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ABSTRACT

We present more applications of the recently introduced ω_1 -strongly compact cardinals in the context of either consistency or reflection results in General Topology, focusing on issues related to normality. In particular, we show that such large cardinal notion provides a new upper bound for the consistency strength of the statement “All normal Moore spaces are metrizable” (NMSC). The proof uses random forcing, as in the original consistency proof of NMSC due to Nykos-Kunen-Solovay (see Fleissner [10]). We establish a compactness theorem for normality (i.e., reflection of non-normality) in the realm of first countable spaces, using the least ω_1 -strongly compact cardinal, as well as two more similar compactness results on related topological properties. We finish the paper by combining the techniques of reflection and forcing to show that our new upper bound for the consistency strength of NMSC can be also obtained via Cohen forcing, using some arguments from Dow-Tall-Weiss [6].

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1. Introduction

In this work, we investigate applications of Set Theory in General Topology. For Set Theory, our standard references are [15] and [19]. The cardinality of a set x is denoted by $|x|$. The cardinality of the continuum, $|\mathbb{R}| = 2^{\aleph_0}$, is denoted by $\mathfrak{c}$. The class of all infinite cardinals is denoted by **CARD**. If $\kappa < \lambda$ are cardinals, $\mathcal{P}_\kappa(\lambda)$ denotes the set $\{x \subseteq \lambda : |x| < \kappa\}$.

[☆] Even though neither of us ever had the opportunity of meeting Kenneth Kunen in person, we would like to express our greatest admiration for his work in both set theory and topology, which contributed decisively to the emergence of set-theoretic topology as a distinct area of mathematics. We are very much indebted to him for his mathematical contributions, as well as for his unparalleled expository skills, as exemplified by his book on Set Theory, which allowed us and many others to enter the fascinating world of forcing.

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For General Topology, our standard reference is [8], where the reader may find the definitions of those topological notions we may have left undefined in this paper.

Throughout this work, we assume the reader is familiar with the usual terminology and machinery of large cardinals; for definitions and details, we refer to [18]. Only for fixing notations, if $\mathcal{U}$ is an ultrafilter over a set I then the ultrapower of $\langle V, \in \rangle$ modulo $\mathcal{U}$ is denoted by $\text{Ult}(V, \mathcal{U})$, with its corresponding elementary embedding $j_{\mathcal{U}} : V \rightarrow \text{Ult}(V, \mathcal{U})$ given by $j_{\mathcal{U}}(x) = [c_x]$, where c_x is the constant function on I of value x . If $\mathcal{U}$ is σ -complete, we denote by $\pi : \text{Ult}(V, \mathcal{U}) \rightarrow M$ the well-known Mostowski collapse (where M is the unique transitive inner model M isomorphic to $\text{Ult}(V, \mathcal{U})$), so that the composition given by $j = \pi \circ j_{\mathcal{U}}$ is an elementary embedding $j : V \rightarrow M$.

The least ordinal which is moved by a non-trivial elementary embedding j is said to be the *critical point* of j and denoted by $\text{cp}(j)$. It is well-known that the critical point of a non-trivial elementary embedding $j : V \rightarrow M$, with M transitive, is a *measurable cardinal* – and, furthermore, there is such a non-trivial elementary embedding if, and only if, there is a measurable cardinal. Recall that an uncountable cardinal κ is a measurable cardinal if there is a κ -complete free ultrafilter on κ .

An uncountable cardinal κ is said to be *strongly compact* if for every set I , every κ -complete filter on I can be extended to a κ -complete ultrafilter on I . It is easy to see that strongly compact cardinals are measurable.

Definition 1.1 ([2]). Let $\theta \leq \kappa$ be uncountable cardinals, which may be singular. κ is said to be *θ -strongly compact* if for every set I , every κ -complete filter on I can be extended to a θ -complete ultrafilter on I .

Notice that, for an uncountable cardinal κ , κ -strong compactness is the same as strong compactness, and because of this one may regard θ -strong compactness, for $\omega_1 \leq \theta < \kappa$, as a degree of strong compactness.

The notion of a θ -strongly compact cardinal was recently introduced by Magidor and the first author ([1,2]). It was established in [1] that, if κ_0 is the least ω_1 -strongly compact cardinal, then κ_0 cannot be a successor cardinal – however, it can be (consistently) singular, whereas its cofinality is at least the first measurable cardinal (in fact, a model of **ZFC** in which κ_0 is singular and its cofinality is precisely the first measurable cardinal is constructed in Section 6 of [2]). Moreover, it is shown in [1] that the Singular Cardinal Hypothesis holds above κ_0 .

It is easy to see that if κ is a θ -strongly compact cardinal then any cardinal $\eta > \kappa$ is θ -strongly compact as well. In particular, the statements “There is a ω_1 -strongly compact cardinal” and “There is a regular ω_1 -strongly compact cardinal” are equiconsistent.

In terms of elementary embeddings, the statement “ κ is θ -strongly compact” (where $\kappa > \theta$ are uncountable cardinals) is equivalent to the following statement (Theorem 4.7 of [2]):

“For every ordinal α greater than or equal to κ , there exists an elementary embedding $j : V \rightarrow M$, with M transitive and critical point $\text{cp}(j) \geq \theta$, such that j is definable in V and there is some $D \in M$ such that

$$j''\alpha := \{j(\beta) : \beta < \alpha\} \subseteq D \quad \text{and} \quad M \models |D| < j(\kappa).”$$

Equivalently, κ is θ -strongly compact if, and only if, for every set I of cardinality greater than or equal to κ there exists an elementary embedding $j : V \rightarrow M$, with M transitive and critical point $\text{cp}(j) \geq \theta$, such that j is definable in V and there is some $D \in M$ such that the pointwise image of I given by $E = j''I = \{j(x) : x \in I\}$ satisfies $E \subseteq D$, with $M \models |D| < j(\kappa)$. Notice that we may assume that $D \subseteq j(I)$ (otherwise we work with $D' = D \cap j(I)$, which includes E as well and also has cardinality less than $j(\kappa)$ in M).

A number of results of [1] were obtained in the context of General Topology. For instance, a topological characterization of ω_1 -strongly compact cardinals was given as follows: A cardinal κ is ω_1 -strongly compact

if, and only if, every product of Lindelöf spaces is κ -Lindelöf.¹ Moreover, after developing some framework for reflection arguments based on ω_1 -strongly compact cardinals, the following results were also established in the paper referred to:

- If κ is ω_1 -strongly compact, then for every first-countable non-metrizable topological space X there is a subspace $Y \subseteq X$ of cardinality less than κ which is also non-metrizable.
- If κ is ω_1 -strongly compact, then for every first-countable topological space X which is not completely regular there is a subspace $Y \subseteq X$ of cardinality less than κ which is also non-completely regular.

Notice that both results assert a *strong reflection* of the negation of a certain topological property.

Definition 1.2.² Let $\mathcal{P}$ be a property which applies to a certain class $\mathcal{C}$ of mathematical structures and let κ be an infinite cardinal. We say that:

- (i) κ is a **reflection cardinal** for $\mathcal{P}$ if every structure in $\mathcal{C}$ of cardinality κ and satisfying $\mathcal{P}$ has a substructure of cardinality less than κ which also satisfies $\mathcal{P}$.
- (ii) κ is a **strong reflection cardinal** for $\mathcal{P}$ if every structure in $\mathcal{C}$ satisfying $\mathcal{P}$ (no restriction on the cardinality) has a substructure of cardinality less than κ which also satisfies $\mathcal{P}$.
- (iii) κ is a **(weakly) compact cardinal** for $\mathcal{P}$ if a structure in $\mathcal{C}$ of cardinality κ satisfies $\mathcal{P}$ whenever all of its substructures of cardinality less than κ satisfy $\mathcal{P}$.
- (iv) κ is a **strongly compact cardinal** for $\mathcal{P}$ if a structure in $\mathcal{C}$ satisfies $\mathcal{P}$ (no restriction on the cardinality) whenever all of its substructures of cardinality less than κ satisfy $\mathcal{P}$.

Accordingly, a large cardinal notion ϕ is said to (strongly) reflect a certain property $\mathcal{P}$ if any cardinal satisfying ϕ is a (strong) reflection cardinal for $\mathcal{P}$. In these terms, the above mentioned results establish that ω_1 -strong compactness strongly reflects non-metrizability and non-complete regularity in the realm of first countable topological spaces.

A non- $\mathcal{P}$ reflection result is a compactness result for $\mathcal{P}$ – in fact, compactness and reflection are easily seen to be dual notions, in the sense that the compactness of a property is the reflection of the negation of the property and vice-versa. Thus, we may also say that an ω_1 -strongly compact cardinal is a strong compactness number for metrizable and complete regularity in the realm of first countable topological spaces.

In this note, we present new applications of ω_1 -strongly compact cardinals to General Topology. We use this large cardinal notion to establish both consistency and reflection/compactness results, focusing on normality and related topological properties. Let us describe the organization of the paper. In the first subsection of Section 2 we recall all topological notions around the Normal Moore Space Conjecture (NMSC) and we introduce a natural measure extension axiom, denoted by $\mathbf{PMEA}_\sigma$. Such axiom is a considerably weaker version of the so-called $\mathbf{PMEA}$ (the *Product Measure Extension Axiom*) – whose consistency follows from strongly compact cardinals (as proved by Kunen, using a result of Solovay) and implies the consistency of NMSC (as proved by Nyikos). Adapting Nyikos' previous arguments, we show that (the consistency of) $\mathbf{PMEA}_\sigma$ is enough to prove the consistency of NMSC – and then, in the second subsection of Section 2, we show that $\mathbf{PMEA}_\sigma$ holds in a forcing model obtained by adding ω_1 -strongly compact many random reals. In Section 3, we apply some machinery from [1] in order to establish that, if we restrict ourselves to first countable topological spaces, then the least ω_1 -strongly compact cardinal κ_0 is

¹ For this result, the notion of κ -Lindelöfness is defined in the following way: a topological space is said to be κ -Lindelöf if every open cover of X has a subcover of size less than κ . Sometimes such property is also defined as κ -compactness.

