{\mathfrak{p} \subseteq R \mid \mathfrak{p} \text{ is a prime ideal}\}$.

(ii) $\mathcal{P} = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathrm{ht}(\mathfrak{p}) = 1\}$.

(iii) $\mathcal{P}_{\mathrm{fg}} = \{\mathfrak{p} \in \mathcal{P} \mid \mathfrak{p} \text{ is finitely generated}\}$.

(iv) $\mathrm{Supp}(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid M_{\mathfrak{p}} \neq 0\}$.

(v) $\mathcal{P}(M) = \{\mathfrak{p} \in \mathcal{P} \mid (M_{\mathrm{tor}})_{\mathfrak{p}} \neq 0\} = \mathrm{Supp}(M_{\mathrm{tor}}) \cap \mathcal{P}$.

Lemma 3.1.2. *Let M be a finitely generated R-module. Then*

$$\mathrm{Supp}(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathrm{Ann}_R(M) \subseteq \mathfrak{p}\}$$

Proof. We prove that the complements of each sets are equal. If $\mathrm{Ann}_R(M) \subsetneq \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Spec}(R)$ then there exists $r \in \mathrm{Ann}_R(M) \cap (R \backslash \mathfrak{p})$ and so $r(m - 0) = 0$ for all $m \in M$, meaning $M_{\mathfrak{p}} = 0$.

If $M_{\mathfrak{p}} = 0$ and $a_1, \ldots, a_n \in M$ generate M, there exist $r_1, \ldots, r_n \in R \backslash \mathfrak{p}$ such that $r_i a_i = 0$ and so $r = r_1 \cdots r_n \in \mathrm{Ann}_R(M) \cap (R \backslash \mathfrak{p})$. □

Definition 3.1.3 (Pseudo-null module)**.** Let R be a ring and M an R-module. We say M is *pseudo-null* if $M_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \mathcal{P}$.

Proposition 3.1.4. *Let M be a finitely generated pseudo-null R-module, the following are equivalent:*

(i) M is torsion.

(ii) $M_{\mathfrak{p}} = 0$ for all $\mathfrak{p}$ prime ideals with $\mathrm{ht}(\mathfrak{p}) \leq 1$.

(iii) Any prime ideal $\mathfrak{p}$ such that $\mathrm{Ann}_R(M) \subseteq \mathfrak{p}$ satisfies $\mathrm{ht}(\mathfrak{p}) \geq 2$.

Proof. (i) $\implies$ (ii): If M is torsion then $M \otimes_R Q(R) = 0$. If $\mathrm{ht}(\mathfrak{p}) = 0$, then $\mathfrak{p}$ is minimal, thus generated by an idempotent, and so all elements of $\mathfrak{p}$ are zero divisors. Thus, $R_{\mathfrak{p}} = 0$ if $R \backslash \mathfrak{p}$ contains a zero divisor or $R \backslash \mathfrak{p}$ is the set of all non-zero divisors. In both cases $M \otimes_R R_{\mathfrak{p}} = 0$.

(ii) $\implies$ (i): If $M_{\mathfrak{p}} = 0$ for all $\mathfrak{p}$ with $\mathrm{ht}(\mathfrak{p}) = 0$ then $M \otimes_R Q(R) = 0$, since $Q(R)$ is the localization at the set of non-zero divisors.

(ii) $\iff$ (iii): Since M is finitely generated by some elements $a_1, \ldots, a_n$, $M_{\mathfrak{p}} = 0$ implies there exist $s_i \in R \backslash \mathfrak{p}$ such that $s_i a_i = 0$. Then $s = s_1 \ldots s_n \in R \backslash \mathfrak{p}$ as it is multiplicatively closed and $s \neq 0$, so $sM = 0$. This implies $s \in \mathrm{Ann}_R(M) \nsubseteq \mathfrak{p}$. Conversely, given $\mathfrak{p}$ such that $\mathrm{ht}(\mathfrak{p}) \leq 1$ there exists some element $s \in \mathrm{Ann}_R(M)$ such that $s \notin \mathfrak{p}$, and so $sm = 0$ implies $M_{\mathfrak{p}} = 0$. □

Remark 3.1.5. The concept of pseudo-null module is usually defined over integrally closed Noetherian domains. In this context, the usual definition given for *pseudo-null* module is one of the conditions (ii) or (iii) in Proposition 3.1.4. This is how it is done in [4] and [21]. However, we have chosen to give a more general definition like in [7], as we will need in this form in the next section.

Example 3.1.6. (i) If R is a Dedekind domain, then M a finitely generated torsion module is pseudo-null if and only if $M = 0$. This arises from the fact that all prime ideals of R have height at most 1 (Proposition 1.1.44).

(ii) Let R be a Noetherian domain and $\mathfrak{m}$ a maximal ideal of R. Suppose $\mathrm{ht}(\mathfrak{m}) \geq 2$, then the R-module $M = R/\mathfrak{m}$ is pseudo-null. Indeed, note that M is torsion since $\mathrm{Ann}_R(M) = \mathfrak{m}$ and R being Noetherian means M is finitely generated as an R-module. Since $\mathrm{Ann}_R(M)$ is maximal and of height greater than 1, it is pseudo-null.

As the name and previous discussion suggests, we can think of pseudo-null modules as being "close" to zero. This motivates the following definition.

Definition 3.1.7 (Pseudo-isomorphism)**.** Let M, N be R-modules and $f : M \to N$ an R-module morphism. We say f is *pseudo-injective* if $\mathrm{Ker}\, f$ is pseudo-null. We say f is *pseudo-surjective* if $\mathrm{Coker}\, f$ is pseudo-null. We say f is a *pseudo-isomorphism* if it is both pseudo-injective and pseudo-surjective.

Lemma 3.1.8. *Let M, N be R-modules and $f : M \to N$ an R-module morphism. f is a pseudo-isomorphism (resp. pseudo-injective, pseudo-surjective) if and only if $f_\mathfrak{p} : M_\mathfrak{p} \to N_\mathfrak{p}$ the morphism induced by localization at $\mathfrak{p}$ is an isomorphism (resp. injective, surjective) for every $\mathfrak{p} \in \mathcal{P}$.*

Proof. The converse implication ($\Leftarrow$) is direct by definition. Implication ($\Rightarrow$) is a direct consequence of the exactness of localization, as $\mathrm{Ker}(f)_\mathfrak{p} = \mathrm{Ker}(f_\mathfrak{p})$ and $\mathrm{Coker}(f)_\mathfrak{p} = \mathrm{Coker}(f_\mathfrak{p})$. □

Lemma 3.1.9. *Let M, N, K be R-modules and $f : M \to N$, $g : N \to K$ be R-module morphisms. If $g \circ f$ is a pseudo-isomorphism then f is pseudo-injective and g is pseudo-surjective.*

Proof. This follows directly from Lemma 3.1.8 and the functoriality of localization. If $g \circ f$ is a pseudo-isomorphism then $g_\mathfrak{p} \circ f_\mathfrak{p}$ is an isomorphism for all $\mathfrak{p} \in \mathcal{P}$ and so $f_\mathfrak{p}$ is injective while $g_\mathfrak{p}$ is surjective. □

The following proposition provides examples of pseudo-isomorphisms.

Proposition 3.1.10. *Let M be a finitely generated torsion R-module and let $r \in R$, $r \neq 0$ such that $\mathrm{Supp}(R/rR) \cap \mathcal{P}(M) = \varnothing$. Then the morphism $\varphi_r : M \to M$ defined by $\varphi_r(x) = rx$ is a pseudo-isomorphism.*

Proof. If $\mathfrak{p} \in \mathcal{P}(M)$ then $M_\mathfrak{p} \neq 0$ so by hypothesis $\mathfrak{p} \notin \operatorname{Supp}(R/rR)$. Lemma 3.1.2 implies $\operatorname{Ann}_R(R/rR) \not\subseteq \mathfrak{p}$, so $r \notin \mathfrak{p}$. Thus, r is a unit in $R_\mathfrak{p}$ and so $(\varphi_r)_\mathfrak{p} : M_\mathfrak{p} \to M_\mathfrak{p}$ is an isomorphism.

If $\mathfrak{p} \in \mathcal{P} \backslash \mathcal{P}(M)$ then $M_\mathfrak{p} = 0$ so $(\varphi_r)_\mathfrak{p}$ is trivially an isomorphism. □

Remark 3.1.11. In general, given a pseudo-isomorphisms there need not exist a pseudo-isomorphism in the reverse direction. We will see later some cases when can this happen (see Corollary 3.1.25). For now, we provide a counterexample.

Example 3.1.12. Consider K and algebraically closed field and $R = K[X, Y]$ the ring of polynomials in two variables over K. Consider the maximal ideal $\mathfrak{m} = (X, Y)$, by Example 3.1.6(ii) the inclusion morphism $\iota : \mathfrak{m} \to R$ is a pseudo-isomorphism as $\operatorname{Ker} \iota = 0$ and $\operatorname{Coker} \iota = R/\mathfrak{m}$. Note that $\operatorname{Hom}_R(R, \mathfrak{m}) \cong \mathfrak{m}$ and each $\varphi : R \to \mathfrak{m}$ is determined by $\varphi(1) = f \in \mathfrak{m}$, so let us call φ_f such a map. Then $\operatorname{Ker} \varphi_f = 0$ and $\operatorname{Im} \varphi_f = (f) \subsetneq \mathfrak{m}$. Thus, $\operatorname{Coker} \varphi_f = R/(f)$ is a torsion module and (f) cannot be a maximal ideal. If f is irreducible then (f) is prime and $\operatorname{Ann}_R(\operatorname{Coker} \varphi_f) = (f)$ is a prime ideal of height 1 (or 0 if $f = 0$), so φ_f is not a pseudo-isomorphism. If f is not irreducible, then there exist $f_1, \ldots, f_n$ irreducible polynomials with $f = f_1 \ldots f_n$ and $\operatorname{Ann}_R(\operatorname{Coker} \varphi_f) = (f) \subseteq (f_1)$ so $\operatorname{Coker} \varphi_f$ is not pseudo-null.

Definition 3.1.13. Let R be a ring and M, N be R-modules. We define the relation $\approx$ over the class of R-modules as $M \approx N$ if and only if there exists a pseudo-isomorphism $f : M \to N$.

As we have said, the Example 3.1.12 shows that $\approx$ is not symmetric for general R modules.

