
Equilibrium distributions and discrete Schur-constant models

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Abstract This paper introduces Schur-constant equilibrium distribution models of dimension n for arithmetic non-negative random variables. Such a model is defined through the (several orders) equilibrium distributions of a univariate survival function. First, the bivariate case is considered and analyzed in depth, stressing the main characteristics of the Poisson case. The analysis is then extended to the multivariate case. Several properties are derived, including the implicit correlation and the distribution of the sum.

Keywords Schur-constant property · Discrete stationary-excess operator · Discrete equilibrium distributions

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1 Introduction

Schur-constant models for discrete survival data have been studied by several authors including Castañer et al. (2015), Lefèvre et al. (2017) and Ta and Van (2017). In this case the discrete setting means that it is valued at $\mathbb{N}_0 = \{0, 1, \dots\}$. There is also a vast literature regarding Schur-constant models for continuous survival data valued in \mathbb{R}_+ ; let us mention, among others, Caramellino and Spizzichino (1994), Chi et al. (2009) and Nelsen (2005). Castañer et al. (2015) discuss the properties of a Schur-constant vector (X_1, \dots, X_n) generated by the survival function of an univariate random variable (r.v.). In that paper, it is shown that this survival function has to be n -monotone. The present paper aims to develop a class of Schur-constant models generated by an admissible univariate survival function. The admissibility condition is the existence of the $(n-1)$ -th equilibrium distribution of the univariate r.v. Because of it, this new model is called *Schur-constant (multivariate) equilibrium distribution model*. Some properties of the bivariate continuous case have been previously studied by Nair and Sankaran (2014). For the sake of completeness, we include in this introduction some definitions and well-known results.

Let (X_1, \dots, X_n) be a vector of n (≥ 2) arithmetic non-negative random variables, called lifetimes. It is said to have a Schur-constant joint survival function if for all $(x_1, \dots, x_n) \in \mathbb{N}_0^n$,

$$P(X_1 \geq x_1, \dots, X_n \geq x_n) = S(x_1 + \dots + x_n), \quad (1.1)$$

where S is an admissible function from \mathbb{N}_0 to $[0, 1]$. In fact, S is the survival function of each of the marginal r.v. X_i and it has to be n -monotone on \mathbb{N}_0 . A function $f(x): \mathbb{N}_0 \rightarrow \mathbb{R}$ is said to be n -monotone if it satisfies

$$(-1)^j \Delta^j f(x) \geq 0, \text{ for } j = 0, \dots, n, \quad (1.2)$$

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where Δ is the forward difference operator (i.e. $\Delta f(x) = f(x+1) - f(x)$) and Δ^j is its j -th iterate.

A general representation (Castañer et al., 2015) valid for any discrete Schur-constant model, putting $\binom{a}{b} = 0$ when $a < b$, is:

$$S(x_1 + \dots + x_n) = E \left[\binom{Z - (x_1 + \dots + x_n) + n - 1}{n - 1} / \binom{Z + n - 1}{n - 1} \right], \quad (1.3)$$

where the variable Z is distributed as $X_1 + \dots + X_n$, i.e., with a probability mass function (p.m.f.) given by

$$P(Z = z) = (-1)^n \Delta^n S(z) \binom{z + n - 1}{n - 1}. \quad (1.4)$$

As a particular case, for $n = 2$,

$$P(Z = z) = \Delta^2 S(z)(z + 1). \quad (1.5)$$

The p.m.f. of any subvector in (X_1, \dots, X_n) can be obtained from the j -th iterated forward difference of S ,

$$P(X_1 = x_1, \dots, X_j = x_j) = (-1)^j \Delta^j S(x_1 + \dots + x_j). \quad (1.6)$$

For bivariate Schur-constant vectors we can obtain an easy expression of the p.m.f. of the marginal. This is the next Lemma 1.1, which will be used in the next section of the paper.