² After M. Magidor's *Patterns of reflection properties*. Talk presented at the Conference *Reflections on Set Theoretic Reflection* - In celebration of Joan Bagaria's 60th Birthday, Montseny, Catalonia, November 2018.

a strong compactness number for both normality and countable paracompactness, and we prove a similar compactness result for a certain subclass of the non- $< \delta$ -collectionwise Hausdorff spaces, using a δ -strongly compact cardinal (for an uncountable δ). In Section 4, we combine the techniques of reflection and forcing to show that our new upper bound for the consistency strength of **NMSC** can be also obtained via Cohen forcing (and with no use of measure theory), using some results from [6].

2. ω_1 -strongly compact cardinals, **PMEA** $_{\sigma}$ and the Normal Moore Space Problem

2.1. A new upper bound for the consistency of **NMSC**

In what follows, **NMSC** denotes the *Normal Moore Space Conjecture*, which asserts that all normal Moore spaces are metrizable. Recall that a *Moore space* is a regular space which is developable, meaning that it has a *development*. A development for a topological space X is a sequence of open covers $\{\mathcal{U}_n : n < \omega\}$ such that, for every $x \in X$, $\{\text{St}(x, \mathcal{U}_n) : n < \omega\}$ is a local base at x , where $\text{St}(x, \mathcal{U}) := \bigcup \{U \in \mathcal{U} : x \in U\}$. Notice that Moore spaces are necessarily first countable.

The Normal Moore Space Problem – that is, to determine whether **NMSC** is true or not – was one of the more influential and longstanding problems of General Topology in the 20th Century, maybe the most celebrated of them all.

PMEA stands for the *Product Measure Extension Axiom*, which declares that for every set X there is a $\mathfrak{c}$ -additive *full measure* on $\mathcal{P}(2^X)$ (meaning that the measure is defined on every subset of 2^X) which extends the usual product measure. Of course, it suffices to ensure the existence of some extending full measure on 2^κ for arbitrarily large κ . So, we may assume that **PMEA** states as follows:

“For every cardinal κ , the product measure on 2^κ extends to a $\mathfrak{c}$ -additive full measure.”

One could also consider parametrized versions of measure extension axioms.

Definition 2.1 ([10]). Let κ, λ, ρ denote infinite cardinals. **PMEA** (κ, λ, ρ) denotes the statement:

“For any family $\mathcal{S}$ of λ many subsets of 2^κ , the product measure on 2^κ can be extended to a ρ -additive measure, $m_{\mathcal{S}}$, with $\mathcal{S} \subseteq \text{dom}(m_{\mathcal{S}})$.”

With this notation,

$$\mathbf{PMEA} \equiv \forall \kappa \in \mathbf{CARD} [\mathbf{PMEA}(\kappa, 2^{2^\kappa}, \mathfrak{c})].$$

Next we include some topological terminology (see [10] and [22]).

Definition 2.2. Let $\mathcal{A} = \{A_i : i \in I\}$ be a family of disjoint subsets of a topological space X .

(i) We say that $\mathcal{A}$ is **normalized** if for every $J \subseteq I$ there are disjoint open sets U and V such that $\bigcup \{A_i : i \in J\} \subseteq U$ and $\bigcup \{A_i : i \notin J\} \subseteq V$.

Equivalently, $\mathcal{A}$ is normalized if for every $J \subseteq I$ there is an open set U_J such that $\bigcup \{A_i : i \in J\} \subseteq U_J$ and $\bigcup \{A_i : i \notin J\} \subseteq X \setminus \overline{U_J}$. In this case we say that the open family $\{U_J : J \subseteq I\}$ **normalizes** $\mathcal{A}$, or that $\mathcal{A}$ is **normalized** by $\{U_J : J \subseteq I\}$.

(ii) If $\mathcal{W} = \{W_i : i \in I\}$ is a family of pairwise disjoint open sets satisfying $A_i \subseteq W_i$ for all $i \in I$ – or, more precisely, $W_i \cap \bigcup \mathcal{A} = A_i$ for all $i \in I$ –, then the family $\mathcal{A}$ is said to be **separated** (by $\mathcal{W}$). We also say that $\mathcal{W}$ is a **disjoint open expansion** of $\mathcal{A}$. Clearly, separated families are normalized.

A topological space is normal if, and only if, all discrete collections of closed sets are normalized. If we require, furthermore, that all discrete collections of closed sets are in fact separated, we get to *collectionwise normality*.

Definition 2.3. A topological space X is **collectionwise normal** if all discrete collections of closed sets have disjoint open expansions.

An equivalent notion is defined if we require that all discrete collections of non-empty sets are separated.

Bing ([3]) showed that, for Moore spaces, metrizability is equivalent to collectionwise normality. So, to prove **NMSC** it suffices to prove that normalized families in Moore spaces are separated.

Nyikos showed that **PMEA** implies **NMSC** by proving the following theorem. Recall that the *local character* of a point $x \in X$ is the smallest cardinality of a local base of such point, and it is denoted by $\chi(x, X)$.

Theorem 2.4 ([21]). *Assuming **PMEA**, normalized families are separated in any topological space X in which every point has local character strictly smaller than $\mathfrak{c}$.*

This result was initially tagged as a “provisional solution” for the Normal Moore Space Problem because **PMEA** is a statement with large cardinal strength – it was already known that the consistency of **PMEA** follows from the consistency of the existence of strongly compact cardinals (an unpublished result due to Kunen), and it is easy to see that the consistency of **PMEA** implies the consistency of the existence of measurable cardinals. However, Fleissner ([9]) showed in 1982 that the consistency of **NMSC** also implies the consistency of the existence of measurable cardinals. So, there is no way of discussing **NMSC** without dealing with large cardinals. The reader may find more information on **PMEA** (as well as references for all results mentioned in this paragraph) in Fleissner’s survey paper on **NMSC** and large cardinals ([10]). At the end of [10], Fleissner says:

“In my opinion, the ‘cleanest’ resolution still possible is for $ZFC + NMSC$ to be equiconsistent with $ZFC + \exists$ strongly compact cardinal. It seems that this result will have to wait for the development of the theory of inner models with strongly compact cardinals.”

In this work we dismiss this possibility by giving a lower large cardinal upper bound for the consistency strength of **NMSC**, namely the existence of an ω_1 -strongly compact cardinal. Notice that in the model of [2] already mentioned in the Introduction – the one where the first ω_1 -strongly compact cardinal κ_0 is singular and has its cofinality given, precisely, by the first measurable cardinal –, one has that the first measurable cardinal is strictly smaller than the first ω_1 -strongly compact cardinal, which is also strictly smaller than the first strongly compact cardinal. So, the consistency strength of the existence of an ω_1 -strongly compact cardinal is strictly stronger than the consistency strength of a measurable, and strictly weaker than the consistency strength of a strongly compact cardinal. However, the question remains open as to whether (modulo ZFC) the consistency of **NMSC** implies the consistency of the existence of an ω_1 -strongly compact cardinal.

The main result of this section is the following:

Theorem 2.5. *If ZFC plus the existence of an ω_1 -strongly compact cardinal is consistent, then so is ZFC plus “All normal Moore spaces are metrizable”.*

Our proof will follow the lines of Kunen’s proof of the consistency of **PMEA** from strongly compact cardinals – but we will use an ω_1 -strongly compact cardinal to establish the consistency of the following measure extension axiom, which is weaker than **PMEA** but strong enough to yield **NMSC**.

Definition 2.6. Let $\mathbf{PMEA}_\sigma$ denote the following statement:

“For every cardinal κ , the product measure on 2^κ extends to a σ -additive full measure.”

Using parameters as in Definition 2.1, one has

$$\mathbf{PMEA}_\sigma \equiv \forall \kappa \in \mathbf{CARD} [\mathbf{PMEA}(\kappa, 2^{2^\kappa}, \aleph_1)].$$

To see that $\mathbf{PMEA}_\sigma$ implies $\mathbf{NMSC}$, one has just to verify the validity of the following theorem, whose proof is a straightforward modification (*mutatis mutandis*) of the original proof of Theorem 2.4 in [21] (and thus will be omitted).

Theorem 2.7. *The measure extension axiom $\mathbf{PMEA}_\sigma$ implies that normalized families are separated in all first countable spaces. In particular, all first countable normal spaces are collectionwise normal under $\mathbf{PMEA}_\sigma$ – and, consequently, $\mathbf{PMEA}_\sigma$ implies that all normal Moore spaces are metrizable.*

2.2. The consistency of $\mathbf{PMEA}_\sigma$ from an ω_1 -strongly compact cardinal

We begin with a technical lemma. In what follows, we work with μ -complete ultrafilters, where μ is the first measurable cardinal. This is justified by the following observation, which is easily verified (see page 267 of [1], or page 1865 of [2]): *if κ is an ω_1 -strongly compact cardinal, then it is, in fact, μ -strongly compact.*

Lemma 2.8. *Let κ be an ω_1 -strongly compact cardinal. For every cardinal λ such that $cf(\lambda) > \kappa$, there are a set I , a μ -complete ultrafilter $\mathcal{U}$ on I (where μ is the first measurable cardinal), and a family of functions $\{f_\alpha : \alpha < \lambda\}$, such that*

- (i) $\{f_\alpha : \alpha < \lambda\} \subseteq {}^I \kappa$.
- (ii) If $\alpha < \beta < \lambda$, then $\{i \in I : f_\alpha(i) < f_\beta(i)\} \in \mathcal{U}$.
- (iii) For all $\alpha < \lambda$ and $\delta < \kappa$, $\{i \in I : f_\alpha(i) \neq \delta\} \in \mathcal{U}$.