Lemma 3.1.14. *The relation $\approx$ is reflexive and transitive.*

Proof. Reflexivity is obvious. Consider $M \approx N$, $N \approx L$, then there exist pseudo-isomorphisms $f : M \to N$ and $g : N \to L$. By Lemma 3.1.8 the morphisms induced by localization at $\mathfrak{p} \in \mathcal{P}$ are isomorphisms, and so is their composition. Using the same Lemma we conclude transitivity. □

3.1.2 Structure of modules over Noetherian Krull domains

For the rest of this section we focus on rings which are Noetherian Krull domains (NKD), that is, rings which are Noetherian integrally closed domains (recall Definition 1.1.51 and Theorem 1.1.52(ii)). We will describe the structure of finitely generated modules over NKD rings up to pseudo-nullity. Before we proceed to the structure theorems, we introduce some preliminary results and notation.

Notation 3.1.15. Let M be an R-module, then we define the torsion-free module $M_{\mathrm{tf}} = M/M_{\mathrm{tor}}$.

Lemma 3.1.16. *A semi-local Dedekind domain is a principal ideal domain.*

Proof. A Dedekind domain is Prüfer, by Proposition 2.1.11 a semi-local Prüfer domain is Bézout. Moreover, a Dedekind domain is Noetherian and a Noetherian Bézout domain is a PID. □

Lemma 3.1.17. *Let R be an NKD, let $\mathfrak{p}_1, \dots, \mathfrak{p}_n \in \mathcal{P}$, $n \geq 1$ and consider the multiplicatively closed set $S = R \setminus \bigcup_{k=1}^{n} \mathfrak{p}_i$. Then $S^{-1}R$ is a principal ideal domain.*

Proof. It is clear that $S^{-1}R$ is a semi-local domain with maximal ideals $\mathfrak{m}_i = \mathfrak{p}_i S^{-1}R$ for $1 \leq i \leq n$. Since R is a domain and $\mathfrak{p}_i$ have height 1, $\mathfrak{m}_i$ are all the non-zero prime ideals of $S^{-1}R$. Note that the localization $(S^{-1}R)_{\mathfrak{m}_i} \cong R_{\mathfrak{p}_i}$ is a DVR as R is a Krull domain and $\mathfrak{p}_i \in \mathcal{P}$. Thus, $S^{-1}R$ is a Dedekind domain (Proposition 1.1.35(iii)) and by Lemma 3.1.16, $S^{-1}R$ is a PID. □

Our first objective is to prove that we can decompose a finitely generated module M into $M_{\text{tor}} \oplus M_{\text{tf}}$ via a pseudo-isomorphism. Since $M_{\text{tor}} \oplus M_{\text{tf}} \cong M_{\text{tor}} \times M_{\text{tf}}$, we start by giving one of the projections.

Proposition 3.1.18. *Let R be a NKD and M a finitely generated R-module. Then there exists an R-module morphism $\pi_{\text{tor}} : M \to M_{\text{tor}}$ such that $\pi_{\text{tor}}|_{M_{\text{tor}}}$ is a pseudo-isomorphism and there exists $r \in R$ such that $\pi_{\text{tor}}|_{M_{\text{tor}}} = (m \mapsto rm)$ (i.e. it is a homothety).*

Proof. If $M_{\text{tor}} = 0$ then $\pi_{\text{tor}} = 0$. If $M_{\text{tor}} \neq 0$, since R is Noetherian and M is finitely generated so is M_{tor}. Then $\operatorname{Ann}_R(M) \neq 0$, and note that $\mathcal{P}(M) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ can be defined as the set of prime ideals of height 1 containing $\operatorname{Ann}_R(M)$ (Lemma 3.1.2). $\mathcal{P}(M)$ is finite, and possibly empty, as R is a Krull domain (Definition 1.1.51(iii)). If it is empty, then M_{tor} is pseudo-null by Proposition 3.1.4, so we may take $\pi_{\text{tor}} = 0$ to be the morphism, since $\operatorname{Ker}(\pi_{\text{tor}}|_{M_{\text{tor}}}) = M_{\text{tor}}$ is pseudo-null.

Suppose it is non-empty, then consider the multiplicatively closed set $S = R \setminus \bigcup_{i=1}^{n} \mathfrak{p}_i$. Then $S^{-1}R$ is a PID by Lemma 3.1.17, and the structure theorem of finitely generated modules over PID's gives $S^{-1}M \cong S^{-1}M_{\text{tor}} \oplus S^{-1}M_{\text{tf}}$, so we may consider the canonical projection $\pi_0 : M \to M_{\text{tor}}$. Now since M is finitely generated and R Noetherian, M is finitely presented (Lemma 1.2.5), so by Proposition 1.2.44 we have

$$S^{-1} \operatorname{Hom}_R(M, M_{\text{tor}}) \cong \operatorname{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}M_{\text{tor}}) \tag{3.1}$$

Using isomorphism (3.1), we conclude there is a morphism $\pi : M \to M_{\text{tor}}$ and $s_0 \in S$ such that, abusing notation, $\pi_0 = s_0^{-1}\pi : S^{-1}M \to S^{-1}M_{\text{tor}}$ is the projection to its direct summand, i.e. $s_0^{-1}\pi|_{S^{-1}M_{\text{tor}}} = \operatorname{Id}_{S^{-1}M_{\text{tor}}}$. Thus, there exists $s_1 \in S$ such that $s_1(\pi|_{M_{\text{tor}}} - s_0 \operatorname{Id}_{M_{\text{tor}}}) = 0$. Define $\pi_{\text{tor}} = s_1\pi$ and then note $\pi_{\text{tor}}|_{M\text{tor}} = s_1 s_0 \operatorname{Id}_{M_{\text{tor}}}$.

Finally, note that $s_0 s_1 \notin \bigcup_{i=1}^{n} \mathfrak{p}_i$ so $\operatorname{Supp}(R/(s_0 s_1)R) \cap \mathcal{P}(M) = \varnothing$ so by Proposition 3.1.10 $\pi_{\text{tor}}|_{M_{\text{tor}}}$ is a pseudo-isomorphism. □

Remark 3.1.19. Note we have used the structure theorem of finitely generated modules over PID's in the proof. Although not explicitly stated in Chapter 1, PID's

that are not fields are Dedekind domains (they have Krull dimension 1), so Theorem 1.2.46 includes them. Finitely generated modules over a field are simply finitely generated vector spaces and so they are free and classified by the dimension/rank.

We can now prove the following decomposition theorem.

Theorem 3.1.20. *Let R be an NKD, let M be a finitely generated R-module. Then there exists a pseudo-isomorphism $f : M \to M_{\mathrm{tor}} \oplus M_{\mathrm{tf}}$.*

Proof. Let $\pi_{\mathrm{tor}} : M \to M_{\mathrm{tor}}$ be the projection morphism from Proposition 3.1.18. Let $\pi_{\mathrm{tf}} : M \to M_{\mathrm{tf}}$ be the canonical projection morphism, as $M_{\mathrm{tf}} = M/M_{\mathrm{tor}}$. Now define the morphism $f = (\pi_{\mathrm{tor}}, \pi_{\mathrm{tf}}) : M \to M_{\mathrm{tor}} \oplus M_{\mathrm{tf}}$. By construction, the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_{\mathrm{tor}} & \longrightarrow & M & \longrightarrow & M_{\mathrm{tf}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_{\mathrm{tor}}|_{M_{\mathrm{tor}}}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{Id}_{M_{\mathrm{tf}}}} & & \\ 0 & \longrightarrow & M_{\mathrm{tor}} & \longrightarrow & M_{\mathrm{tor}} \oplus M_{\mathrm{tf}} & \longrightarrow & M_{\mathrm{tf}} & \longrightarrow & 0 \end{array} \tag{3.2}$$

Using the snake lemma, we get

$$0 \to \operatorname{Ker} \pi_{\mathrm{tor}}|_{M_{\mathrm{tor}}} \to \operatorname{Ker} f \to 0 \to \operatorname{Coker} \pi_{\mathrm{tor}}|_{M_{\mathrm{tor}}} \to \operatorname{Coker} f \to 0 \to 0$$

so $\pi_{\mathrm{tor}}|_{M_{\mathrm{tor}}}$ being a pseudo-isomorphism implies f is a pseudo-isomorphism. □

Now we proceed to study the finitely generated torsion modules through pseudo-isomorphisms. We start proving a uniqueness lemma.

Lemma 3.1.21. *Let R be an NKD. Let $(\mathfrak{p}_i)_{i=1}^s$, $(\mathfrak{q}_j)_{j=1}^t$ be finite families of height 1 prime ideals of R, and $(n_i)_{i=1}^s$, $(m_j)_{j=1}^t$ be families of positive integers. Consider the torsion R-modules $M = \bigoplus_{i=1}^s R/\mathfrak{p}_i^{n_i}$ and $N = \bigoplus_{j=1}^t R/\mathfrak{q}_j^{m_j}$. Then if there exists a pseudo-isomorphism $f : M \to N$, then $s = t$ and there exists a permutation $\sigma \in S_t$ such that $\mathfrak{p}_i = \mathfrak{q}_{\sigma(i)}$, $n_i = m_{\sigma(i)}$ for all $1 \leq i \leq t$.*

Proof. Let $f : M \to N$ be a pseudo-isomorphism, by Lemma 3.1.8 for any $\mathfrak{p} \in \mathcal{P}$ the induced morphism $f_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is an isomorphism. Since R is an NKD, $R_{\mathfrak{p}}$ is a DVR, in particular, a PID. Furthermore, $\mathfrak{p}_i R_{\mathfrak{p}} \neq R_{\mathfrak{p}}$ if and only if $\mathfrak{p}_i = \mathfrak{p}$, as all proper ideals of $R_{\mathfrak{p}}$ are powers of $\mathfrak{p}R_{\mathfrak{p}}$. Thus we restrict to $\mathfrak{p} \in (\mathfrak{p}_i)_{i=1}^s$, otherwise $f_{\mathfrak{p}}$ is zero. This already shows that $(\mathfrak{p}_i)_{i=1}^s$ and $(\mathfrak{p}_j)_{j=1}^t$ contain the same prime ideals. To see that they contain the same amount of copies of each ideal and that the powers coincide up to reordering, we simply apply the uniqueness of the structure theorem of finitely generated modules over PID's to each $f_{\mathfrak{p}}$. □

Before the main theorem, we require an additional technical lemma.