Lemma 1.1 *If the vector (X_1, X_2) is Schur-constant, the probability mass function of X_1 fulfills*

$$P(X_1 = x + 1) = P(X_1 = x) - \frac{P(Z = x)}{x + 1}, \quad (1.7)$$

for $x \geq 0$, with

$$P(X_1 = 0) = E \left(\frac{1}{Z + 1} \right).$$

Proof The survival function of X_1 , for $n = 2$, is obtained from (1.3) when $x_2 = 0$,

$$S(x) = E \left[\frac{(Z - x + 1)_+}{Z + 1} \right] = \sum_{z=x}^{\infty} \frac{z - x + 1}{z + 1} P(Z = z). \quad (1.8)$$

From definition, $P(X_1 = x) = -\Delta S(x)$. Then, considering (1.8),

$$P(X_1 = x) = \sum_{z=x}^{\infty} \frac{1}{z + 1} P(Z = z). \quad (1.9)$$

From (1.9), for $x = 0$, $P(X_1 = 0) = E \left(\frac{1}{Z + 1} \right)$. Now, let us apply again the definition of the survival function so that $\Delta^2 S(x) = P(X_1 = x) - P(X_1 = x + 1)$. Using this expression and (1.5), the desired result (1.7) is obtained. \square

On the basis of these definitions, we develop our analysis. The paper is organized as follows. In Section 2, we introduce the discrete Schur-constant bivariate equilibrium distribution model and provide its characteristics. We prove that in the Poisson bivariate Schur-constant equilibrium distribution model, Z is also Poisson distributed. In Section 3, we extend the analysis to the multivariate case. The paper ends with some concluding remarks.

2 Schur-constant bivariate equilibrium distribution model

In renewal theory, the asymptotic distribution of the age or residual life in a renewal process is known as the univariate equilibrium distribution (Cox, 1962). Let X be a non negative discrete random variable representing the lifetime of a component with finite mean μ and survival function $S(x) = P(X \geq x), x \geq 0$. The (first order) equilibrium distribution is defined through the probability mass function, as

$$p^*(x) = \frac{S(x+1)}{\mu}, \text{ for } x \in \mathbb{N}_0, \quad (2.10)$$

or, via the survival function, as

$$S^*(x) = \sum_{h=x+1}^{\infty} \frac{S(h)}{\mu}, \text{ for } x \in \mathbb{N}_0. \quad (2.11)$$

Expression (2.11) is well defined as survival function as long as it is a decreasing function and $S^*(0) = 1$. This latter condition can be easily checked taking into account that $E(X) = \sum_{h=1}^{\infty} S(h)$.

Much attention has been paid to the equilibrium distributions (of first and higher orders) associated with a given distribution function, but most studies are for continuous univariate random variables (see for instance Deshpande et al. (1986), Sunoj (2004) and the references therein). Regarding the discrete univariate case, two papers should be mentioned: Li (2011) and Willmot et al. (2005). In addition, Lefèvre and Loisel (2010) introduce a specific discrete version of the stationary-excess operator for discrete non-negative random variables. This discrete stationary-excess operator H maps any non-negative discrete random variable X to an associated discrete non-negative random variable X_H whose probability mass function is

$$P(X_H = x) = \frac{P(X \geq x+1)}{E(X)}, \quad x \in \mathbb{N}_0.$$

Notice that the equilibrium distribution can be also obtained by applying this discrete stationary-excess operator.

In this section, we discuss a (first order) bivariate equilibrium distribution of discrete random variables. It represents in fact a family of discrete Schur-constant distributions, that we call *Schur-constant bivariate equilibrium distributions*. Although it has been recently analyzed by Gupta (2012) and Nair and Sankaran (2014) in a continuous setting, as far as we know, the bivariate discrete case has not previously been evaluated. The asymptotic joint distribution of age and residual life is defined by the survival function

$$G(x, y) = \frac{1}{\mu} \sum_{h=x+y+1}^{\infty} S(h) = S^*(x+y). \quad (2.12)$$

Now we prove, in the next proposition, that this survival function corresponds to a Schur-constant survival function.