Proof. κ is also a μ -strongly compact cardinal, as we have just mentioned, and we have $\lambda > \kappa$; so, as in the proof of (1) $\Rightarrow$ (2) of Theorem 4.7 of [2] we may define an μ -complete ultrafilter $\mathcal{U}$ on $\mathcal{P}_\kappa(\lambda)$ which is also a *fine measure*, meaning that $X_\alpha := \{x \in \mathcal{P}_\kappa(\lambda) : \alpha \in x\}$ belongs to $\mathcal{U}$, for all $\alpha < \lambda$. With $\mathcal{U}$ we may now define, via the usual ultrapower followed by the Mostowski collapse procedure, an elementary embedding $j : V \rightarrow M$, with M transitive, critical point greater than or equal to μ , and such that there exists a set $D \in M$ satisfying

$$j''\lambda := \{j(\alpha) : \alpha < \lambda\} \subseteq D \text{ and } M \models |D| < j(\kappa).$$

Notice that, necessarily, one has $j(\kappa) > \lambda$ (since, in M , $j(\kappa) > |D|$ and D includes a copy of λ in V).

Take $I = \mathcal{P}_\kappa(\lambda)$. We will use the ultrapower construction to exhibit a family of functions as desired. A key feature we will exploit is the easily verifiable fact that ordinals below $j(\kappa)$ may be represented, in the ultrapower $\text{Ult}(V, \mathcal{U})$, by functions with domain I and codomain κ .

We work in V . Recall that j is definable in V . Let

$$z = \{\beta < \kappa : j(\beta) < \lambda\}.$$

As $|z| \leq \kappa < cf(\lambda)$, $j''z = (j''\kappa) \cap \lambda$ is a bounded subset of λ .

Let $\xi = \sup(j''\kappa \cap \lambda) + 1$. Notice that the interval of ordinals $[\xi, \lambda[$ has size λ and is disjoint from $j''\kappa$.

Using that $\xi + \lambda = \lambda$ in the ordinal arithmetic, we may fix, for every ordinal $\alpha < \lambda$, a function $f_\alpha \in {}^I \kappa$ such that, in M ,

$$\pi([f_\alpha]) = \xi + \alpha < \lambda < j''\kappa$$

where π is the Mostowski collapse isomorphism $\pi : \text{Ult}(V, \mathcal{U}) \rightarrow M$.

Clause (i) of the lemma is already satisfied. Notice that if $\alpha < \beta < \lambda$ then $\xi + \alpha < \xi + \beta$ and so $[f_\alpha] < [f_\beta]$ in the ultrapower $\text{Ult}(V, \mathcal{U})$; this gives us, precisely, clause (ii). To prove (iii), suppose towards a contradiction that there are $\alpha < \lambda$ and $\delta < \kappa$ such that $\{i \in I : f_\alpha(i) = \delta\} \in \mathcal{U}$. It follows that $[f_\alpha] = [c_\delta]$ in the ultrapower $\text{Ult}(V, \mathcal{U})$, where c_δ is the function on I with constant value δ . So,

$$\pi([f_\alpha]) = \pi([c_\delta]) = j(\delta) \in j''\kappa$$

which is an absurd since by construction we have $\pi([f_\alpha]) = \xi + \alpha \in [\xi, \lambda[$ and we have already observed that such interval is disjoint from $j''\kappa$. $\square$

The above functions will be used to pull the *Solovay measure* on a certain $\mathcal{P}(I)$ back to 2^λ , where λ is an arbitrary cardinal whose cofinality is greater than some fixed ω_1 -strongly compact cardinal.

When it comes to the construction of a forcing extension (via random forcing) in which $\mathbf{PMEA}_\sigma$ holds, our exposition will follow closely the outline of Fleissner's paper in the *Handbook of Set Theoretic Topology* [10], where the original forcing arguments – due to Kunen – for establishing

$$\text{Con}(\mathbf{ZFC} + \text{There is a strongly compact cardinal}) \Rightarrow \text{Con}(\mathbf{ZFC} + \mathbf{PMEA})$$

were first displayed in print (since Kunen has never published them). Indeed, we get the desired consistency result – i.e., the consistency of $\mathbf{PMEA}_\sigma$ – by proving (with due care) the ω_1 -strongly compact version of the corresponding result on strongly compact cardinals (Theorem 3.4 of [10]).

Let us recall some terminologies and notation for random forcing. If κ is an infinite cardinal, then $\langle B_\kappa^*, \subseteq \rangle$ is the forcing notion for *adding κ -many random reals*, where B_κ is the family of all Borel subsets of 2^κ and B_κ^* comprehends those with positive (product) measure. Given any $p \in \text{Fn}(\kappa, 2)$, let $[p] = \{f \in 2^\kappa : p \subseteq f\}$. Recall that each $b \in B_\kappa$ is a *cylinder* over some countable subset $\text{supp}(b)$ of κ (i.e., $\text{supp}(b) \in [\kappa]^{\leq \aleph_0}$). This means that there is some $b' \subseteq 2^{\text{supp}(b)}$ with the following property: for all $f \in 2^\kappa$,

$$f \in b \iff f \upharpoonright \text{supp}(b) \in b'.$$

In certain convenient cases, a function $f \in 2^\kappa$ will be written as a κ -sequence $f = \langle f(\alpha) : \alpha < \kappa \rangle$.

Given a statement σ of the forcing language of $B^* = B_\kappa^*$, we use the notation $\llbracket \sigma \rrbracket$ to denote a fixed element b of B^* which forces σ and has maximal value of measure. If σ does not hold in any extension then no condition forces σ and we may pick $\llbracket \sigma \rrbracket$ to be a null set. This procedure resembles the machinery (and the notation) of forcing with Boolean values – but notice that we are forcing with posets, not with Boolean algebras.

Finally, recall that, given an ultrafilter $\mathcal{U}$ on a set I , the assertion “ $\mathcal{U}$ is κ -complete” is equivalent to the assertion “the non-trivial, two-valued measure given by the characteristic function $\chi_{\mathcal{U}} : \mathcal{P}(I) \rightarrow \{0, 1\}$ is κ -additive”.

The following theorem is, essentially, Theorem 3.3 of Fleissner's paper [10], which is, in turn, a version of Theorem 7 of Solovay's paper [26] (where it is used to obtain, consistently, a real valued measurable cardinal from a measurable cardinal). For more details, we refer to Lemma 22.9 of [15] and Theorem 17.5 of [18]. We have stated this theorem in its more general form, namely for any cardinal κ greater than the continuum. Nevertheless, in the present paper, it will be only applied (combined with Lemma 2.8) for $\kappa = \mu$, where μ is the first measurable cardinal.

Theorem 2.9. *Let $\kappa > \mathfrak{c}$ be a cardinal, and let $\mathcal{U}$ be a κ -complete free ultrafilter over a set I . Let B^* be the forcing notion for adding any number of random reals. If $G \subseteq B^*$ is a generic filter over V , then in the generic extension $V[G]$ there is a κ -additive real measure ν (the so-called Solovay measure) which is defined for every subset of I in $V[G]$ (that is, $\nu : \mathcal{P}(I)^{V[G]} \rightarrow [0, 1]$) and extends the measure $\chi_{\mathcal{U}}$ (that is, $\nu \upharpoonright \mathcal{P}(I)^V = \chi_{\mathcal{U}}$).*

Proof. (Sketch) Let us only outline the proof (since the complete arguments are easily adaptable from the mentioned references) and exhibit the construction of the measure ν . We use “ $\Vdash \sigma$ ” to mean that every condition forces σ .

Let $b \in B^*$ and $\dot{A}$ be a name for a subset of I (i.e., $\Vdash \dot{A} \subseteq \check{I}$).

As $\kappa > \mathfrak{c}$ and $\mathcal{U}$ is κ -complete, we may let $\nu_b(\dot{A})$ be the unique $r \in [0, 1]$ such that

$$\left\{ i \in I : \frac{m(\llbracket \check{i} \in \dot{A} \rrbracket \cap b)}{m(b)} = r \right\} \in \mathcal{U}$$

It is easy to check that:

1. $\nu_b(\check{I}) = 1$ and $\nu_b(\check{\emptyset}) = 0$.
More generally, if in V we have $Z \in \mathcal{U}$ then $\nu_b(\check{Z}) = 1$ and if $Z \notin \mathcal{U}$ then $\nu_b(\check{Z}) = 0$.
2. If $\dot{A}$ and $\dot{B}$ are names for subsets of I such that $b \Vdash \dot{A} \subseteq \dot{B}$, then one has $\nu_b(\dot{A}) \leq \nu_b(\dot{B})$.
It follows that if $b \Vdash \dot{A} = \dot{B}$, then one has $\nu_b(\dot{A}) = \nu_b(\dot{B})$.
3. If $\beta < \kappa$ and $\dot{A}$ and $\dot{A}_\alpha$, for $\alpha < \beta$, are names such that

$$b \Vdash \dot{A} = \{\dot{A}_\alpha : \alpha < \beta\} \text{ is a pairwise disjoint family of subsets of } I$$

$$\text{then } \nu_b(\bigcup \dot{A}) = \sum_{\alpha < \beta} \nu_b(\dot{A}_\alpha).$$

Now, consider $G \subseteq B^*$ a generic filter and let A be a subset of I in $V[G]$. Let $\dot{A}$ be a name for A . We define the Solovay measure ν by putting

$$\nu(A) = \lim_{b \in G} \nu_b(\dot{A}).$$

One can check (using the Radon-Nikodym Theorem) that ν is well-defined in $V[G]$ – that is, the limit exists and it does not depend on the choice of a particular name for a given subset of I – and that ν is a κ -additive real measure, which is defined on every subset of I and extends the measure $\chi_{\mathcal{U}}$, as desired. $\square$

We may now complete the proof of the main theorem of this section (Theorem 2.5). What remains to be done will be settled by the following theorem:

Theorem 2.10. *Suppose κ is ω_1 -strongly compact cardinal in V and let $V[G]$ be a forcing extension obtained after adding κ -many random reals over V . Then $\mathbf{PMEA}_\sigma$ holds in $V[G]$.*

Proof. We follow the argument given in Theorem 3.4 of [10], but we give more details since there are some small, subtle modifications in the proof. Nevertheless, the strategy is the same, namely use the functions of Lemma 2.8 to pull the Solovay measure on $\mathcal{P}(I)$, for some I , back to 2^λ – where λ is arbitrarily large.