Lemma 3.1.22. *Let M be a finitely presented torsion R-module and let N be an R-module. Define the multiplicatively closed set $S = R \setminus \bigcup_{\mathfrak{p} \in \mathcal{P}(M)} \mathfrak{p}$. If there exists an isomorphism $f : S^{-1}M \to S^{-1}N$ of $S^{-1}R$-modules then there exist a pseudo-isomorphism $g : M \to N$.*

Proof. Using the isomorphism from Proposition 1.2.44

$$S^{-1}\operatorname{Hom}_R(M,N) \cong \operatorname{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$$

and the isomorphism $f : S^{-1}M \to S^{-1}N$ given by hypothesis, we can obtain a morphism $g : M \to N$ and $s \in S$ such that $g = sf$. To show g is a pseudo-isomorphism, we proceed similarly to Proposition 3.1.10. If $\mathfrak{q} \in \mathcal{P}\backslash\mathcal{P}(M)$ then $M_\mathfrak{q} = 0$. Furthermore, $\mathfrak{p}R_\mathfrak{q} = R_\mathfrak{q}$ if $\mathfrak{p} \nsubseteq \mathfrak{q}$, so $N_\mathfrak{q} = 0$ too and $f_\mathfrak{q} = 0$ is an isomorphism. Now if $\mathfrak{q} \in \mathcal{P}(M)$ then $(S^{-1}R)_\mathfrak{q} = R_\mathfrak{q}$ and $s \in R\backslash\mathfrak{q}$ so it is invertible. Thus, $f_\mathfrak{q} = s(f_0)_\mathfrak{q}$ is an isomorphism as f_0 is and s is invertible. Using Lemma 3.1.8 we conclude f is a pseudo-isomorphism. □

Theorem 3.1.23. *Let R be an NKD, let M be a finitely generated torsion R-module. Then there exist families $(\mathfrak{p}_i)_{i=1}^k$ and $(n_i)_{i=1}^k$ with $\mathfrak{p}_i \in \mathcal{P}(M)$ and n_i positive integers for $1 \leq i \leq k$, such that there exists a pseudo-isomorphism $f : M \to N$ where $N = \bigoplus_{i=1}^k R/\mathfrak{p}_i^{n_i}$. Furthermore, the families $(\mathfrak{p}_i)_{i=1}^k$ and $(n_i)_{i=1}^k$ are unique up to reordering.*

Proof. Uniqueness has been proved in Lemma 3.1.21.

For existence, we may assume $M \neq 0$ an we proceed in a similar way as in Proposition 3.1.18. Recall $\mathcal{P}(M)$ is finite and define the multiplicatively closed set $S = R\backslash\bigcup_{i=1}^n \mathfrak{p}_i$ where $\mathcal{P}(M) = \{\mathfrak{p}_i\}_{i=1}^n$. Again, $S^{-1}R$ is a semi-local Dedekind domain, so a PID (Lemma 3.1.16), with maximal ideals $\mathfrak{m}_i = \mathfrak{p}_i S^{-1}R$, $1 \leq i \leq n$. Now $S^{-1}M$ is a finitely generated torsion $S^{-1}R$-module, so we may apply the structure theorem of finitely generated modules of a PID to obtain families $(\mathfrak{m}_j)_{j=1}^k$, $(m_j)_{j=1}^k$ and an isomorphism

$$f_0 : S^{-1}M \to \bigoplus_{j=1}^k S^{-1}R/\mathfrak{m}_j^{m_j} = \bigoplus_{j=1}^k S^{-1}R/\mathfrak{p}_j^{m_j}S^{-1}R \cong S^{-1}\left(\bigoplus_{j=1}^k R/\mathfrak{p}_j^{m_j}\right) \tag{3.3}$$

Define $N = \bigoplus_{j=1}^k R/\mathfrak{p}_j^{m_j}$ and so we have an isomorphism $f_0 : S^{-1}M \to S^{-1}N$. Using Lemma 3.1.22 we conclude there exists a pseudo-isomorphism $f : M \to N$. □

Remark 3.1.24. Note that the $\mathfrak{p}_i$ appearing in N are exactly all the prime ideals of $\mathcal{P}(M)$ (maybe with repetitions). Indeed, $M \approx N$ implies $M_\mathfrak{p} \cong N_\mathfrak{p}$ for all $\mathfrak{p} \in \mathcal{P}$ so, in particular, $\mathcal{P}(M) = \operatorname{Supp}(M) \cap \mathcal{P} = \operatorname{Supp}(N) \cap \mathcal{P} = \mathcal{P}(N)$. Here we used $M = M_{\text{tor}}$, $N = N_{\text{tor}}$.

Corollary 3.1.25. *Let R be an NKD. The restriction of the relation $\approx$ to the class of finitely generated torsion R-modules is an equivalence relation.*

Proof. Recall from Lemma 3.1.14 that the relation $\approx$ is reflexive and transitive in general. Assume R is an NKD and M, N are finitely generated R-torsion modules, thus finitely presented (Lemma 1.2.5). Suppose $M \approx N$, then by Theorem 3.1.23 there exists a module $E = \bigoplus_{i=1}^k R/\mathfrak{p}_i^{n_i}$ with $\mathfrak{p}_i \in \mathcal{P}(N)$ and n_i positive integers

for $1 \leq i \leq k$ such that $N \approx E$. By transitivity $M \approx E$ and by the previous remark $\mathcal{P}(M) = \mathcal{P}(N)$. Defining the multiplicatively closed set $S = R \setminus \bigcup_{\mathfrak{p} \in \mathcal{P}(M)} \mathfrak{p}$, the proof of Theorem 3.1.23 guarantees the existence of isomorphisms $S^{-1}M \cong S^{-1}E \cong S^{-1}N$. Applying Lemma 3.1.22 we conclude $N \approx M$. □

3.2 Structure of admissible modules

For this section, recall R is a commutative ring with unit and recall all the notation introduced in Definition 3.1.1. Furthermore, we will be working with rings that are not necessarily domains, so we make use of the definitions of generalized valuation rings and their Prüfer counterparts as introduced in Definition 1.1.21.

The main objective of this section is to introduce *admissible* modules and prove the structure theorem for finitely presented admissible modules, proved in [7, Theorem 2.3] by Burns, Daoud, and Liang. We start by introducing the concept of admissible module and its interaction with the Noetherian property.

Definition 3.2.1 (Admissible module)**.** Let M be a finitely generated R-module. We say M is *admissible* if it satisfies the following properties:

(A1) Let $\mathfrak{p} \in \operatorname{Spec}(R)$ and $\mathcal{F} = \{\mathfrak{q} \in \operatorname{Spec}(R) \mid \mathfrak{q} \subseteq \bigcup_{\mathfrak{a} \in \mathcal{P}(M)} \mathfrak{a}\}$. If $\mathfrak{p}$ is maximal over $\mathcal{F}$, then $R_{\mathfrak{p}}$ is a valuation ring.

(A2) The set $\mathcal{P}(M)$ is finite and $\mathcal{P}(M) \subseteq \mathcal{P}_{\mathrm{fg}}$.

Remark 3.2.2. If $\mathcal{P}(M)$ is finite, property (A1) is equivalent to: if $\mathfrak{p} \in \mathcal{P}(M)$, then $R_{\mathfrak{p}}$ is a valuation ring. This is a consequence of the prime avoidance lemma 1.1.38.

Remark 3.2.3. Note that if R is Noetherian, $\mathcal{P} = \mathcal{P}_{\mathrm{fg}}$ and so property (A2) simplifies to $\mathcal{P}(M)$ being finite.

Proposition 3.2.4. *Let R be a Noetherian domain and M a finitely generated R-module. The following are equivalent:*

(i) M is admissible.

(ii) $R_{\mathfrak{p}}$ is a valuation ring for all $\mathfrak{p} \in \mathcal{P}(M)$.

Proof. (i) $\implies$ (ii): By (A2), we have $\mathcal{P}(M)$ is finite. The implication follows from Remark 3.2.2.

(i) $\implies$ (ii): Property (A2) follows from Proposition 1.1.50. Then by Remark 3.2.2, the hypothesis implies (A1). □

Remark 3.2.5. In particular, over an NKD all finitely generated modules are admissible.

Rings over which all finitely presented modules are admissible are of special interest.

Definition 3.2.6. We say a ring R is *admissible* if $Q(R)$ is semisimple and evey finitely presented torsion R-module is admissible.

Remark 3.2.7. The choice of hypothesis for admissible rings will become clear later in the context of Theorem 3.2.20.

Example 3.2.8. We provide some varied examples of admissible rings.

(1) By Remark 3.2.5, if R is a NKD then all finitely generated modules are admissible and $Q(R)$ is a field, so R is admissible.

(2) Let R be a UFD, so $Q(R)$ is field, so semisimple. Then R is a Krull domain where all height 1 prime ideals are principal by Theorem 1.1.52(vi), so $\mathcal{P} = \mathcal{P}_{\text{fg}}$. Let M be a finitely generated torsion R-module, then $\mathcal{P}(M) = \text{Supp}(M) \cap \mathcal{P} = \{\mathfrak{p} \in \mathcal{P} \mid \text{Ann}_R(M) \subseteq \mathfrak{p}\}$ by Lemma 3.1.2. By Definition 1.1.51 of Krull domain, this implies $\mathcal{P}(M)$ is finite, since any element of R is contained in finitely many primes of $\mathcal{P}$. Thus, by Remark 3.2.2, R is admissible.

(3) Consider $R = \mathbb{Z}_p[[X_1, \dots]] = \varprojlim \mathbb{Z}_p[[X_1, \dots, X_n]]$ the ring of power series in infinitely many variables. Then, by a result of Nishimura [22, Theorem 1], R is a UFD if $R_n = \mathbb{Z}_p[[X_1, \dots, X_n]]$ is a UFD for any n. Each R_n is a UFD by [4, VII §3.9, Proposition 8] as $\mathbb{Z}_p$ is a DVR.

Following a similar structure as in the previous section, our first objective is to prove that for a finitely presented admissible R-module M, we have $M \approx M_{\text{tor}} \oplus M_{\text{tf}}$. Since we do not assume R to be a domain, we will require some additional conditions on the total quotient ring $Q(R)$. We start with a preliminary lemma.