Proposition 2.1 *The random vector (X^*, Y^*) with survival function G given by (2.12) is Schur-constant and its marginal distribution is the equilibrium distribution of X given by (2.11).*

Proof It is enough to put $y = 0$ in expression (2.12) to check that X^* is the equilibrium distribution of X . With respect to the Schur-constancy property, from (2.12), we obtain

$$\Delta S^*(x) = -\frac{S(x+1)}{\mu} \leq 0, \quad (2.13)$$

and

$$\begin{aligned} \Delta^2 S^*(x) &= -\frac{S(x+2)}{\mu} + \frac{S(x+1)}{\mu} = \frac{1}{\mu} [P(X \geq x+1) - P(X \geq x+2)] \\ &= \frac{P(X = x+1)}{\mu} \geq 0. \end{aligned} \quad (2.14)$$

Then, from (2.13) and (2.14), $S^*(x)$ is always 2-monotone. \square

Notice that, from Proposition 2.1, the survival function $S^*(x)$, given by (2.11) is always 2-monotone although $S(x)$ may not be.

Now we derive several useful formulas. As (X^*, Y^*) is Schur-constant,

$$P(X^* = x, Y^* = y) = \Delta^2 S^*(x + y) = \frac{P(X = x + y + 1)}{\mu}, \quad (2.15)$$

and the p.m.f of the sum $Z = X^* + Y^*$, by (1.5) and (2.14), is

$$P(Z = z) = \frac{P(X = z + 1)(z + 1)}{\mu}. \quad (2.16)$$

In fact, Z turns out to be the (first) length-biased type transform of X (as defined in Lefèvre and Loisel, 2013).

The basic characteristics of (X^*, Y^*) are available in terms of the expectations of simple functions of X and this is our next proposition.

Proposition 2.2 *Let (X^*, Y^*) be as in Proposition 2.1. Then,*

$$\mu^* = E(X^*) = \frac{1}{2} \left(\frac{E(X^2)}{\mu} - 1 \right), \quad (2.17)$$

$$V(X^*) = \frac{4\mu E(X^3) - 3(E(X^2))^2 - \mu^2}{12\mu^2}. \quad (2.18)$$

Proof Considering (2.10), we have

$$\mu^* = E(X^*) = \frac{1}{\mu} \sum_{x=0}^{\infty} xS(x+1). \quad (2.19)$$

It is a simple exercise that $\Delta\left(\frac{x^2-x}{2}\right) = x$, so that we can apply summation by parts to the last sum in (2.19). This leads to

$$\begin{aligned} \sum_{x=0}^{\infty} xS(x+1) &= \frac{1}{2} \left(\sum_{x=0}^{\infty} P(X = x+1)(x+1)^2 - \sum_{x=0}^{\infty} P(X = x+1)(x+1) \right) \\ &= \frac{1}{2} \left(\sum_{x=0}^{\infty} P(X = x)x^2 - \sum_{x=0}^{\infty} P(X = x)x \right). \end{aligned} \quad (2.20)$$

Substituting this in (2.19), we obtain (2.17). Let us derive now a formula for the second ordinary moment of X^* ,

$$E((X^*)^2) = \sum_{x=0}^{\infty} x^2 \frac{S(x+1)}{\mu}. \quad (2.21)$$

We know that $\Delta\left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{6}\right) = x^2$. Considering this, summation by parts yields

$$E((X^*)^2) = \frac{1}{\mu} \left(\frac{E(X^3)}{3} - \frac{E(X^2)}{2} + \frac{\mu}{6} \right). \quad (2.22)$$

By (2.17) and (2.22), we obtain formula (2.18). \square

Castañer et al. (2015) presented simple formulas for the Pearson correlation coefficient, ρ , between any two variables in a Schur-constant vector. One of them allows to compute ρ in terms of the expectation and the variance of one of the marginal random variables. But in these Schur-constant discrete bivariate equilibrium distributions, it is convenient to relate ρ to the main characteristics of the original variable X , the building block of the model.

Proposition 2.3 *In terms of $E(X^*)$ and $V(X^*)$,*

$$\rho = \frac{V(X^*) - (E(X^*))^2 - E(X^*)}{2V(X^*)}, \quad (2.23)$$

and, in terms of the ordinary moments of X ,

$$\rho = \frac{2\mu E(X^3) - 3(E(X^2))^2 + \mu^2}{4\mu E(X^3) - 3(E(X^2))^2 - \mu^2}. \quad (2.24)$$

Proof Formula (2.23) is direct from Proposition 6.2. in Castañer et al. (2015). Let us now establish (2.24). We could evaluate $E(X^*Y^*)$ directly using (2.15). However, a simpler method consists in using (2.23) in which $E(X^*)$ is substituted by (2.17) and $V(X^*)$ is substituted by (2.18). Simple algebra yields the desired formula. \square

The corresponding formulas (2.17), (2.18) and (2.24) for continuous random variables can be found in Nair and Sankaran (2014).