Let $B^* = B_\kappa^*$ and let $G \subseteq B^*$ be a generic filter. We want to check that, in $V[G]$, for every set X there is a σ -additive measure which is defined on all subsets of 2^X and extends the product measure. Of course, it

suffices to check the existence of such a measure for sets X of arbitrarily large cardinality, and in particular we may assume, without loss of generality, that $X = \lambda \geq cf(\lambda) > \kappa$.

As κ is an ω_1 -strongly compact cardinal and $cf(\lambda) > \kappa$, we know that there are a set I , a μ -complete ultrafilter $\mathcal{U}$ on I (where μ is the first measurable cardinal), and a family of functions $\{f_\alpha : \alpha < \lambda\} \subseteq {}^I\kappa$ satisfying clauses (ii) and (iii) of Lemma 2.8 – that is, if $\alpha < \beta < \lambda$ then $f_\alpha \neq f_\beta \pmod{\mathcal{U}}$ and, moreover, for any $\delta \in \kappa$ the set of i 's such that $f_\alpha(i) \neq \delta$ is $\mathcal{U}$ -large.

Let $g = \bigcap G$ be the generic function from κ into 2 added by the forcing (so that $V[G] = V[g]$). In $V[g]$, we define, for every $\alpha < \lambda$, a function $\varphi_\alpha : I \rightarrow 2$ such that, for every $i \in I$,

$$\varphi_\alpha(i) = 1 \iff g(f_\alpha(i)) = 1.$$

Thus, the elements of the subset of I coded by φ_α are precisely the i 's such that $f_\alpha(i)$ is an element of the generic subset of κ added by the forcing. Notice that $\varphi_\alpha = g \circ f_\alpha$ for every $\alpha < \lambda$.

Next, we define $\phi : I \rightarrow 2^\lambda$ such that, for every $i \in I$ and $\alpha < \lambda$,

$$\phi(i)(\alpha) = \varphi_\alpha(i).$$

Let ν be the Solovay measure defined on all subsets of I and constructed in $V[g]$ from I and $\mathcal{U}$ as in Theorem 2.9. ν is μ -additive, and as μ is still uncountable in the extension $V[G]$ (since random forcing is c.c.c. and therefore preserves cardinals), ν is σ -additive.

Let $\gamma : \mathcal{P}(2^\lambda) \rightarrow [0, 1]$ be given by $\gamma = \nu \circ (\phi^{-1})''$, that is, for every $Y \subseteq 2^\lambda$ we have

$$\begin{aligned} \gamma(Y) &= \nu((\phi^{-1})''Y) \\ &= \nu(\{i \in I : \phi(i) \in Y\}) \\ &= \nu(\{i \in I : \langle \phi(i)(\alpha) : \alpha < \lambda \rangle \in Y\}) \\ &= \nu(\{i \in I : \langle \varphi_\alpha(i) : \alpha < \lambda \rangle \in Y\}) \\ &= \nu(\{i \in I : \langle g(f_\alpha(i)) : \alpha < \lambda \rangle \in Y\}) \end{aligned}$$

As pre-images preserve all basic set theoretic properties, γ is a σ -additive measure, defined on all subsets of 2^λ ; it remains to verify that γ extends the product measure on 2^λ .

Let $p \in Fn(\lambda, 2)$ and let $Y = [p]$ be the corresponding basic clopen set, with $|p| = n \geq 1$ and $\text{dom}(p) = \{\alpha_i : 1 \leq i \leq n\}$. We have to show that $\gamma([p]) = 2^{-n}$. By definition, $\gamma([p]) = \nu(D)$, where

$$\begin{aligned} D &= \{i \in I : \langle g(f_\alpha(i)) : \alpha < \lambda \rangle \in [p]\} \\ &= \{i \in I : g(f_{\alpha_1}(i)) = p(\alpha_1), \dots, g(f_{\alpha_n}(i)) = p(\alpha_n)\} \end{aligned}$$

In the forcing language, given $i \in I$ and a name $\dot{D}$ for D , the formula “ $\check{i} \in \dot{D}$ ” is equivalent to

$$“\dot{g}(\check{f}_{\alpha_1}(i)) = \check{p}(\alpha_1) \wedge \dots \wedge \dot{g}(\check{f}_{\alpha_n}(i)) = \check{p}(\alpha_n)”$$

where $\dot{g}$ is a name for the generic function g .

By the clause (ii) of Lemma 2.8, we may pick $J \in \mathcal{U}$ such that, for any $i \in J$, $f_{\alpha_1}(i), \dots, f_{\alpha_n}(i)$ are pairwise distinct. So, for $i \in J$, it is easy to see that $\llbracket \check{i} \in \dot{D} \rrbracket = [q_i]$, where $q_i \in Fn(\kappa, 2)$ is given by

$$q_i = \{\langle f_{\alpha_1}(i), p(\alpha_1) \rangle, \dots, \langle f_{\alpha_n}(i), p(\alpha_n) \rangle\}$$

It follows that, considering the forcing measure algebra $m = m_\kappa$, we have

$$m(\llbracket \check{i} \in \dot{D} \rrbracket) = 2^{-n} \text{ for every } i \in J.$$

Now, let $b \in B^*$ be an arbitrary condition. We know that $\text{supp}(b)$ is a countable set. Let

$$H = \{i \in J : \{f_{\alpha_1}(i), \dots, f_{\alpha_n}(i)\} \cap \text{supp}(b) = \emptyset\}.$$

As $\mathcal{U}$ is μ -complete, clause (iii) of 2.8 ensures that $H \in \mathcal{U}$; this follows from the observation that, in fact,

$$H = J \cap \left(\bigcap_{\delta \in \text{supp}(b)} \{i \in I : f_{\alpha_1}(i) \neq \delta\} \right) \cap \dots \cap \left(\bigcap_{\delta \in \text{supp}(b)} \{i \in I : f_{\alpha_n}(i) \neq \delta\} \right).$$

If $i \in H$, the conditions $\llbracket \check{i} \in \dot{D} \rrbracket = [q_i]$ and b are, borrowing the probabilistic terminology, *independent sets* – since the event “ $h \in b$ ”, for any given $h \in 2^\kappa$, does not depend on “ $h \in [q_i]$ ” since the domain of q_i is disjoint from $\text{supp}(b)$. So, for any $i \in H$ we have that $m(\llbracket \check{i} \in \dot{D} \rrbracket \cap b) = 2^{-n} \cdot m(b)$ – and this ensures that H is a subset of

$$\left\{ i \in I : \frac{m(\llbracket \check{i} \in \dot{D} \rrbracket \cap b)}{m(b)} = 2^{-n} \right\}$$

and so the displayed set is also an element of $\mathcal{U}$.

By definition, it follows that for any $b \in B^*$ one has $\nu_b(\dot{D}) = 2^{-n}$, and therefore $\gamma([p]) = \nu(D) = 2^{-n}$, as desired. $\square$

Together with Theorem 2.7, the preceding theorem allows us to conclude that **NMSC** holds in any model obtained by adding κ many random reals, where κ is an ω_1 -strongly compact cardinal. This finishes the proof of Theorem 2.5.

Let N be the model from [2] in which the first ω_1 -strongly compact cardinal $\kappa = \kappa_0$ is singular, with cofinality equal to the first measurable cardinal μ . If λ is the least strongly inaccessible cardinal greater than κ , then V_λ is a model of ZFC in which the cardinal κ is ω_1 -strongly compact and there are no strongly compact cardinals. By adding κ -many random reals over V_λ we then obtain a forcing extension $V_\lambda[H]$ in which there are no strongly compact cardinals, μ is an uncountable cardinal and such that

$$V_\lambda[H] \models [\mu < \mathfrak{c}] \wedge \mathbf{PMEA}_\mu$$

– where, as expected, $\mathbf{PMEA}_\mu$ denotes the measure extension axiom

$$\mathbf{PMEA}_\mu \equiv \forall \kappa \in \mathbf{CARD} [\mathbf{PMEA}(\kappa, 2^{2^\kappa}, \mu)].$$

3. ω_1 -strongly compact cardinals and reflection

In [1], Magidor and the first author used second-order reflection over relational structures, provided by ω_1 -strongly compact cardinals, to give several applications to General Topology. Among other results, they showed that, for the class of all first countable topological spaces, the first ω_1 -strongly compact cardinal is a strong compactness cardinal (in the sense defined in the Introduction) for the properties of metrizability (Theorem 6.4, [1]) and complete regularity (Theorem 6.6, [1]). These results were obtained by applying the following theorem, which characterizes ω_1 -strongly compact cardinals in terms of reflection. Given a formula $\varphi(\vec{Y}, \vec{y})$, where $\vec{Y}$ is a countable sequence $\langle Y_i^{k_i} \rangle$ of relational variables of second order, with Y_i having arity $k_i < \omega$ for all $i \in I$, and $\vec{y}$ is a countable sequence of variables of first order, we say that κ **reflects** $\varphi(\vec{Y}, \vec{y})$ if for every structure $\mathcal{A}$ (with universe A), every sequence $\vec{R} = \langle R_i^{k_i} \rangle$ with $R_i^{k_i} \subseteq A^{k_i}$ and every sequence

$\vec{a} = \langle a_j \rangle_j$ of the same length as $\vec{y}$ of elements of A , the following holds: whenever $\mathcal{A} \models \varphi(\vec{R}, \vec{a})$, there exists a substructure $\mathcal{B} \subseteq \mathcal{A}$ of cardinality less than κ with $\{a_j\}_j \subseteq B$ and such that $\mathcal{B} \models \varphi(\langle R_i^{k_i} \cap B^{k_i} \rangle, \vec{a})$, where B is the universe of $\mathcal{B}$.