Lemma 3.2.9. *Let M be a finitely generated R-module satisfying property (A1) and with $\mathcal{P}(M) \neq \varnothing$. Consider the multiplicatively closed set $S = R \setminus \bigcup_{\mathfrak{p} \in \mathcal{P}(M)} \mathfrak{p}$.*

(i) $S^{-1}R$ is a generalized Prüfer ring.

(ii) If $Q(R)$ is semisimple, $S^{-1}R$ is a direct product of finitely many Prüfer domains.

Proof. We may assume $\mathcal{P}(M) \neq \varnothing$ otherwise $S^{-1}R = 0$.

Observe that the maximal ideals of $S^{-1}R$ are $\mathfrak{m}S^{-1}R$ where $\mathfrak{m} \subseteq R$ is a prime ideal maximal over the family $\mathcal{F}$ defined in property (A1). Thus, since M satisfies (A1) the localization $(S^{-1}R)_{\mathfrak{m}S^{-1}R}$ is a valuation ring implying that $S^{-1}R$ is a generalized Prüfer ring.

If $Q(R)$ is semisimple. Then we can apply Theorem 2.1.33 and conclude (ii). □

Remark 3.2.10. Since R is not a domain, we cannot guarantee that $S^{-1}R$ is a domain, so in (i) we can only assert that it is generalized Prüfer.

Theorem 3.2.11. *Let R be a ring with $Q(R)$ semisimple. Let M be a finitely presented R-module satisfying property (A1). Then there exists a pseudo-isomorphism $M \xrightarrow{\approx} M_{\text{tor}} \oplus M_{\text{tf}}$.*

Remark 3.2.12. The statement of the theorem includes admissible modules. However, property (A2) is not necessary for the proof, so we have opted for the more general statement.

Proof. If $\mathcal{P}(M) = \varnothing$ then M_{tor} is pseudo-null and so $\pi : M \to M_{\text{tf}}$ is a pseudo-isomorphism and so $f : M \to M_{\text{tor}} \oplus M_{\text{tf}}$ such that $f(m) = 0 + \pi(m)$ is a pseudo-isomorphism. If $\mathcal{P}(M) \neq \varnothing$. Define the multiplicatively closed set $S = R \setminus \bigcup_{\mathfrak{p} \in \mathcal{P}(M)} \mathfrak{p}$. Then by Lemma 3.2.9 $S^{-1}R$ is a direct product of Prüfer rings. By Theorem 2.1.33(iv), the module $S^{-1}M_{\text{tf}}$ is projective, so there exists an isomorphism $f : S^{-1}M \to S^{-1}M_{\text{tor}} \oplus S^{-1}M_{\text{tf}}$ of $S^{-1}R$-modules. We may apply Lemma 3.1.22 to obtain $M \approx M_{\text{tor}} \oplus M_{\text{tf}}$. □

We can now proceed to work on describing the structure of finitely presented admissible torsion modules. Before, recall the following results on modules over products of rings.

Lemma 3.2.13. *Let R_i be $1 \leq i \leq n$ be rings and $R = \prod_{i=1}^{n} R_i$. Let M be a module over R, then there exist R_i-modules M_i such that $M \cong \prod_{i=1}^{n} M_i$. Furthermore, if N is another R-module with components N_i, then $M \oplus N \cong \prod_{i=1}^{n} M_i \oplus N_i$.*

Remark 3.2.14. In the context of the lemma, $\prod$ denotes the cartesian product with the obvious module structure described in the proof.

Proof. It is enough to check for a product of two rings $R = R_1 \times R_2$. Consider the submodules $M_1 = (1,0)M$, $M_2 = (0,1)M$ as sets. We can endow M_i with an R_i-module structure via $r_1 m_1 := (r_1, 0)m_1$ and $r_2 m_2 := (0, r_2)m_2$. Then the set (M_1, M_2) can be endowed with an R-module structure via $(r_1, r_2)(m_1, m_2) = (r_1 m_1, r_2 m_2)$ and componentwise addition. Clearly M and (M_1, M_2) are isomorphic. This is obviously compatible with the direct sum as $(r_1, r_2)(M \oplus N) = (r_1, r_2)M \oplus (r_1, r_2)N$. □

With this technical understanding, we can proceed to a first initial result.

Theorem 3.2.15 (Burns, Daoud, and Liang [7])**.** *Let $Q(R)$ be semisimple. Let M be a finitely presented torsion R-module satisfying (A1) and with $\mathcal{P}(M)$ finite. Then there exists a module N direct sum of cyclically presented modules and a pseudo-isomorphism $M \approx N$.*

Remark 3.2.16. This condition is slightly weaker than admissibility as we do not fully require (A2) for the proof.

Proof. If $\mathcal{P}(M) = \varnothing$ then M is pseudo-null and so $M \approx 0$. Assume $\mathcal{P}(M) \neq \varnothing$, by Lemma 3.2.9, $S^{-1}R$ is a finite direct product of Prüfer rings, where $S = R \setminus \bigcup_{\mathfrak{p} \in \mathcal{P}(M)} \mathfrak{p}$. Let $R_1, \ldots, R_n$ be rings such that $S^{-1}R = \prod_{i=1}^{n} R_i$. Since $\mathcal{P}(M)$ is finite, the ring $S^{-1}R$ is semi-local. Thus, each component R_i is a semi-local Prüfer ring by the description of the maximal ideals of R given in Proposition 1.1.39. By Corollary 2.1.25, semi-local Prüfer rings are semi-local EDR domains. Theorem 2.1.28

provides a decomposition of each R_i component of $S^{-1}M$ in a finite direct sum of cyclically presented R_i-modules, thus, there exist C_i, $1 \leq i \leq n$ cyclically presented R-modules such that $S^{-1}M \cong \bigoplus_{i=1}^{n} S^{-1}C_i = S^{-1}N$. Using Lemma 3.1.22 we can raise the isomorphism to a pseudo-isomorphism $M \approx N$. □

Remark 3.2.17. The final isomorphism of the proof may be exploited to show that $\approx$ is an equivalence relation on the class of modules satisfying the hypotheses of Theorem 3.2.15, for the finite direct sum of finitely presented modules is finitely presented.

The former theorem already provides an interesting description of admissible torsion modules. However, we have yet to make use of all hypotheses of admissibility. The following proposition sheds a light on how we may employ the remaining hypothesis.

Proposition 3.2.18. *Let R be a ring with $Q(R)$ semisimple. Let M be an admissible R-module with $\mathcal{P}(M) \neq \varnothing$ and consider the multiplicatively closed set $S = R \setminus \bigcup_{\mathfrak{p} \in \mathcal{P}(M)} \mathfrak{p}$. Then $S^{-1}R$ is a direct product of finitely many Dedekind domains.*

Proof. By Lemma 3.2.9, $S^{-1}R$ is a finite direct product of Prüfer domains. As M is admissible, $\mathcal{P}(M) \subseteq \mathcal{P}_{\mathrm{fg}}$ by property (A2). Therefore, $R \setminus S$ is a union of finitely many finitely generated prime ideals. In particular, $S^{-1}R$ is semi-local. We now claim that all prime ideals of $S^{-1}R$ are finitely generated. We prove the claim later in Lemma 3.2.19. By Cohen's Theorem 1.1.40, $S^{-1}R$ is Noetherian, and so are all its direct product components. By Lemma 2.1.3, these components are Dedekind domains. □

To finish the previous proof we follow on the claim.

Lemma 3.2.19. *Let R be a ring and $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ finitely generated height 1 prime ideals of R. Consider $S = R \setminus \bigcup_{i=1}^{n} \mathfrak{p}_i$ and suppose $S^{-1}R$ is a finite direct product of domains. Then all prime ideals of $S^{-1}R$ are finitely generated.*

Proof. Let $S^{-1}R = \prod_{j=1}^{m} R_i$ where R_i are domains. Note that $S^{-1}R$ is semi-local with maximal ideals $\mathfrak{m}_i = \mathfrak{p}_i S^{-1}R$. By Proposition 1.1.39, all prime ideals of $S^{-1}R$ are of the form $\mathfrak{q}_j(\mathfrak{p}) := R_1 \times \cdots \times R_{j-1} \times \mathfrak{p} \times R_j \times \cdots \times R_m$ where $\mathfrak{p}$ is a prime ideal of R_j for some $1 \leq j \leq m$. Note that $\mathfrak{q}_j(0)$ is a prime ideal since R_i are domains, and it is obviously finitely generated by $e_j = (1, \ldots, 1, 0, 1, \ldots, 1)$ where the 0 sits in the jth component. Suppose $\mathfrak{p} \neq 0$ is a non-zero prime ideal, then $\mathfrak{q}_j(0) \subsetneq \mathfrak{q}_j(\mathfrak{p})$. Thus, there exist prime ideals $\mathfrak{q}_0, \mathfrak{q}_1$ of R such that $\mathfrak{q}_j(0) = \mathfrak{q}_0 S^{-1}R$ and $\mathfrak{q}_j(\mathfrak{p}) = \mathfrak{q}_1 S^{-1}R$. Furthermore, $\mathfrak{q}_j(\mathfrak{p})$ is contained in some maximal ideal $\mathfrak{m}_i$ for some $1 \leq i \leq n$ (by prime avoidance lemma 1.1.38), so $\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subseteq \mathfrak{p}_i$. Since $\mathfrak{p}_i$ has height 1, $\mathfrak{q}_1 = \mathfrak{p}_i$ and so $\mathfrak{q}_j(\mathfrak{p})$ is finitely generated. □

We can now proceed as in Theorem 3.2.15 and use Proposition 3.2.18 to exploit the structure of finitely generated modules over Dedekind domains.