Then, we are ready to introduce the concept of a *generator* of a Schur-constant equilibrium distribution model. A non-negative discrete random variable X with equilibrium distribution (2.15) is the generator of a Schur-constant equilibrium distribution (X_1, X_2) when $P(X_1 \geq x_1, X_2 \geq x_2) = S^*(x_1 + x_2)$.

Now, we apply these results to two specific models generated by well-known distributions, the Poisson one and the geometric one.

2.1 Poisson Schur-constant bivariate equilibrium model

Let us consider that the Schur-constant equilibrium distribution model (X_1, X_2) is generated by a r.v. X that follows a Poisson distribution with parameter λ ; i.e., $P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$. This model is an interesting and particular one, because of its properties. Using (2.16), it is immediate to conclude that $Z = X_1 + X_2$ also follows a Poisson distribution of parameter λ ,

$$P(Z = z) = \frac{\lambda^{z+1} e^{-\lambda} (z+1)}{\lambda(z+1)!} = \frac{\lambda^z e^{-\lambda}}{z!}, \quad z = 0, 1, \dots \quad (2.25)$$

Proposition 2.4 *Let (X_1, X_2) be a Schur-constant bivariate equilibrium model generated by a r.v. X and $Z = X_1 + X_2$. Then, $Z =_d X$ if and only if X follows a Poisson distribution.*

Proof It is well known from Panjer (1981) that the Poisson distribution is the only one that fulfills a recursive equation on its probabilities of the form

$$P(X = k + 1) = P(X = k) \frac{b}{k + 1}, \quad k = 0, 1, 2, \dots, \quad (2.26)$$

being $b = \lambda$. Now, the proof of the implication that $Z =_d X$ implies X is Poisson simply needs to use (2.26) combined with (2.16). For the other implication, see the remark preceding Proposition 2.4. \square

The distribution of X_1 can be found using the property that X_1 is the equilibrium distribution of X . Since X is Poisson distributed, $S(x) = \sum_{h=x}^{\infty} \frac{e^{-\lambda} \lambda^h}{h!}$ and then

$$P(X_1 = x) = \frac{1 - \frac{\Gamma(x+1, \lambda)}{x!}}{\lambda}, \quad x = 0, 1, \dots, \quad (2.27)$$

being $\Gamma(\cdot, \cdot)$ the incomplete Gamma function. The same result can be obtained using (1.7) and Proposition 2.4. The correlation coefficient between X_1 and X_2 in this model, computed using (2.24), is $\rho = \frac{-\lambda}{6+\lambda}$.

2.2 Geometric Schur-constant bivariate equilibrium model

Let us consider a geometric r.v. X with survival function $S(x) = q^x$. We use X as a generator of a Schur-constant equilibrium vector (X_1, X_2) . Then, from Proposition 2.1 the r.v. X_1 is the equilibrium distribution of X . Taking into account that the mean of the generator X is $\mu = q/(1 - q)$, from (2.11),

$$S^*(x) = \sum_{h=x+1}^{\infty} \frac{q^h(1-q)}{q} = q^x.$$

We immediately see that $X_1 =_d X$. The variance of the generator X is $V(X) = q/(1 - q)^2$, so the Pearson correlation coefficient ρ computed from (2.23) is equal to 0. This last result is an obvious one because X_1 and X_2 are independent r.v. (Castañer et al., 2015).

3 Schur-constant multivariate equilibrium distribution model

In this section we generalize the results of the previous section. The first order equilibrium distribution of a discrete random variable X has been used in Section 2 to define a special class of bivariate Schur-constant models. In the present section, this analysis is extended to the general multivariate case through the use of the $n - th$ order equilibrium distribution.