Theorem 3.1. *[[1], Theorem 6.1] Let Φ be the class of formulas of the form $\forall \vec{X} \exists \vec{x} \psi$, where $\vec{X}$ is a countable sequence $\langle X_i^{k_i} \rangle$ of relational variables of second order, with X_i having arity $k_i < \omega$ for all $i \in I$, $\vec{x}$ is a countable sequence of variables of first order, and ψ is a formula of the infinitary language $\mathcal{L}_{\omega_1, \omega}$ with only relation and constant symbols, without quantifiers, and which may have countably many free variables of first order and second order relational variables of any given arity. Then, a cardinal κ reflects every formula in the class Φ if, and only if, κ is ω_1 -strongly compact.*

The proofs of the reflection results mentioned above regarding non-metrizability and non-complete regularity, for first countable spaces, obey the same pattern: given a first countable topological space X which does not satisfy a certain property $\mathcal{P}$, a certain relational structure is considered, so that a formula φ which declares that “ X does not satisfy $\mathcal{P}$ ” is shown to be (equivalent to) a formula of type Φ , and, consequently, the reflection Theorem 3.1 applies.

The topology τ of a topological space X is an element of $\mathcal{P}(\mathcal{P}(X))$, so at first glance one should need third-order formulas in order to express topological properties. For first countable spaces, however, there is a trick which allows to capture the topology of the space via a countable sequence of second order relational symbols. More precisely: for X first countable, for every $x \in X$ we fix $\{U_n^x : n < \omega\}$ a countable local base at x and then we define a sequence $\langle R_n \rangle_{n < \omega}$ of binary relations on X in the following way: for each $x, y \in X$,

$$R_n(x, y) \iff y \in U_n^x.$$

Using this sequence of relations, we are able to express a number of topological statements. In what follows, we present formulas for a number of basic topological notions we will use in this paper. These formulas will be used as “blocks” in the construction of some more complicated formulas. Whenever a topological notion is expressed by a formula which belongs to the described class Φ , we will be able to apply the previous reflection characterization theorem.

If A is a second-order variable, or a second-order unary relational symbol, then

- $\mathring{A}(x)$ abbreviates the formula

$$\bigvee_{n < \omega} \forall y (R_n(x, y) \rightarrow A(y)).$$

The intended meaning of $\mathring{A}(x)$ is “ x is an interior point of A ”.

- $\overline{A}(x)$ abbreviates the formula

$$\neg \left(\bigvee_{n < \omega} \forall y (R_n(x, y) \rightarrow \neg A(y)) \right)$$

- or, equivalently,

$$\bigwedge_{n < \omega} \exists y (R_n(x, y) \wedge \neg A(y)).$$

The intended meaning of $\overline{A}(x)$ is “ x is not a interior point of $X \setminus A$ ” – which is equivalent to “ x is an adherent point of A ”, or “ x is in the closure of A ”.

- $\text{Open}(A)$ abbreviates the formula

$$\forall x(A(x) \rightarrow \overset{\circ}{A}(x)),$$

whose intended meaning is “all points of A are interior points” – which is equivalent to “ A is an open set”.

- $\text{Closed}(A)$ abbreviates the formula

$$\forall x(\overline{A}(x) \rightarrow A(x)),$$

whose intended meaning is “ A contains all of its adherent points” – which is equivalent to “ A is a closed set”.

In the following three subsections, we present some reflection/compactness results. In particular, we will show that κ_0 , the first ω_1 -strongly compact cardinal, is a strong compactness number for both normality and countable paracompactness. We also prove a similar compactness result for a certain subclass of the non- $< \delta$ -collectionwise Hausdorff spaces, using a δ -strongly compact cardinal (for an uncountable δ). In due course we will explain the reasons why countable paracompactness and $\aleph_0$ -collectionwise Hausdorffness (the simplest case of $< \delta$ -collectionwise Hausdorffness) are usually regarded as topological properties which relate to normality.

3.1. Reflection of non-normality in the realm of first countable spaces

We shall prove the following theorem:

Theorem 3.2. *If κ_0 is the first ω_1 -strongly compact cardinal, then for every first-countable non-normal topological space X there is a subspace $Y \subseteq X$ of cardinality less than κ_0 which is also non-normal.*

Proof. As already explained, it suffices to show that the formula which declares that “ X is not normal” (for a given first countable topological space X) is (equivalent to a formula which is) in the class Φ of Theorem 3.1.

If a first countable topological space X is not normal, there are closed sets F and G which cannot be separated by open sets. We add these “counterexample sets” as symbols of our language – more precisely, we consider the structure

$$\mathcal{X} = \langle X, \langle R_n \rangle_{n < \omega}, F, G \rangle$$

– where the binary relations R_n are defined using fixed countable local bases for each point, as described in the previous subsection.

So, let φ be the following formula, which declares that X is not normal:

$$\begin{aligned} \forall U \forall V [& (\neg \text{Open}(U)) \vee (\neg \text{Open}(V)) \vee (\neg(F \subseteq U)) \vee (\neg(G \subseteq V)) \\ & \vee (U \cap V \neq \emptyset)] \end{aligned}$$

That is, writing explicitly:

$$\begin{aligned} \forall U \forall V \Big[& \left(\neg \left[\forall x (U(x) \rightarrow \left[\bigvee_{n < \omega} \forall y (R_n(x, y) \rightarrow U(y)) \right]) \right] \right) \\ & \vee \left(\neg \left[\forall x (V(x) \rightarrow \left[\bigvee_{n < \omega} \forall y (R_n(x, y) \rightarrow V(y)) \right]) \right] \right) \\ & \vee \left(\exists x (F(x) \wedge \neg U(x)) \right) \vee \left(\exists x (G(x) \wedge \neg V(x)) \right) \vee \left(\exists x (U(x) \wedge V(x)) \right) \Big] \end{aligned}$$

After applying some prenex operations and changing some variables, we get to the following formula (where $\vec{y} = \langle y_n \rangle_{n < \omega}$ and $\vec{z} = \langle z_n \rangle_{n < \omega}$)³:

$$\begin{aligned} \forall U \forall V \Big[& \left(\exists x_1 \exists \vec{y} \left[\bigwedge_{n < \omega} U(x_1) \wedge R_n(x_1, y_n) \wedge \neg U(y_n) \right] \right) \\ & \vee \left(\exists x_2 \exists \vec{z} \left[\bigwedge_{n < \omega} V(x_2) \wedge R_n(x_2, z_n) \wedge \neg V(z_n) \right] \right) \\ & \vee \left(\exists x_3 (F(x_3) \wedge \neg U(x_3)) \right) \\ & \vee \left(\exists x_4 (G(x_4) \wedge \neg V(x_4)) \right) \\ & \vee \left(\exists x_5 (U(x_5) \wedge V(x_5)) \right) \Big] \end{aligned}$$

And then, finally, we obtain the following prenex formula:

$$\begin{aligned} \forall U \forall V \exists x_1 \exists x_2 \exists x_3 \exists x_4 \exists x_5 \exists \vec{y} \exists \vec{z} \\ \Big[& \left(\bigwedge_{n < \omega} U(x_1) \wedge R_n(x_1, y_n) \wedge \neg U(y_n) \right) \vee \left(\bigwedge_{n < \omega} V(x_2) \wedge R_n(x_2, z_n) \wedge \neg V(z_n) \right) \\ & \vee \left(F(x_3) \wedge \neg U(x_3) \right) \vee \left(G(x_4) \wedge \neg V(x_4) \right) \vee \left(U(x_5) \wedge V(x_5) \right) \Big] \end{aligned}$$

The preceding formula is in Φ , which concludes the proof of Theorem 3.2. $\square$

As a corollary we obtain the corresponding compactness result:

Corollary 3.3. *Let κ_0 be the first ω_1 -strongly compact cardinal and X be a first countable topological space. If all subspaces of X of size less than κ_0 are normal, then X is normal.*

3.2. Reflection of non-countable paracompactness in the realm of first countable spaces

A topological space is said to be *countably paracompact* if every countable open cover of X has a locally finite open refinement. Such property was introduced by Dowker in the early 50's ([5]), and since then it has been a central notion in General Topology. Countable paracompactness is often considered as a topological property which relates to normality. In fact, as Nyikos remarked (see [23]), “*countable paracompactness generally goes hand in hand with normality*”. This recognized relationship between the two properties is mostly due to the extensive literature regarding the so-called *Dowker spaces*, namely normal, non-countably paracompact topological spaces. These spaces were characterized by Dowker ([5]) as being precisely the normal spaces whose product with the unit interval fails to be normal. Constructing a Dowker space proved

³ The precise way one deals with countable sequences in the language is given in the Section 6.1 of [1].

to be very difficult, and the first **ZFC** example of a Dowker space (due to Mary Ellen Rudin, [24]) was only found in the early 1970's.