Theorem 3.2.20 (Burns, Daoud, and Liang [7]). *Let $Q(R)$ be semisimple. Let M be a finitely presented admissible torsion R-module. Then there exist height 1 prime ideals $\mathfrak{p}_i \in \mathcal{P}$ and $e_i \geq 1$ natural numbers for $1 \leq i \leq n$ such that $M \approx \bigoplus_{i=1}^{n} R/\mathfrak{p}_i^{e_i}$.*

Proof. If $\mathcal{P}(M) = \varnothing$ then M is pseudo-null and so $M \approx 0$. Assume $\mathcal{P}(M) \neq \varnothing$, let $S = R \setminus \bigcup_{i=1}^{n} \mathfrak{p}_i$, by Proposition 3.2.18, the ring $S^{-1}R$ is a finite direct product of Dedekind domains. By the structure theorem of finitely generated torsion modules over Dedekind domains, i.e. Theorem 1.2.46, each component of $S^{-1}M$ is isomorphic to a direct sum of modules of the form $R'/\mathfrak{q}^e$ where R' is a component of $S^{-1}R$, $\mathfrak{q}$ is a prime ideal of R' and $e \geq 1$. Using the structure of prime ideals over products of rings, we conclude that there exist prime ideals $\mathfrak{p}_i \in \mathcal{P}$ and $e_i \geq 1$ natural numbers for $1 \leq i \leq n$ such that $S^{-1}M \cong \bigoplus_{i=1}^{n} S^{-1}R/\mathfrak{p}_i^{e_i} S^{-1}R$. Using Lemma 3.1.22 we obtain the desired pseudo-isomorphism. □

Remark 3.2.21. If R is an NKD, then the previous theorem recovers Theorem 3.1.23.

We finish this section providing a generalization of the last two theorems. Recall form Theorem 3.2.11 that given a finitely presented R-module M it is enough for it to satisfy (A1) and $Q(R)$ semisimple to guarantee a pseudo-isomorphism $M \approx M_{\text{tor}} \oplus M_{\text{tf}}$. We can work on a description of the torsion modules under these hypotheses exploiting Warfield's structure theorem.

Theorem 3.2.22 (Burns, Daoud, and Liang [7]). *Let M be a finitely presented torsion R-module satisfying (A1). Then there exist R-modules N and E such that $M \oplus N \approx E$ and E is a finite direct sum of cyclically presented R-modules.*

Proof. If $\mathcal{P}(M) = \varnothing$ then M is pseudo-null and so $M \oplus 0 \approx 0$. Assume $\mathcal{P}(M) \neq \varnothing$ and consider $S = R \setminus \bigcup_{\mathfrak{p} \in \mathcal{P}(M)} \mathfrak{p}$. By Lemma 3.2.9 the ring $S^{-1}R$ is a generalized Prüfer ring. Thus, by Warfield's structure theorem 2.2.33, there exists an $S^{-1}R$-module $S^{-1}N$ and a direct sum of cyclically presented R-modules E such that $S^{-1}M \oplus S^{-1}N \cong S^{-1}E$. Let $\varphi : S^{-1}M \oplus S^{-1}N \to S^{-1}E$ be the mentioned isomorphism and φ^{-1} its inverse. Consider the following maps:

$$\psi : S^{-1}M \xrightarrow{(\mathrm{Id},0)} S^{-1}M \oplus S^{-1}N \xrightarrow{\varphi} S^{-1}E,$$
$$\psi' : S^{-1}E \xrightarrow{\varphi^{-1}} S^{-1}M \oplus S^{-1}N \xrightarrow{\pi_1} S^{-1}M$$

where π_1 is the canonical projection onto $S^{-1}M$.

Since both M and E are finitely presented, Proposition 1.2.44 guarantees the following isomorphisms:

$$S^{-1}\operatorname{Hom}_R(M, E) \cong \operatorname{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}E),$$
$$S^{-1}\operatorname{Hom}_R(E, M) \cong \operatorname{Hom}_{S^{-1}R}(S^{-1}E, S^{-1}M)$$

so there exist R-module morphisms $\theta : M \to E$ and $\theta' : E \to M$ and elements $s, s' \in S$ such that $\psi = s^{-1}\theta$ and $\psi' = (s')^{-1}\theta'$. Take $\widetilde{N} := \operatorname{Ker}\theta'$ and note that

$\theta' \circ \theta = ss' \mathrm{Id}_M$ since $\psi' \circ \psi = \mathrm{Id}_{S^{-1}M}$. Then by Proposition 3.1.10, $\theta' \circ \theta$ is a pseudo-isomorphism and so by Lemma 3.1.9 we have $\operatorname{Ker} \theta$ and $\operatorname{Coker} \theta'$ are pseudo-null. Furthermore, note that

$$ss' \mathrm{Id}_{\theta(M)} : \theta(M) \xrightarrow{\theta'|_{\theta(M)}} M \xrightarrow{\theta^*} \theta(M)$$

and so by the same argument we conclude $\operatorname{Ker} \theta'|_{\theta(M)} = \theta(M) \cap \widetilde{N}$ is pseudo-null. Notice that $S^{-1}\theta(M) = \varphi(S^{-1}M)$ while $S^{-1}\widetilde{N} = \varphi(S^{-1}N)$ by definition of θ and θ'. The inclusion $\theta(M) + \widetilde{N} \to E$ induces an isomorphism $S^{-1}(\theta(M) + \widetilde{N}) \to S^{-1}E$ as localization is exact. Thus, by Lemma 3.1.22, the inclusion is a pseudo-isomorphism. Then we have the short exact sequence

$$0 \longrightarrow \theta(M) \cap \widetilde{N} \xrightarrow{x \mapsto (x,x)} \theta(M) \oplus \widetilde{N} \xrightarrow{(x,y) \mapsto x-y} \theta(M) + \widetilde{N} \longrightarrow 0$$

where the last morphism $(x, y) \mapsto x - y$ is a pseudo-isomorphism. Again by Lemma 3.1.22, the isomorphism $S^{-1}\widetilde{N} = \varphi(S^{-1}N) \cong S^{-1}N$ induces a pseudo-isomorphism $\chi : N \to \widetilde{N}$. We can concatenate the following pseudo-isomorphisms:

$$M \oplus N \xrightarrow{(\theta,\chi)} \theta(M) \oplus \widetilde{N} \xrightarrow{(x,y) \mapsto x-y} \theta(M) + \widetilde{N} \longrightarrow E$$

obtaining the desired pseudo-isomorphism. □

3.3 Connection to Iwasawa theory

The aim of this section is to provide a connection between Theorem 3.2.20 on the decomposition of torsion admissible modules and non-Noetherian Iwasawa theory. To see the relevance of Theorem 3.2.20 in this number theoretical context, it is convenient to first relate its analog over Noetherian Krull domains, Theorem 3.1.23, and classic Iwasawa theory.

Throughout these sections, we continue to assume R to be a commutative unital ring.

3.3.1 Classic Iwasawa theory

The main focus of classic Iwasawa theory is the study of $\mathbb{Z}_p$-extensions and $\mathbb{Z}_p[[T]]$-modules. Throughout this subsection, k will always denote a number field unless stated otherwise.

Definition 3.3.1. Let k be a number field and k_∞ a Galois extension k_∞/k. We say k_∞/k is a $\mathbb{Z}_p$-extension if its Galois group $\mathrm{Gal}(k_\infty/k) \cong (\mathbb{Z}_p, +)$.

Remark 3.3.2. We can see $\mathbb{Z}_p$ extensions arise in the following way. Let $k_0 = k$ and consider a tower of fields

$$k = k_0 \subseteq k_1 \subseteq \cdots \subseteq k_\infty = \bigcup_{n \geq 0} k_n$$

with $\mathrm{Gal}(k_n/k) \cong \mathbb{Z}/p^n\mathbb{Z}$ and $\mathrm{Gal}(k_\infty/k) = \varprojlim \mathrm{Gal}(k_n/k)$. These fields k_n are uniquely determined by the non-trivial closed subgroups of $\mathbb{Z}_p$, which are the ideals $p^n\mathbb{Z}_p$, and the index $[k_n : k] = p^n$. This is [30, Proposition 13.1].

The main example of a $\mathbb{Z}_p$-extension arises from the cyclotomic extensions of a number field.

Example 3.3.3. Let k be a number field and μ_{p^∞} be the group of all p^n-th roots of unity for $n \geq 0$. Then the extension $k(\mu_{p^\infty})/k$ is an infinite Galois extension. Furthermore, $\mathrm{Gal}(k(\mu_{p^\infty})/k) \cong \Gamma \times \Delta$ where $\Gamma \cong \mathbb{Z}_p$ and $\Delta \cong \mathbb{Z}/(p-1)\mathbb{Z}$ if p is odd or $\Delta \cong \mathbb{Z}/2\mathbb{Z}$ if $p = 2$. This is due to the existence of an injective representation $\chi : \mathrm{Gal}(k(\mu_{p^\infty})/k) \to \mathbb{Z}_p^\times$ and $\mathbb{Z}_p^\times \cong \mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z}$ if p is odd and $\mathbb{Z}_2^\times \cong \mathbb{Z}_2 \times \mathbb{Z}/2\mathbb{Z}$ if $p = 2$. Thus, we have a a subfield $k_\infty \subseteq k(\mu_{p^\infty})$ such that $\mathrm{Gal}(k_\infty/k) = \Gamma$ and so it is a $\mathbb{Z}_p$-extension. This is called the cyclotomic $\mathbb{Z}_p$-extension of k.

Iwasawa's idea was to utilize $\mathbb{Z}_p$-extensions to study the ideal class groups $\mathrm{Cl}(k_n)$ of the k_n in Remark 3.3.2. To do this, the appearance and study of $\mathbb{Z}_p[[T]]$-modules was crucial. We introduce some of the main ideas in the sequel.

Definition 3.3.4. Let k_∞/k be a $\mathbb{Z}_p$ extension and a decomposition into fields k_n as in Remark 3.3.2. We denote by L_n the maximal unramified p-extension of k_n, by $H_n \subseteq L_n$ the maximal abelian extension inside L_n, and $H_\infty = \bigcup_{n\geq 0} H_n$.

Remark 3.3.5. The fields H_n are called the p-primary part of the Hilbert class field of k_n. Class field theory relates the ideal class group of each k_n with H_n. Mainly, $\mathrm{Cl}(k_n)(p) \cong \mathrm{Gal}(H_n/k_n)$, where $\mathrm{Cl}(k_n)(p)$ is the p-primary component of the ideal class group, i.e. the unique p-Sylow subgroup of $\mathrm{Cl}(k_n)$.