Let X be a non negative discrete random variable with finite mean μ and survival function $S(x)$. The second order equilibrium distribution can be defined by its survival function

$$S^{2*}(x) = \sum_{h=x+1}^{\infty} \frac{S^*(h)}{\mu_{1:1}}, \text{ for } x \in \mathbb{N}_0, \quad (3.28)$$

or, alternatively, by its p.m.f

$$p^{2*}(x) = \frac{S^*(x+1)}{\mu_{1:1}}, \text{ for } x \in \mathbb{N}_0, \quad (3.29)$$

being $\mu_{1:1}$ the first order moment of X^* (the first order equilibrium distribution of X). Likewise, we can define recursively the $n - th$ order equilibrium distribution,

$$S^{n*}(x) = \sum_{h=x+1}^{\infty} \frac{S^{(n-1)*}(h)}{\mu_{n-1:1}}, \text{ for } x \in \mathbb{N}_0, \quad (3.30)$$

and

$$p^{n*}(x) = \frac{S^{(n-1)*}(x+1)}{\mu_{n-1:1}}, \text{ for } x \in \mathbb{N}_0, \quad (3.31)$$

being $\mu_{i:1} < \infty$ the first order moment of the $i - th$ order equilibrium distribution of X , with $\mu_{0:1} = \mu$, and $S^{(0)*}(x) = S(x)$.

Following the same procedure as in Section 2, let us define the random vector (X_1, \dots, X_n) with survival function

$$P(X_1 \geq x_1, \dots, X_n \geq x_n) = S^{(n-1)*}(x_1 + \dots + x_n). \quad (3.32)$$

Our next proposition is the parallel of Proposition 2.1 in the bivariate case. Now, we prove that the survival function (3.32) corresponds to a Schur-constant survival function.

Proposition 3.1 *The vector (X_1, \dots, X_n) with survival function (3.32) is Schur-constant and the marginal X_1 is the $(n - 1) - th$ order equilibrium distribution of X ((3.30) and (3.31)).*

Proof We know from Proposition 2.1 that it is true for $n = 2$, because $S^*(x)$ is 2-monotone. We assume now that it is true for $n - 1$, i.e., $S^{(n-2)*}$ is $(n - 1)$ -monotone. From (3.30),

$$(-1)^n \Delta^n S^{(n-1)*}(x) = (-1)^n \Delta^{(n-1)} \Delta S^{(n-1)*}(x) = (-1)^{(n-1)} \Delta^{(n-1)} \frac{S^{(n-2)*}(x+1)}{\mu_{n-2:1}}. \quad (3.33)$$

Since $S^{(n-2)*}$ is $(n - 1)$ -monotone, expression (3.33) is equal or greater than zero, which in turn implies that $S^{(n-1)*}$ is n -monotone. Putting $x_2 = \dots = x_n = 0$ in (3.32), $P(X_1 \geq x_1) = S^{(n-1)*}(x_1)$, and thus, the marginal distribution X_1 is the $(n - 1)$ -th order equilibrium distribution of X . \square

A Schur-constant vector of n components can be defined using a survival function not necessarily n -monotone, using the previous proposition; we say then that such vector follows a *Schur-constant multivariate equilibrium distribution model*. We can also extend the concept of a generator that has been previously defined for the bivariate models. A non-negative random variable X with $(n - 1)$ -th order equilibrium distribution (3.30) is the generator of a Schur-constant multivariate equilibrium distribution (X_1, \dots, X_n) if $P(X_1 \geq x_1, \dots, X_n \geq x_n) = S^{(n-1)*}(x_1 + \dots + x_n)$. By exchangeability, all of the X_i 's have the same ordinary moments, and the same Pearson correlation coefficient. In this model we are able to obtain all of them from the ordinary moments of the original distribution X , as long as X_i is the $(n - 1)$ -th order equilibrium distribution of X . Let us first establish the following lemma.

Lemma 3.1 *The set of discrete functions $F(x)$ such that $\Delta F(x) = x^{n-1}$ is given by $P_n(x) + C$, C being a constant and $P_n(x)$ a polynomial of degree n , such that $P_n(x) = \sum_{r=0}^n a_r(n)x^r$ with $a_0(n) = 0$ and $a_r(n)$, for $r = 1, \dots, n$, that can be computed recursively,*

$$\begin{aligned} a_n(n) &= \frac{1}{n}, \\ a_{n-h+1}(n) &= \frac{-1}{n-h+1} \sum_{r=n-h+2}^n \binom{r}{n-h} a_r(n), \quad h = 2, \dots, n. \end{aligned} \quad (3.34)$$