We shall prove the following theorem:

Theorem 3.4. *If κ_0 is the first ω_1 -strongly compact, then for every first-countable topological space X which is not countably paracompact there is a subspace $Y \subseteq X$ of cardinality less than κ_0 which also lacks the property of being countably paracompact.*

Proof. The proof is, again, by checking that the formula which declares that “ X is not countably paracompact” (for a given first countable topological space X) is in the class Φ of Theorem 3.1.

It is well-known that a topological space X is countably paracompact if, and only if, for every decreasing sequence $\langle F_n : n < \omega \rangle$ of closed sets satisfying $\bigcap_{n < \omega} F_n = \emptyset$ there is a sequence $\langle W_n : n < \omega \rangle$ of open sets such that $F_n \subseteq W_n$ for all $n < \omega$ and $\bigcap_{n < \omega} \overline{W_n} = \emptyset$ ([14]).

So, if a space X is not countably paracompact, there is a “bad” decreasing sequence of closed sets with empty intersection – meaning, there is some decreasing sequence $\langle F_n : n < \omega \rangle$ of closed sets satisfying $\bigcap_{n < \omega} F_n = \emptyset$ such that, for every sequence $\langle W_n : n < \omega \rangle$ of open sets, either there is some $m < \omega$ such that $F_m \not\subseteq W_m$ or $\bigcap_{n < \omega} \overline{W_n} \neq \emptyset$.

We consider the structure $\mathcal{X} = \langle X, \langle R_n \rangle_{n < \omega}, \langle F_n \rangle_{n < \omega} \rangle$ – where the binary relations R_n are defined using fixed countable local bases for each point, as always.

The formula φ which declares that “ X is not countably paracompact” is the following (where $\vec{W} = \langle W_n \rangle_{n < \omega}$):

$$\forall \vec{W} \left[\left(\neg \left[\bigwedge_{n < \omega} \text{Open}(W_n) \right] \right) \vee \left(\neg \left[\bigwedge_{n < \omega} F_n \subseteq W_n \right] \right) \vee \left(\exists x \left[\bigwedge_{n < \omega} \overline{W_n}(x) \right] \right) \right]$$

That is, writing explicitly:

$$\begin{aligned} \forall \vec{W} \left[\left(\bigvee_{n < \omega} \neg \left[\forall x \left(W_n(x) \rightarrow \left[\bigvee_{m < \omega} \forall y (R_m(x, y) \rightarrow W_n(y)) \right] \right) \right] \right) \right. \\ \vee \left(\bigvee_{n < \omega} \left[\exists x [F_n(x) \wedge \neg W_n(x)] \right] \right) \\ \left. \vee \left(\exists x \left[\bigwedge_{n < \omega} \left[\neg \left(\bigvee_{m < \omega} \forall y (R_m(x, y) \rightarrow \neg W_n(y)) \right) \right] \right] \right) \right]. \end{aligned}$$

After applying some prenex operations and changing some variables, we obtain the following formula (where $\vec{y} = \langle y_m \rangle_{m < \omega}$ and $\vec{z} = \langle z_m \rangle_{m < \omega}$):

$$\begin{aligned} \forall \vec{W} \left[\left(\exists x_1 \exists \vec{y} \left[\bigvee_{n < \omega} \left[\bigwedge_{m < \omega} W_n(x_1) \wedge R_m(x_1, y_m) \wedge \neg W_n(y_m) \right] \right] \right) \right. \\ \vee \left(\exists x_2 \left[\bigvee_{n < \omega} [F_n(x_2) \wedge \neg W_n(x_2)] \right] \right) \\ \left. \vee \left(\exists x_3 \exists \vec{z} \left[\bigwedge_{n < \omega} \left[\bigwedge_{m < \omega} [R_m(x_3, z_m) \wedge W_n(z_m)] \right] \right] \right) \right] \end{aligned}$$

And then, finally, we come to the following prenex formula:

$$\forall \vec{W} \exists x_1 \exists x_2 \exists x_3 \exists \vec{y} \exists \vec{z}$$

$$\begin{aligned} & \left[\left(\bigvee_{n < \omega} \left[\bigwedge_{m < \omega} W_n(x_1) \wedge R_m(x_1, y_m) \wedge \neg W_n(y_m) \right] \right) \right. \\ & \vee \left(\bigvee_{n < \omega} [F_n(x_2) \wedge \neg W_n(x_2)] \right) \\ & \left. \vee \left(\bigwedge_{n < \omega} \left[\bigwedge_{m < \omega} [R_m(x_3, z_m) \wedge W_n(z_m)] \right] \right) \right] \end{aligned}$$

The formula above is in Φ , and this concludes the proof of Theorem 3.4. $\square$

Corollary 3.5. *Let κ_0 be the first ω_1 -strongly compact cardinal and X be a first countable topological space. If all subspaces of X of size less than κ_0 are countably paracompact, then X is countably paracompact.*

3.3. Reflecting non- $< \delta$ -collectionwise Hausdorffness for spaces X with $\hat{\chi}(X) < \delta$ (for uncountable δ)

In this subsection we change our presentation a little bit and work with an uncountable cardinal δ , which is not necessarily ω_1 .

Following [6] and [7], one can use $\chi(x, X)$ (the local character of a point x of a topological space X) to define two more cardinal functions. $\chi(X)$, the *character* of X , is $\sup\{\chi(x, X) : x \in X\} + \omega$, thus the character of X is the smallest infinite cardinal κ such that every point of the space has a local base whose size is not larger than κ . A slightly different definition gives us $\hat{\chi}(X)$, which is the least infinite cardinal κ such that $\chi(x, X) < \kappa$ for all $x \in X$.

We shall establish a reflection result for a topological notion which is clearly related to normality when considered in its simplest case. More specifically, we consider an uncountable cardinal δ and prove a reflection theorem for topological spaces X satisfying $\hat{\chi}(X) < \delta$ and which are not $< \delta$ -collectionwise Hausdorff (see definitions below). The reflection is based on the use of δ -strongly compact cardinals.

The simplest possible case (of our reflection result) would be the one where $\delta = \omega_1$ and thus we would be reflecting non- $\aleph_0$ -collectionwise Hausdorffness for first countable spaces, using ω_1 -strongly compact cardinals. Such result is still a particular case of the main theorem of this subsection, but alone it is not particularly interesting since there are simpler methods of reflection available for this simplest case (as it will be explained in its due time).

It is a well-known topological fact that normality is enough to provide disjoint open expansions for countable discrete families of closed sets. So, normality is, in fact, equivalent to $\aleph_0$ -collectionwise normality.

Definition 3.6. Let X be a topological space and κ be an infinite cardinal. X is said to be **κ -collectionwise normal** (or **κ -cwN**) if all discrete families consisting of no more than κ disjoint closed sets have disjoint open expansions.

For a T_1 topological space X and $A \subseteq X$, $\{\{x\} : x \in A\}$ is a discrete family of closed sets if, and only if, A is a closed and discrete subspace of X . In this context, if $\mathcal{F}$ is a discrete family of singletons and $\mathcal{W}$ is a disjoint open expansion of $\mathcal{F}$ we may abuse of the language and also say that $\mathcal{W}$ is a disjoint open expansion of $\bigcup \mathcal{F}$.

Definition 3.7. Let X be a T_1 topological space.

(i) X is said to be **collectionwise Hausdorff** (or **cwH**) if every closed discrete subset of X has a disjoint open expansion.

(ii) If κ is an infinite cardinal, X is said to be **κ -collectionwise Hausdorff** (or **κ -cwH**) if every non-empty closed discrete subset of X whose size is not larger than κ has a disjoint open expansion.

(iii) ([20]) If κ is an infinite cardinal, X is said to be $< \kappa$ -**collectionwise Hausdorff** (or $< \kappa$ -**cwH**) if every non-empty closed discrete subset of X whose size is strictly less than κ has a disjoint open expansion.

An $\aleph_0$ -collectionwise Hausdorff topological space is Hausdorff. Notice that regular spaces are $\aleph_0$ -collectionwise Hausdorff (one has just to adapt the usual argument for “normal $\Rightarrow \aleph_0$ -cwN”). In particular, $\aleph_0$ -collectionwise Hausdorffness is directly implied by normality.

Collectionwise Hausdorffness plays an important role in the separable case of the Normal Moore Space Conjecture. If $B \subseteq \mathbb{R}$ is an uncountable Q -set – i.e., all of its subsets are relative G_δ –, then there is a canonical construction of a separable normal Moore space $M(B)$ which is not collectionwise Hausdorff, thus not metrizable (see [27] for details). In fact, in this canonical construction – which is a subspace of the well-known Niemytzki plane –, the uncountable set B is a closed and discrete subset which has no disjoint open expansions (since no uncountable set has disjoint open expansions in a separable space). The consistency of the existence of uncountable Q -sets follows from Martin’s Axiom plus the negation of the Continuum Hypothesis.⁴ Notice that the existence of a separable normal space with an uncountable closed discrete subset implies, alone, that $2^{\aleph_0} = 2^{\aleph_1}$ (due to the so-called Jones’ Lemma ([16]); in this very same paper Jones proved that $2^{\aleph_0} < 2^{\aleph_1}$ implies that every separable normal Moore space is metrizable).