Proposition 3.3.6. *Let k_∞/k be a $\mathbb{Z}_p$ extension with $\Gamma = \mathrm{Gal}(k_\infty/k) \cong \mathbb{Z}_p$. Then $\mathrm{Gal}(H_\infty/k_\infty)$ is a $\mathbb{Z}_p[[\Gamma]]$-module, where $\mathbb{Z}_p[[\Gamma]]$ is the completed group ring of Γ.*

Proof (Sketch). Recall the extensions H_n/k_n are abelian by definition. Let $\Gamma_n = \mathrm{Gal}(k_n/k)$ and $X_n = \mathrm{Gal}(H_n/k_n)$. We can extend any $\gamma \in \Gamma_n$ to be $\bar{\gamma} \in \mathrm{Gal}(H_n/k)$. Since X_n is normal in $\mathrm{Gal}(H_n/k)$, conjugation by $\bar{\gamma}$ induces an action on X_n. Since X_n is abelian, any choice of representative $\bar{\gamma}$ when inducing produces the same action, so Γ_n acts over X_n. Thus, X_n is a $\mathbb{Z}_p[\Gamma_n]$-module. Since $X = \mathrm{Gal}(H_\infty/k_\infty) = \varprojlim \mathrm{Gal}(H_n/k_n)$ we have X is a $\mathbb{Z}_p[[\Gamma]]$-module. □

Iwasawa's original approach (see for instance [16, §1-§5]) consisted in directly studying these $\mathbb{Z}_p[[\Gamma]]$-modules. It is common to refer to $\Lambda = \mathbb{Z}_p[[\Gamma]]$ as the Iwasawa algebra and Λ-modules as Iwasawa modules. Moreover, Iwasawa's approach was greatly simplified by Serre [27], who realized that $\mathbb{Z}_p[[\Gamma]] \cong \mathbb{Z}_p[[T]]$ for some variable T.

Proposition 3.3.7 (Serre)**.** *There is a non-canonical isomorphism $\mathbb{Z}_p[[\Gamma]] \cong \mathbb{Z}_p[[T]]$ given by $\gamma \mapsto 1 + T$ where γ is a fixed generator of the procyclic group Γ.*

Proof. See [21, Proposition 5.3.5]. □

Remark 3.3.8. Here γ refers to a topological generator of Γ, in the sense that the subgroup generated by γ is dense in Γ. Since $\mathbb{Z}_p$ is topologically cyclic, i.e. procyclic group, we know such a generator exists.

Thus, the Galois group $X = \mathrm{Gal}(H_\infty/k_\infty)$ can be studied as a $\mathbb{Z}_p[[T]]$-module, from now on, Λ-module. The advantage of this description is that $\mathbb{Z}_p[[T]]$ is a Noetherian Krull domain. Recall Theorem 3.1.23 provided a description of the finitely generated torsion modules of a NKD.

Proposition 3.3.9. *Let k_∞/k be a $\mathbb{Z}_p$-extension and consider H_∞ as in Definition 3.3.4. The Galois group $X = \mathrm{Gal}(H_\infty/k_\infty)$ is a finitely generated torsion Λ-module.*

Proof. See [21, Proposition 11.1.4]. □

Using a description of the height 1 prime ideals of $\Gamma = \mathbb{Z}_p[[T]]$, it is possible to further specify Theorem 3.1.23.

Definition 3.3.10. Let $(R, \mathfrak{m})$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$. A *Weierstrass polynomial* is a monic polynomial $f(x) \in R[x]$ such that all its coefficients $a_0, \dots a_{n-1} \in \mathfrak{m}$ with $n = \deg(f)$.

Theorem 3.3.11. *Let M be a finitely generated Iwasawa Λ-module. Then there exist irreducible Weierstrass polynomials $f_j \in \mathbb{Z}_p[T] \subset \Lambda$ and naturals r, m_i, n_j, s, t such that*

$$M \approx \Lambda^r \oplus \bigoplus_{i=1}^{s} \Lambda/p^{m_i} \oplus \bigoplus_{j=1}^{t} \Lambda/f_j^{n_j} \tag{3.4}$$

is a pseudo-isomorphism and the numbers r, m_i, n_j, s, t and ideals $f_j\Lambda$ are uniquely determined.

Proof. By [21, Lemma 5.3.7], the prime ideals of height 1 in Λ are precisely $p\Lambda$ and $f\Lambda$ for any f Weierstrass polynomial. By Theorem 3.1.20, there is a pseudo-isomorphism $M \approx M_{\mathrm{tf}} \oplus M_{\mathrm{tor}}$. Applying Theorem 3.1.23 provides the pseudo-isomorphism on the torsion part of the module. The fact that the torsion free part is in fact pseudo-isomorphic to a free module Λ^r is not true in general for NKDs, but it is true for Λ-modules, see [21, Propositions 5.1.8, 5.1.9]. □

Remark 3.3.12. Recall that by Proposition 3.3.7, the isomorphism $\Lambda = \mathbb{Z}_p[[\Gamma]] \cong \mathbb{Z}_p[[T]]$ is not canonical, so the choice of Weierstrass polynomials in the decomposition given by Theorem 3.3.11 depends on the choice of generator.

Using Theorem 3.3.11 we can define the following objects.

Definition 3.3.13. With the notation in Theorem 3.3.11, we define

(a) $\mathrm{rank}(M) = r$ the Λ-rank of M,

(b) $\mu(M) = \sum_{i=1}^{s} m_i$ the Iwasawa μ-invariant of M,

(c) $\lambda(M) = \sum_{j=1}^{t} n_j \deg(f_j)$ the Iwasawa λ-invariant of M.

(d) $\mathrm{char}_\Lambda(M) = \prod_{i=1}^{s} p^{m_i}\Lambda \prod_{j=1}^{t} f^{n_i}\Lambda$ the characteristic ideal of M.

Remark 3.3.14. The rank of an Iwasawa module and its Iwasawa invariants are determined up to pseudo-isomorphism and are independent of the choice of generator in the isomorphism $\mathbb{Z}_p[[\Gamma]] \cong \mathbb{Z}_p[[T]]$. See [21, Remark after Definition 5.3.9] for more details and properties of the invariants.

Remark 3.3.15. The characteristic ideal is dependent on the pseudo-isomorphism and the chosen generator, in contrast to the invariants.

Using these definitions and Theorem 3.3.11, it is possible to prove the following remarkable result due to Iwasawa [16, Theorem 11].

Theorem 3.3.16 (Iwasawa)**.** *Let k_∞/k be a $\mathbb{Z}_p$-extension. Consider the intermediate fields k_n/k as defined in Remark 3.3.2. Let $|\mathrm{Cl}(k_n/k)(p)| = p^{e_n}$ be the size of the p-primary component of $\mathrm{Cl}(k_n/k)$. Then there exist $\lambda, \mu, n_0 \geq 0$ and ν integers independent of n such that $e_n = \lambda n + \mu p^n + \nu$ for all $n \geq n_0$.*

Proof. See [21, Theorem 11.1.6]. □

Remark 3.3.17. The proof of Theorem 3.3.16 relies heavily on the Λ-module structure of $\mathrm{Gal}(H_\infty/k_\infty)$ and the fact that $\mathrm{Gal}(H_n/k_n) \cong \mathrm{Gal}(H_n k_\infty/k_\infty)$ to exploit it. This last isomorphism can be found in [21, Lemma 11.1.5]. This motivates the importance of Theorem 3.1.23 and its specific version Theorem 3.3.11 to the field of Iwasawa theory.

The characteristic ideal also plays a central role in Iwasawa theory. We are severely lacking in concepts and notation to properly introduce the result known as the Main Conjecture of Iwasawa theory, however, we introduce the statement and give some guiding remarks to understand the role of the characteristic ideal. For a more complete picture and further references see [21, XI §6].

Theorem 3.3.18. *Let k be a totally real number field, p a prime, and $d = [k(\mu_{2p}) : k]$. There exists a unique continuous function $\zeta_{k,p} : \mathbb{Z}_p \setminus \{1\} \to \mathbb{Q}_p$ such that*

$$\zeta_{k,p}(1-n) = \zeta_k(1-n) \prod_{\mathfrak{p}|p} (1 - \mathfrak{N}(\mathfrak{p})^{n-1}) \tag{3.5}$$

for all $n > 1$ with $d \mid n$, where $\zeta_k(s)$ is the Dedekind zeta function of k and $\mathfrak{N}$ denotes the absolute norm. Furthermore, the function $\zeta_{k,p}(s)$ is p-adic analytic with at most a simple pole at $s = 1$.

The number theoretic interest in this p-adic function lies in the following result.

Theorem 3.3.19. *Let k be a number field,*

$$\lim_{s\to 1}(s-1)\zeta_{k,p}(s) = \frac{2^n R_p h}{w\sqrt{|d_k|}} \prod_{\mathfrak{p}|p} (1 - \mathfrak{N}(\mathfrak{p})^{-1}) \tag{3.6}$$

where R_p is called the p-adic regulator, h is the class number, d_k is the discriminant of k, w is the number of roots of unity contained in k, and $n = [k : \mathbb{Q}]$.

Remark 3.3.20. Formula (3.6) relating the p-adic zeta function value at $s = 1$ and the class number h of k is called the p-adic analytic class number formula.

Finally, we introduce the result known as the Main Conjecture of Iwasawa theory, relating the characteristic ideal and the p-adic zeta function. For simplicity, we consider p an odd prime. First we must consider the following preliminary result.

Theorem 3.3.21. *Let k be a totally real number field and k_∞ its cyclotomic $\mathbb{Z}_p$-extension. Fix a generator $\gamma \in \Gamma = \mathrm{Gal}(k_\infty/k)$ and consider $\kappa : \Gamma \to \mathbb{Z}_p^\times$ the p-part of the cyclotomic character and denote $q = \kappa(\gamma)$. There exist a unique power series $G_{k,p} \in \mathbb{Z}_p[[T]]$ such that*

$$\zeta_{k,p}(s) = \frac{G_{k,p}(q^{1-s} - 1)}{q^{1-s} - 1} \tag{3.7}$$

The Main Conjecture of Iwasawa theory relates $G_{k,p}$ with the characteristic ideal of the Iwasawa module $X = \mathrm{Gal}(H_\infty/k_\infty)$. Recall H_∞ is the maximal abelian unramified p-extension of k_∞ as defined in Definition 3.3.4.

Theorem 3.3.22 (Main Conjecture)**.** *Let k be a totally real number field, k_∞ be its cyclotomic $\mathbb{Z}_p$-extension. Let $\Lambda = \mathbb{Z}_p[[T]]$ be the Iwasawa algebra with a fixed generator γ.*

$$G_{k,p}\Lambda = \mathrm{char}_\Lambda(\mathrm{Gal}(H_\infty/k_\infty)) \tag{3.8}$$

where $G_{k,p} \in \Lambda$ is as defined in Theorem 3.3.21 and H_∞ as in Definition 3.3.4.