Proof Let us consider $F(x) = P_n(x) + C$ being $P_n(x) = \sum_{r=0}^n a_r(n)x^r$ with $a_0(n) = 0$, a polynomial of degree n on x . The difference of any of these functions is exactly the same and equals

$$\begin{aligned} \Delta P_n(x) &= \sum_{r=0}^n a_r(n) \Delta x^r = \sum_{r=0}^n a_r(n) \left(\sum_{s=0}^r \binom{r}{s} x^s - x^r \right) \\ &= \sum_{r=0}^{n-1} a_r(n) \sum_{s=0}^r \binom{r+1}{s} x^s = \sum_{s=0}^{n-1} x^s \sum_{r=s}^{n-1} \binom{r+1}{s} a_{r+1}(n). \end{aligned} \quad (3.35)$$

From (3.35), we observe that $\Delta P_n(x)$ is a new polynomial of degree $n - 1$ and coefficients b_s , that can be derived from the coefficients of the original polynomial, $a_r(n)$,

$$b_s = \sum_{r=s+1}^n \binom{r}{s} a_r(n), \quad s = 0, \dots, n - 1. \quad (3.36)$$

From (3.35) and (3.36) and imposing the condition that $\Delta P_n(x) = x^{n-1}$, that is $b_{n-1} = 1$, and $b_s = 0$ for $s = 0, \dots, n - 2$, a linear system of n equations on $a_r(n)$ is defined. The value of $a_n = \frac{1}{n}$ is obtained when $s = n - 1$. When $s = n - h$, the corresponding equation is

$$0 = \sum_{r=n-h+1}^n \binom{r}{n-h} a_r(n). \quad (3.37)$$

Isolating $a_{n-h+1}(n)$ from (3.37) the statement of the lemma is obtained. \square

Table 1 below gives the values of $a_r(n)$, for the first values of n .

Table 1 Some coefficients $a_r(n)$

$r \backslash n$	2	3	4	5	6	7	8	9	10
1	-1/2	1/6	0	-1/30	0	1/42	0	-1/30	0
2	1/2	-1/2	1/4	0	-1/12	0	1/12	0	-3/20
3		1/3	-1/2	1/3	0	-1/6	0	2/9	0
4			1/4	-1/2	5/12	0	-7/24	0	1/2
5				1/5	-1/2	1/2	0	-7/15	0
6					1/6	-1/2	7/12	0	-7/10
7						1/7	-1/2	2/3	0
8							1/8	-1/2	3/4
9								1/9	-1/2
10									1/10

Proposition 3.2 The j -th ordinary moment of the i -th order equilibrium distribution, $\mu_{i:j} = E[(X^{i*})^j]$, $j = 1, 2, \dots$, $i = 1, 2, \dots$, is given by the recurrence formula

$$\mu_{i:j} = \frac{1}{\mu_{i-1:1}} \sum_{r=1}^{j+1} \mu_{i-1:r} a_r(j+1), \quad (3.38)$$

with $a_r(\cdot)$ as in Lemma 3.1, and $\mu_{0:r} = E(X^r)$.

Proof By definition,

$$\mu_{i:j} = \frac{1}{\mu_{i-1:1}} \sum_{x=0}^{\infty} x^j S^{(i-1)*}(x+1). \quad (3.39)$$

From Lemma 3.1, $\Delta\left(\sum_{r=1}^{j+1} a_r(j+1)x^r\right) = x^j$, where the coefficients $a_r(\cdot)$ are given by expression (3.34). Therefore, the sum in (3.39) can be computed by applying summation by parts,

$$\sum_{x=0}^{\infty} p^{(i-1)*}(x) \sum_{r=1}^{j+1} a_r(j+1)x^r = \sum_{r=1}^{j+1} a_r(j+1) \sum_{x=0}^{\infty} p^{(i-1)*}(x)x^r. \quad (3.40)$$

□

In Li (2011) an equivalent expression relating the n -th factorial moments is found.

We focus now on the study of the Pearson correlation coefficient, ρ , in this Schur-constant multivariate equilibrium model. We see that ρ is a function of some ordinary moments of X .