Reflection results on non- $\aleph_0$ -collectionwise Hausdorffness, for first countable spaces, are easily achievable using *elementary submodels*. Indeed: if F is a countable closed discrete subset of a first countable topological space (X, τ) which has no disjoint open expansions, we may consider a countable elementary submodel M such that $X, \tau, F \in M$ and work with the *topology of elementary submodels* X_M , as introduced in [17]. X_M is the topology on $X \cap M$ which is generated by the base $\{U \cap M : U \in \tau \cap M\}$. As X is supposed to be first countable, X_M is a subspace of X (see Theorem 1.3 of the paper referred to) and so X has a countable subspace which also lacks the property of being $\aleph_0$ -collectionwise Hausdorff (as witnessed by $F \cap M$ for X_M). However, it is not true, in general, that X_M is a subspace of X , thus the method of elementary submodels will not provide, in general, *subspaces* reflecting non- $< \delta$ -collectionwise Hausdorffness for non-first countable spaces. In what follows, we show that, restricting ourselves to spaces X with $\hat{\chi}(X) < \delta$, then the reflection based on δ -strongly compact cardinals still works.

First, one can – similarly as in [1], Theorem 6.1 – prove the following:

Theorem 3.8. *Let δ be an uncountable cardinal, and let Φ' be the class of formulas of the form $\forall \vec{X} \exists \vec{x} \psi$, where $\vec{X}$ is a sequence $\langle X_i^{k_i} \rangle$ of length $< \delta$ of relational variables of second order, with X_i having arity $k_i < \omega$ for all $i \in I$, $\vec{x}$ is a sequence of length $< \delta$ of variables of first order, and ψ is a formula of the infinitary language $\mathcal{L}_{\delta, \omega}$ with only relation and constant symbols, without quantifiers, and which may have $< \delta$ -many free variables of first order and second order relational variables of any given arity. If κ is a δ -strongly compact cardinal, then it reflects every formula in the class Φ' .*

And now we are able to directly apply the preceding theorem to prove the main result of this subsection:

Theorem 3.9. *If κ is a δ -strongly compact cardinal, then for every topological space X satisfying $\hat{\chi}(X) < \delta$ which is not $< \delta$ -collectionwise Hausdorff there is a subspace $Y \subseteq X$ of cardinality less than κ which also lacks the property of being $< \delta$ -collectionwise Hausdorff.*

Proof. Let X be a non- $< \delta$ -cwH topological space with $\hat{\chi}(X) < \delta$. Under these assumptions, there is a closed discrete subset F of size $< \delta$ which has no disjoint open expansion. For every $x \in X$ we fix $\{U_\alpha^x : \alpha < \hat{\chi}(X)\}$ a local base at x and then we define a sequence $\langle R_\alpha \rangle_{\alpha < \hat{\chi}(X)}$ of binary relations on X in the following way: for each $x, y \in X$,

⁴ In fact, if $\kappa < \mathfrak{p}$ – where $\mathfrak{p}$ is the *pseudo-intersection number* (see [4]), which is known to satisfy $\aleph_1 < \mathfrak{p} = \mathfrak{c}$ under $\mathbf{MA} + \neg \mathbf{CH}$ –, then all κ -sized subsets of $\mathbb{R}$ are Q -sets; the reader may find a proof of this fact in [25].

$$R_\alpha(x, y) \iff y \in U_\alpha^x.$$

We consider the structure $\mathcal{X} = \langle X, \langle R_\alpha \rangle_{\alpha < \hat{\chi}(X)}, F \rangle$.

Let us write down the formula to be reflected. The formula φ which declares that “ X is not $<\delta$ -cwH” (as witnessed by F), in the language of the structure $\mathcal{X}$, is the following (where $\vec{U} = \langle U_\alpha \rangle_{\alpha < \hat{\chi}(X)}$ is a sequence of second order variables and $\vec{y} = \langle y_\beta \rangle_{\beta < \hat{\chi}(X)}$ is a sequence of first order variables):

$$\begin{aligned} \forall \vec{U} \left[\left(\exists x_1 \exists \vec{y} \left[\bigvee_{\alpha < \hat{\chi}(X)} \left[\bigwedge_{\beta < \hat{\chi}(X)} U_\alpha(x_1) \wedge R_\beta(x_1, y_\beta) \wedge \neg U_\alpha(y_\beta) \right] \right) \right. \\ \vee \left(\bigvee_{\beta \neq \alpha} \exists x_2 \exists x_3 \left[\left(U_\beta(x_2) \wedge \neg U_\alpha(x_2) \right) \wedge \left(U_\beta(x_3) \wedge U_\alpha(x_3) \right) \right] \right) \\ \vee \left(\exists x_4 \left[F(x_4) \wedge \left(\bigwedge_{\alpha < \hat{\chi}(X)} \neg U_\alpha(x_4) \right) \right] \right) \\ \left. \vee \left(\exists x_5 \exists x_6 \left[(x_5 \neq x_6) \wedge F(x_5) \wedge F(x_6) \wedge \left(\bigvee_{\alpha < \hat{\chi}(X)} U_\alpha(x_5) \wedge U_\alpha(x_6) \right) \right] \right) \right] \end{aligned}$$

The intended meaning of the subformula in the first line (inside the square brackets) is “ $\{U_\alpha : \alpha < \hat{\chi}(X)\}$ is not a family of open sets”; in the second line, “ $\{U_\alpha : \alpha < \hat{\chi}(X)\}$ is not a disjoint family”; in the third line, “ $\{U_\alpha : \alpha < \hat{\chi}(X)\}$ does not cover F ”; and, in the fourth and last line, “there is some U_α such that $|U_\alpha \cap F| \geq 2$ ”. So, the intended meaning of φ is “Whenever $\mathcal{U} = \{U_\alpha : \alpha < \hat{\chi}(X)\}$ is a disjoint family of open sets which covers F , $\mathcal{U}$ is not a disjoint open expansion of F ” – as desired.

The above formula is equivalent to the following prenex formula:

$$\begin{aligned} \forall \vec{U} \exists x_1 \exists \vec{y} \exists x_2 \exists x_3 \exists x_4 \exists x_5 \exists x_6 \\ \left[\left(\bigvee_{\alpha < \hat{\chi}(X)} \bigwedge_{\beta < \hat{\chi}(X)} \left[U_\alpha(x_1) \wedge R_\beta(x_1, y_\beta) \wedge \neg U_\alpha(y_\beta) \right] \right) \right. \\ \vee \left(\bigvee_{\beta \neq \alpha} \left[\left(U_\beta(x_2) \wedge \neg U_\alpha(x_2) \right) \wedge \left(U_\beta(x_3) \wedge U_\alpha(x_3) \right) \right] \right) \\ \vee \left(F(x_4) \wedge \left[\bigwedge_{\alpha < \hat{\chi}(X)} \neg U_\alpha(x_4) \right] \right) \\ \left. \vee \left((x_5 \neq x_6) \wedge F(x_5) \wedge F(x_6) \wedge \left[\bigvee_{\alpha < \hat{\chi}(X)} U_\alpha(x_5) \wedge U_\alpha(x_6) \right] \right) \right] \end{aligned}$$

The above formula is in the class Φ' , hence, by Theorem 3.8, it is reflected by δ -strongly compact cardinals. So, there is a substructure $\mathcal{Y} = \langle Y, \langle R_\alpha \cap Y^2 \rangle_{\alpha < \hat{\chi}(X)}, F \cap Y \rangle$ of $\mathcal{X}$ such that $|Y| < \kappa$ and

$$\mathcal{Y} \models \varphi[\langle R_\alpha \cap Y^2 \rangle_{\alpha < \hat{\chi}(X)}, F \cap Y].$$

As the sequence of relations $\langle R_\alpha \cap Y^2 \rangle_{\alpha < \hat{\chi}(X)}$ generate the subspace topology on Y , and $F \cap Y$ is a closed discrete subset of Y , it follows that Y is a subspace whose size is smaller than κ and which is not $<\delta$ -collectionwise Hausdorff (as witnessed by $F \cap Y$), as desired. $\square$

Corollary 3.10. *Let κ_0 be the first δ -strongly compact cardinal and let X be a topological space with $\hat{\chi}(X) < \delta$. If all subspaces of X of size less than κ_0 are $<\delta$ -collectionwise Hausdorff, then X is $<\delta$ -collectionwise Hausdorff.*

We finish this section by posing a question. Let us recap the following five reflection theorems, three of them established in [1] and two of them in this paper. All of these theorems are direct applications of Theorem 6.1 of [1].

- (Theorem 6.3 of [1]) If κ is ω_1 -strongly compact, then for every graph G that is not countably chromatic there is a subgraph H of G of cardinality less than κ that is not countably chromatic.
- (Theorem 6.4 of [1]) If κ is ω_1 -strongly compact, then for every first-countable non-metrizable topological space X there is a subspace $Y \subseteq X$ of cardinality less than κ which is also non-metrizable.
- (Theorem 6.6 of [1]) If κ is ω_1 -strongly compact, then for every first-countable topological space X which is not completely regular there is a subspace $Y \subseteq X$ of cardinality less than κ which is also non-completely regular.
- (Theorem 3.2 of this paper) If κ is ω_1 -strongly compact, then for every first-countable non-normal topological space X there is a subspace $Y \subseteq X$ of cardinality less than κ which is also non-normal.
- (Theorem 3.4 of this paper) If κ is ω_1 -strongly compact, then for every first-countable topological space X which is not countable paracompact there is a subspace $Y \subseteq X$ of cardinality less than κ which is also non-countably paracompact.

Question 3.11. Do any of the five consequences of κ being ω_1 -strong compact stated above (or even all of them together) imply that κ is ω_1 -strong compact?

Of course, one could also ask whether the reflection of non- $< \delta$ -collectionwise Hausdorffness for spaces X with $\hat{\chi}(X) < \delta$, for uncountable δ , characterizes δ -strongly compactness (in view of Theorem 3.9).