This conjecture was first proven by Mazur and Wiles in [18] when $k/\mathbb{Q}$ is an abelian extension. It was later proved by Wiles for totally real fields in [31].

3.3.2 Non-Noetherian Iwasawa theory

So far we have presented the picture of Iwasawa theory when considering $\mathbb{Z}_p$-extensions. This can be extended to consider more general settings such as $\mathbb{Z}_p^d$-extensions for $d \geq 1$. In this case, the Iwasawa algebra can be seen to be isomorphic to $\Lambda_d \cong \mathbb{Z}_p[[T_1, \ldots, T_d]]$. As noted in Example 3.2.8(iii), these rings Λ_d are Noetherian UFD's, so they are Noetherian Krull domains, and Theorem 3.1.23 still applies, so it is possible to define the notion of characteristic ideal.

Following this trend, one may consider the ring $\mathbb{Z}_p^{\mathbb{N}}$, referring to a countably infinite product of the ring $\mathbb{Z}_p$, and consider $\mathbb{Z}_p^{\mathbb{N}}$-extensions. In this case, the completed group ring would be $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^{\mathbb{N}}]]$ and we can consider a non-canonical isomorphism

$$\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^{\mathbb{N}}]] \cong \varprojlim_n \mathbb{Z}_p[[X_1, \ldots, X_n]] = \mathbb{Z}_p[[X_1, \ldots]] \tag{3.9}$$

by choosing generators of each $\mathbb{Z}_p$ and the isomorphism in Proposition 3.3.7.

We must first identify whether there actually are $\mathbb{Z}_p^{\mathbb{N}}$-extensions for some field. In the classical example, we worked over cyclotomic extensions of a number field as the units $\mathbb{Z}_p^\times$ of $\mathbb{Z}_p$ contained a copy of $\mathbb{Z}_p$, thus obtaining a $\mathbb{Z}_p$-extension. The following theorem motivates a change in setting.

Theorem 3.3.23. *Let k be a local field of characteristic $p > 0$ and $q = p^n$ the number of elements of its residue field. Then there is an algebraic and topological isomorphism*

$$k^\times \cong \mathbb{Z} \oplus \mathbb{Z}/(q-1)\mathbb{Z} \oplus \mathbb{Z}_p^{\mathbb{N}} \tag{3.10}$$

Furthermore, if $\mathcal{O}$ is the ring of integers of k, then its units are $\mathcal{O}^\times \cong \mathbb{Z}_p^{\mathbb{N}}$.

Proof. See [20, Proposition II.5.7] for the isomorphism of $k^\times$ and [20, Proposition II.5.3] for the isomorphism with the units $\mathcal{O}^\times$. □

Thus, rather than considering number fields, i.e finite extensions of $\mathbb{Q}$; the natural setting for $\mathbb{Z}_p^{\mathbb{N}}$-extensions are function fields, i.e. finite extensions of $\mathbb{F}_p[T]$ for some variable T.

Our next objective is to provide an example of such a $\mathbb{Z}_p^{\mathbb{N}}$-extension. Nevertheless, we are lacking notions which are beyond the scope of this discussion, so we briefly introduce some new concepts and mostly work by analogy with the classical example of cyclotomic $\mathbb{Z}_p$-extensions. The example we present is discussed in more detail by Bars in [8, Appendix].

Henceforth, let k denote a global function field of characteristic $p > 0$ whose base field of constants is $\mathbb{F}_q$ with q a power of p. We fix a place ∞ of k. Recall that a place of k is an equivalence class of absolute values of our field k (up to the usual equivalence of norms). For instance, in the case of $\mathbb{Q}$ the places are denoted ∞ for the real absolute value and p for each prime valuation of $\mathbb{Q}$. We denote by $R = \{f \in k \mid v_\infty(f) \geq 0\}$ the ring of integers associated to the place ∞ through its valuation v_∞.

Let H denote the Hilbert class field of k that is totally split at ∞ and let $H^+/H/k$ be the normalizing field of k. We now want to construct the analog of the cyclotomic extension $k(\mu_{p^\infty})$ in the classical case. To do so, we require the notion of Drinfeld module, which is the object that plays the same role as the cyclotomic polynomials in a number field.

Definition 3.3.24. We define a sgn-normalized rank 1 Drinfeld module Φ over k as a non-trivial $\mathbb{F}_q$-algebra embedding

$$\Phi : R \to H^+\{\tau\} \tag{3.11}$$

where τ is such that $\tau a = a^q \tau$ for all $a \in H^+$ and $\tau^0 = \mathrm{id}$.

Now fix a prime ideal $\mathfrak{p}$ of R, we define $\Phi[\mathfrak{p}^n]$ as the set of zeros of $\Phi(x)$, $x \in \mathfrak{p}^n$. We can consider the extension

$$k(\Phi[\mathfrak{p}^\infty]) = \bigcup_{n \geq 1} k(\Phi[\mathfrak{p}^n]) \tag{3.12}$$

called the $\Phi[\mathfrak{p}^\infty]$-torsion extension of k with respect to the place ∞. This newly defined field $k(\Phi[\mathfrak{p}^\infty])$ can be considered the function field analog to $k(\mu_{p^\infty})$ in the number field setting. Similarly, it can be seen that $\mathrm{Gal}(k(\Phi[\mathfrak{p}^\infty])/K) \cong \Delta \times \Gamma$ where Δ is a finite group of order coprime to p and $\Gamma \cong \mathbb{Z}_p^{\mathbb{N}}$. Thus, the subfield

$$k_\infty \coloneqq k(\Phi[\mathfrak{p}^\infty])^\Delta \tag{3.13}$$

is a $\mathbb{Z}_p^{\mathbb{N}}$-extension of k. Although in a very qualitative manner, we have succeeded in constructing a $\mathbb{Z}_p^{\mathbb{N}}$-extension for any global function field.

Now that we know $\mathbb{Z}_p^{\mathbb{N}}$-extensions exist and appear in a relatively natural manner over global function fields, we can start studying the Iwasawa modules of these extensions, i.e. the modules of $\Lambda = \mathbb{Z}_p[[X_1, \dots]]$ that provide an action over abelian extensions.

The main difference with classical Iwasawa theory is that these modules are not Noetherian. Thus, we lack a general structure theorem to describe Λ-modules. Nevetheless, in the particular case of finitely presented torsion modules, Theorem 3.2.20 due to Burns, Daoud, and Liang gives us a structure result for these modules, since Λ is admissible (recall Example 3.2.8). This enables us to define a notion of generalized characteristic ideal in this setting.

Remark 3.3.25. Before the definition, let us recall and rewrite the structure theorems of finitely presented admissible torsion modules. Let M be such an R-module, by Theorem 3.2.20 we have a pseudo-isomorphism

$$M \approx \bigoplus_{i=1}^{n} R/\mathfrak{q}_i^{e_i} \approx \bigoplus_{\mathfrak{p} \in \mathcal{P}(M)} \bigoplus_{i=1}^{n(\mathfrak{p})} R/\mathfrak{p}^{e(\mathfrak{p})_i} \tag{3.14}$$

with $\mathfrak{q}_i \in \mathcal{P}$ and $e_i \geq 1$. The second pseudo-isomorphism is simply a reordering of the first expression, taking into account that $\mathcal{P}(M)$ is finite by admissibility and any prime ideal $\mathfrak{p} \in \mathcal{P}$ outside of the support of M produces a pseudo-null module, so we can restrict $\mathcal{P}$ to $\mathcal{P}(M)$.

Another version of this pseudo-isomorphism is given by Theorem 3.2.15, where we can consider C_i cyclically presented R-modules for $1 \leq i \leq m$ for some finite m such that

$$M \approx \bigoplus_{i=1}^{m} C_i \cong \bigoplus_{i=1}^{m} R/L_i \tag{3.15}$$

and L_i are principal ideals such that $C_i \cong R/L_i$ (recall Definition 2.2.3 of cyclically presented ideals).

Definition 3.3.26 (Generalized characteristic ideal)**.** Let R be a ring and $Q(R)$ be semisimple. Let M be a finitely presented admissible torsion R-module. With the notation of Remark 3.3.25, we define the *generalized characteristic ideals* of M as

$$\mathrm{char}_R(M) = \prod_{\mathfrak{p} \in \mathcal{P}(M)} \mathfrak{p}^{\sum_{i=1}^{n(\mathfrak{p})} e(\mathfrak{p})_i}, \qquad \mathrm{Char}_R(M) = \prod_{i=1}^{m} L_i \tag{3.16}$$

Remark 3.3.27. The key difference between both definitions of generalized characteristic ideals is that one is principal and the other a product of prime ideals. Furthermore, both need not coincide, as they are only pseudo-isomorphic.

We present some basic properties of these generalized characteristic ideals, given by Burns, Daoud, and Liang, and some relations between both definitions.

In the sequel, we say an R-module M is quadratically presented if it admits a presentation $R^n \to R^n \to M \to 0$ for some $n \geq 0$. Recall that the k-th Fitting ideal of a finitely generated module M with n generators is $\mathrm{Fit}_R^k(M)$ the ideal generated by the $(n-k) \times (n-k)$ minors of a presentation matrix of M.

Proposition 3.3.28. *Let R be a ring with $Q(R)$ semisimple. Let M be a finitely presented admissible torsion R-module.*

(i) The ideal $\mathrm{char}_R(M)$ is independent of the chosen pseudo-isomorphism in Theorem 3.2.20 and $\mathrm{char}_R(M)_\mathfrak{p} = \mathrm{Char}_R(M)_\mathfrak{p}$ for all $\mathfrak{p} \in \mathcal{P}$.

(ii) Let G be a finite abelian group and A a $\mathbb{Z}_p$-algebra that is a Krull domain. Let $R = A[G]$. Then $\mathrm{char}_R(M) \subseteq \mathrm{Char}_R(M)$ and the equality holds if and only if $\mathrm{char}_R(M)$ is principal. Furthermore, $\mathrm{Char}_R(M)/\mathrm{char}_R(M)$ is pseudo-null and if M is quadratically presented then $\mathrm{Char}_R(M) = \mathrm{Fit}_R^0(M)$.