Proposition 3.3 Consider the vector (X_1, \dots, X_n) with survival function (3.32). In terms of expectation and variance of the $(n-1)$ -th order equilibrium distribution of X , the Pearson correlation coefficient is given by

$$\rho = \frac{V(X^{(n-1)*}) - \left(E\left(X^{(n-1)*}\right)\right)^2 - E(X^{(n-1)*})}{2V(X^{(n-1)*})}. \quad (3.41)$$

Proof It is direct from Proposition 6.2. in Castañer et al. (2015). □

Thanks to (3.38), an equivalent expression of ρ as a function of the ordinary moments of X can be deduced. For instance, let us make explicit the formula when $n = 3$ and $n = 4$. Let (X_1, X_2, X_3) be a Schur-constant multivariate equilibrium distribution model defined by X . The Pearson correlation coefficient can be computed as

$$\rho = \frac{1}{2} - \frac{(\mu - E(X^3)) (2\mu - 3E(X^2) + E(X^3))}{2\mu^2 + 2(E(X^3))^2 + 3E(X^2)(E(X^2) - E(X^4)) + \mu(-3E(X^2) - 4E(X^3) + 3E(X^4))}.$$

If we consider now the Schur-constant multivariate equilibrium distribution model (X_1, X_2, X_3, X_4) ,

$$\rho = \frac{1}{2} - \frac{5(6\mu - 11E(X^2) + 6E(X^3) - E(X^4))(2\mu - E(X^2) - 2E(X^3) + E(X^4))}{2(D - 4\mu J)},$$

being

$$D = 36\mu^2 + 65(E(X^2))^2 + 20(E(X^3))^2 - 70E(X^2)E(X^4) + 5(E(X^4))^2 + 24E(X^2)E(X^5) + 8E(X^3)E(X^5),$$

and

$$J = 21E(X^2) + 8E(X^3) - 15E(X^4) + 4E(X^5).$$

As an example, Tables 2 and 3 include the values of the Pearson correlation coefficient in a Schur-constant multivariate equilibrium distribution model built from a Poisson distributed random variable of mean λ , as a function of n , the number of elements of the random vector.

Table 2 ρ in a Poisson Schur-constant multivariate equilibrium model (X_1, \dots, X_n)

n	2	3	4	5
ρ	$\frac{-\lambda}{6 + \lambda}$	$\frac{-\lambda}{12 + 2\lambda}$	$\frac{-\lambda}{20 + 3\lambda}$	$\frac{-\lambda}{30 + 4\lambda}$

Table 3 ρ in a Poisson Schur-constant multivariate equilibrium model (X_1, \dots, X_n) for different values of λ

$n \backslash \lambda$	0.01	0.5	1	5	10	100
2	-0.00166	-0.07692	-0.14286	-0.45455	-0.62500	-0.94340
3	-0.00083	-0.03846	-0.07143	-0.22727	-0.31250	-0.47170
4	-0.00050	-0.02326	-0.04348	-0.14286	-0.20000	-0.31250
5	-0.00033	-0.01563	-0.02941	-0.10000	-0.14286	-0.23256

One interesting property of these models is that the probability mass function of $Z = X_1 + \dots + X_n$ is easily calculated from the survival function of X . This is our next proposition.

Proposition 3.4 Consider the vector (X_1, \dots, X_n) with survival function (3.32). The p.m.f. of $Z = X_1 + \dots + X_n$ is

$$P(Z = z) = \frac{P(X = z + n - 1)}{\mu \cdot \mu_{1:1} \cdots \mu_{(n-2):1}} \binom{z + n - 1}{n - 1}. \tag{3.42}$$

Proof From Proposition 3.1, the survival function of X_1 is $S^{(n-1)*}(x)$, and from its definition (3.30),

$$\Delta^n S^{(n-1)*}(x) = (-1)^n \frac{P(X = x + n - 1)}{\mu \cdot \mu_{1:1} \cdots \mu_{(n-2):1}}. \tag{3.43}$$

Inserting (3.43) in (1.4) yields (3.42). □

4 Conclusions

Discrete Schur-constant models (of dimension n) can be constructed using univariate survival functions that are n -monotone. This paper introduces a class of Schur-constant models that are generated by a wider class of univariate survival functions, i.e., those survival functions such that their $(n - 1)$ -th equilibrium distribution exists.

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