Proof. See [7, Proposition 2.11(i)]. □

It can also be seen that the generalized characteristic ideal $\mathrm{char}_R(M)$ is multiplicative with respect to exact sequences.

Lemma 3.3.29. *Let M, M', M'' be R-modules with M', M'' finitely presented admissible torsion R-modules, given an exact sequence $0 \to M' \to M \to M'' \to 0$ we have that M is also finitely presented, admissible and torsion and $\mathrm{char}_R(M) = \mathrm{char}_R(M')\,\mathrm{char}_R(M'')$.*

Proof. See [7, Proposition 2.11(ii)]. □

With these notions of generalized characteristic ideals and its properties (many of which we have omitted in this document, see [7, §2]), it is possible to derive Iwasawa theoretical results in the non-Noetherian setting. In particular, [7] apply their new theory to study the inverse limits with respect to corestriction of the p-completions of the degree one Weil-étale cohomology groups of $\mathbb{G}_m$ over finite extensions K/k of k a function field of a smooth projective curve over $\mathbb{F}_p$. This is way beyond the scope for this work, so we refer the interested reader to their paper [7].

Finally, we end this section with some remarks regarding the extension of the study of non-Noetherian Iwasawa algebras and a comment on previous work in this field.

Remark 3.3.30. Prior to Burns, Daoud, and Liang in [7] and their structure theorem for admissible modules, a different approach to define a characteristic ideal in the Noetherian setting was given by Bandini, Bars, and Longhi in [3, Definition 3.10].

They called their object *pro-characteristic ideal*, and used it in [2] and [1] to study a specific type of $\mathbb{Z}_p^{\mathbb{N}}$-extensions, called Carlitz cyclotomic $\mathbb{Z}_p^{\mathbb{N}}$-extensions. In [1], Anglès et al. prove an Iwasawa main conjecture for these kind of extensions.

Under certain restrictions, Burns, Daoud, and Liang show in [7, Proposition 3.4] that the notions of pro-characteristic ideal and $\mathrm{char}_R(M)$ coincide.

Remark 3.3.31. In addition to studying $\mathbb{Z}_p^{\mathbb{N}}$-extensions, Burns, Daoud, and Liang also prove that fixed a finite abelian group G such that $p \nmid |G|$, the rings $\mathbb{Z}_p[[X_1, \dots]][G]$ and $\mathbb{Z}_p[[X_1, \dots, X_n]][G]$ for all $n \geq 1$ are also admissible. This is [7, Lemma 3.1(ii)]. Thus, the structure theorem 3.2.20 for admissible torsion modules can also be employed to study $\mathbb{Z}_p^{\mathbb{N}} \times G$-extensions, which are also natural to consider given the example showcased in this subsection.

Bibliography

[1] Bruno Anglès, Andrea Bandini, Francesc Bars, and Ignazio Longhi. "Iwasawa main conjecture for the Carlitz cyclotomic extension and applications". English. In: *Math. Ann.* 376.1-2 (2020), pp. 475–523. ISSN: 0025-5831. DOI: 10.1007/s00208-019-01875-8. URL: ddd.uab.cat/record/240658.

[2] Andrea Bandini, Francesc Bars, and Ignazio Longhi. "Aspects of Iwasawa theory over function fields". English. In: *t-motives: Hodge structures, transcendence and other motivic aspects*. Zürich: European Mathematical Society (EMS), 2020, pp. 375–416. ISBN: 978-3-03719-198-9. DOI: 10.4171/198-1/7. URL: hdl.handle.net/11568/925063.

[3] Andrea Bandini, Francesc Bars, and Ignazio Longhi. "Characteristic ideals and Iwasawa theory". English. In: *New York J. Math.* 20 (2014), pp. 759–778. ISSN: 1076-9803.

[4] Nicolas Bourbaki. *Elements of mathematics. Commutative algebra. Chapters 1–7. Transl. from the French.* English. 2nd printing. Berlin etc.: Springer-Verlag, 1989. ISBN: 3-540-19371-5.

[5] Demetrios Brizolis. "A theorem on ideals in Prüfer rings of integral-valued polynomials". English. In: *Commun. Algebra* 7 (1979), pp. 1065–1077. ISSN: 0092-7872. DOI: 10.1080/00927877908822391.

[6] Michel Broué. *From rings and modules to Hopf algebras. One flew over the algebraist's nest*. English. Cham: Springer, 2024. ISBN: 978-3-031-50062-6. DOI: 10.1007/978-3-031-50062-6.

[7] David Burns, Alexandre Daoud, and Dingli Liang. *On Non-Noetherian Iwasawa Theory*. Preprint, arXiv:2401.02946 [math.NT] (2024). 2024. URL: https://arxiv.org/abs/2401.02946.

[8] David Burns and Fabien Trihan. "On geometric Iwasawa theory and special values of zeta functions". English. In: *Arithmetic geometry over global function fields. Selected notes based on the presentations at five advanced courses on arithmetic geometry at the Centre de Recerca Matemàtica, CRM, Barcelona, Spain, February 22 – March 5 and April 6–16, 2010*. Basel: Birkhäuser / Springer, 2014, pp. 119–181. ISBN: 978-3-0348-0852-1. DOI: 10.1007/978-3-0348-0853-8_2.

[9] I. S. Cohen. "Commutative rings with restricted minimum condition". English. In: *Duke Math. J.* 17 (1950), pp. 27–42. ISSN: 0012-7094. DOI: 10.1215/S0012-7094-50-01704-2.

[10] Shizuo Endo. "On semi-hereditary rings". English. In: *J. Math. Soc. Japan* 13 (1961), pp. 109–119. ISSN: 0025-5645. DOI: 10.2969/jmsj/01320109.

[11] Robert M. Fossum. *The divisor class group of a Krull domain*. English. Vol. 74. Ergeb. Math. Grenzgeb. Springer-Verlag, Berlin, 1973.

[12] László Fuchs and Luigi Salce. *Modules over non-Noetherian domains*. English. Vol. 84. Math. Surv. Monogr. Providence, RI: American Mathematical Society (AMS), 2001. ISBN: 0-8218-1963-1.

[13] Leonard Gillman and Melvin Henriksen. "Rings of continuous functions in which every finitely generated ideal is principal". English. In: *Trans. Am. Math. Soc.* 82 (1956), pp. 366–391. ISSN: 0002-9947. DOI: 10.2307/1993054.

[14] R. Gilmer. "Power series rings over a Krull domain". English. In: *Pac. J. Math.* 29 (1969), pp. 543–549. ISSN: 1945-5844. DOI: 10.2140/pjm.1969.29.543.

[15] Robert Gilmer and William W. Smith. "Finitely generated ideals of the ring of integer-valued polynomials". English. In: *J. Algebra* 81 (1983), pp. 150–164. ISSN: 0021-8693. DOI: 10.1016/0021-8693(83)90213-2.

[16] Kenkichi Iwasawa. "On Γ-extensions of algebraic number fields". English. In: *Bull. Am. Math. Soc.* 65 (1959), pp. 183–226. ISSN: 0002-9904. DOI: 10.1090/S0002-9904-1959-10317-7.

[17] Hideyuki Matsumura. *Commutative ring theory. Transl. from the Japanese by M. Reid.* English. Paperback ed. Vol. 8. Camb. Stud. Adv. Math. Cambridge etc.: Cambridge University Press, 1989. ISBN: 0-521-36764-6.

[18] B. Mazur and A. Wiles. "Class fields of abelian extensions of $\mathbb{Q}$". English. In: *Invent. Math.* 76 (1984), pp. 179–330. ISSN: 0020-9910. DOI: 10.1007/BF01388599. URL: https://eudml.org/doc/143124.

[19] Masayoshi Nagata. *Local rings*. English. Vol. 13. Intersci. Tracts Pure Appl. Math. Interscience Publishers, New York, NY, 1962.

[20] Jürgen Neukirch. *Algebraic number theory. Transl. from the German by Norbert Schappacher*. English. Vol. 322. Grundlehren Math. Wiss. Berlin: Springer, 1999. ISBN: 3-540-65399-6.

[21] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg. *Cohomology of number fields*. English. 2nd ed. Vol. 323. Grundlehren Math. Wiss. Berlin: Springer, 2008. ISBN: 978-3-540-37888-4.

[22] H. Nishimura. "On the unique factorization theorem for formal power series". English. In: *J. Math. Kyoto Univ.* 7 (1967), pp. 151–160. ISSN: 0023-608X. DOI: 10.1215/kjm/1250524274.

[23] Joseph J. Rotman. *Advanced modern algebra*. English. Upper Saddle River, NJ: Prentice Hall/Pearson Education, 2002. ISBN: 0-13-087868-5.

[24] Joseph J. Rotman. *An introduction to homological algebra*. English. 2nd ed. Universitext. Berlin: Springer, 2009. ISBN: 978-0-387-68324-9. DOI: 10.1007/b98977.

[25] P. Samuel. *Lectures on unique factorization domains. Notes by M. Pavman Murthy*. English. Vol. 30. Lect. Math. Phys., Math., Tata Inst. Fundam. Res. Bombay: Tata Institute of Fundamental Research, 1965.

[26] Heinz-Werner Schülting. "Über die Erzeugendenanzahl invertierbarer Ideale in Prüferringen". German. In: *Commun. Algebra* 7 (1979), pp. 1331–1349. ISSN: 0092-7872. DOI: 10.1080/00927877908822406.

[27] Jean-Pierre Serre. *Classes des corps cyclotomiques*. French. Semin. Bourbaki 11 (1958/59), No. 174, 11 p. (1959). 1959. URL: https://eudml.org/doc/109603.

[28] R. B. jun. Warfield. "Decomposability of finitely presented modules". English. In: *Proc. Am. Math. Soc.* 25 (1970), pp. 167–172. ISSN: 0002-9939. DOI: 10.2307/2036549.

[29] R. B. jun. Warfield. "Purity and algebraic compactness for modules". English. In: *Pac. J. Math.* 28 (1969), pp. 699–719. ISSN: 1945-5844. DOI: 10.2140/pjm.1969.28.699.

[30] Lawrence C. Washington. *Introduction to cyclotomic fields*. English. Vol. 83. Grad. Texts Math. Springer, Cham, 1982.

[31] A. Wiles. "The Iwasawa conjecture for totally real fields". English. In: *Ann. Math. (2)* 131.3 (1990), pp. 493–540. ISSN: 0003-486X. DOI: 10.2307/1971468.

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