4. The NMSC again

In the 1990's, Dow, Tall and Weiss ([6] and [7]) presented new proofs of the consistency of **NMSC**, assuming the consistency of a supercompact cardinal. They declared that this assumption of a much stronger large cardinal hypothesis is

“... compensated for by a proof that needs no measure theory – indeed can be done with Cohen reals –, generalizes easily to higher cardinals, and is conceptually transparent” ([6], page 34).

The authors remark that the result also holds if, instead of the consistency of a supercompact cardinal, one only assumes the consistency of a strongly compact cardinal ([6], Remark 3.1). We shall next show that the result does hold by just assuming that κ is a regular ω_1 -strongly compact cardinal. By a recent result of Gitik [13], it is consistent, modulo the existence of a supercompact cardinal and a strongly compact cardinal above it, that the first ω_1 -strongly compact cardinal is regular and strictly smaller than the first strongly compact cardinal.

Suppose X is a first-countable (normal) topological space and $\{A_i\}_{i \in I}$ is a discrete collection of closed sets that cannot be separated. This fact can be expressed over the structure

$$\mathcal{X} = \langle X, \langle R_n \rangle_{n < \omega}, \langle A_i \rangle_{i \in I} \rangle$$

by a formula

$$\forall \{W_i\}_{i \in I} \varphi(\{W_i\}_{i \in I}, \{A_i\}_{i \in I})$$

where φ is a formula of $\mathcal{L}_{\omega, |I|+}$ asserting that if $\{W_i\}_{i \in I}$ is a family of pairwise-disjoint open sets, then $A_i \not\subseteq W_i$, for some $i \in I$.

Let κ be an ω_1 -strongly compact cardinal. Let $j : V \rightarrow M$ be a definable elementary embedding such that $j''X \subseteq D$, for some $D \in M$ such that $M \models |D| < j(\kappa)$. We may assume that $D \subseteq j(X)$, for otherwise we may work with $D \cap j(X)$ instead of D . Let

$$j(\mathcal{X}) = \langle j(X), \langle j(R_n) \rangle_{n < \omega}, \langle A_i^* \rangle_{i \in j(I)} \rangle.$$

We may turn D into a substructure $\mathcal{D}$ of $j(\mathcal{X})$ by restricting all the relations $j(R_n)$ and A_i^* to D , for all $n < \omega$ and all $i < j(I)$. Namely,

$$\mathcal{D} = \langle D, \langle j(R_n) \cap D^2 \rangle_{n < \omega}, \langle A_i^* \cap D \rangle_{i \in j(I)} \rangle.$$

Claim 4.1. *In M ,*

$$\mathcal{D} \models \forall \{W_i\}_{i < j(I)} \varphi(\{W_i\}_{i \in j(I)}, \{A_i^* \cap D\}_{i \in j(I)}).$$

Proof. Otherwise, for some $\{W_i\}_{i \in j(I)}$ in M with $W_i \subseteq D$, all $i < j(I)$, we have that $\mathcal{D}$ satisfies that $\{W_i\}_{i \in j(I)}$ is a family of open sets that separates $\{A_i^* \cap D\}_{i \in j(I)}$. Since $j''X \subseteq D$, again we may turn $j''X$ into a substructure $\mathcal{E}$ of $\mathcal{D}$. Namely, in V , let

$$\mathcal{E} = \langle j''X, \langle j(R_n) \cap (j''X)^2 \rangle_{n < \omega}, \langle A_{j(i)}^* \cap j''X \rangle_{i \in I} \rangle.$$

It is easily seen that $\mathcal{E}$ satisfies that $\{W_{j(i)} \cap j''X\}_{i \in I}$ is a family of open sets that separates $\{A_{j(i)}^* \cap j''X\}_{i \in I}$. By elementarity of j , $\mathcal{X}$ satisfies that $\{j^{-1}[W_{j(i)} \cap j''X]\}_{i \in I}$ is a family of open sets that separates $\{j^{-1}[A_{j(i)}^* \cap j''X]\}_{i \in I}$, and therefore it separates $\{A_i\}_{i \in I}$. But this contradicts the fact that $\mathcal{X}$ satisfies $\forall \{W_i\}_{i \in I} \varphi(\{W_i\}_{i \in I}, \{A_i\}_{i \in I})$. This proves the claim. $\square$

Thus,

$$M \models \exists \mathcal{D} \subseteq j(\mathcal{X}) (|\mathcal{D}| < j(\kappa) \wedge \mathcal{D} \models \forall \{W_i\}_{i \in j(I)} \varphi(\{W_i\}_{i \in j(I)}, \{A_i^* \cap D\}_{i \in j(I)}))$$

where D is the universe of $\mathcal{D}$. So, by elementarity,

$$V \models \exists \mathcal{D} \subseteq \mathcal{X} (|\mathcal{D}| < \kappa \wedge \mathcal{D} \models \forall \{W_i\}_{i \in I} \varphi(\{W_i\}_{i \in I}, \{A_i \cap D\}_{i \in I})).$$

We have thus proved the following:

Theorem 4.2. *If κ_0 is the first ω_1 -strongly compact cardinal, then for every first-countable (normal) non-collectionwise normal topological space X there is a subspace $Y \subseteq X$ of cardinality less than κ_0 which also lacks the property of being collectionwise normal.*

For any set S , let $Fn(S, 2)$ be the poset consisting of partial finite functions from S into $\{0, 1\}$, ordered by $\supseteq$. Thus, for a cardinal κ , $Fn(\kappa, 2)$ is the standard forcing notion that adds κ -many Cohen subsets of ω . Let κ be an ω_1 -strongly compact cardinal and G be a $Fn(\kappa, 2)$ -generic filter over V . We claim that the conclusion of Theorem 4.2 also holds in $V[G]$, namely, for every first-countable (normal) non-collectionwise normal topological space X there is a subspace $Y \subseteq X$ of cardinality less than κ which also lacks the property of being collectionwise normal. The proof proceeds similarly as before. In $V[G]$, let $\mathcal{X} = \langle X, \langle R_n \rangle_{n < \omega}, \langle A_i \rangle_{i \in I} \rangle$ be a first-countable (normal) space, where $\langle A_i \rangle_{i \in I}$ is a discrete family of closed sets that cannot be separated. Thus, in $V[G]$,

$$\mathcal{X} \models \forall \{W_i\}_{i \in I} \varphi(\{W_i\}_{i \in I}, \{A_i\}_{i \in I})$$

where φ is as before. Let $\dot{X} = \langle \dot{X}, \langle \dot{R}_n \rangle_{n < \omega}, \langle \dot{A}_i \rangle_{i \in I} \rangle$ be a $Fn(\kappa, 2)$ -name in V whose interpretation by G is $\mathcal{X}$. We may assume that X is a cardinal λ , and therefore, since $Fn(\kappa, 2)$ is homogeneous, we may assume that $\dot{X} = \check{\lambda}$.

Let $j : V \rightarrow M$ be a definable elementary embedding such that $j''\check{\lambda} \subseteq D$, for some $D \in M$ such that $M \models |D| < j(\kappa)$ and with $D \subseteq j(\check{\lambda})$. Note that D is an $Fn(j(\kappa), 2)$ -name.

The partial ordering $Fn(j(\kappa), 2)$ is isomorphic to $Fn(j''\kappa, 2) \times Fn(j(\kappa) - j''\kappa, 2)$ (via the map $f \mapsto \langle f \restriction j''\kappa, f \restriction (j(\kappa) - j''\kappa) \rangle$). Now let H be an $Fn(j(\kappa) - j''\kappa, 2)$ -generic filter over V . We claim that $G^* := j''G \times H$ is $Fn(j(\kappa), 2)$ -generic over M . For suppose A is a maximal antichain of $Fn(j(\kappa), 2)$ in M . Since $Fn(j(\kappa), 2)$ is ccc, A is countable. Let $A_0 := A \restriction Fn(j''\kappa, 2)$ and $A_1 := A \restriction Fn(j(\kappa) - j''\kappa, 2)$. Let $\langle p_n : n < \omega \rangle$ be an enumeration of A_0 . Since the domain of each p_n is a finite subset of $j''\kappa$, each p_n is in the range of j , and therefore so is $\langle p_n : n < \omega \rangle$. So, by elementarity of j , $j^{-1}(\langle p_n : n < \omega \rangle) = \langle j^{-1}(p_n) : n < \omega \rangle$ is a maximal antichain of $Fn(\kappa, 2)$. Thus, some $j^{-1}(p_n)$ belongs to G , and therefore p_n belongs to $j''G$. Also, since A_1 is a maximal antichain of $Fn(j(\kappa) - j''\kappa, 2)$, some $q \in Fn(j(\kappa) - j''\kappa, 2)$ belongs to H . Then, $\langle p_n, q \rangle \in A \cap G^*$.

By the Silver criterion (see, e.g., [6], Proposition 2.1) we can now lift the embedding j to an elementary embedding

$$j : V[G] \rightarrow M[G^*].$$

Thus, in $M[G^*]$,

$$j(\mathcal{X}) \models \forall \{W_i\}_{i \in j(I)} \varphi(\{W_i\}_{i \in j(I)}, \{A_i^*\}_{i \in j(I)})$$

where $\{A_i^*\}_{i \in j(I)} = j(\{A_i\}_{i \in I})$ is a discrete collection of closed subsets of $j(X)$.

In $M[G^*]$, let Y be the interpretation of D by G^* , and let

$$\mathcal{Y} = \langle Y, \langle j(R_n) \cap Y^2 \rangle_{n < \omega}, \langle A_i^* \cap Y \rangle_{i \in j(I)} \rangle.$$

Claim 4.3. In $M[G^*]$,

$$\mathcal{Y} \models \forall \{W_i\}_{i < j(I)} \varphi(\{W_i\}_{i \in j(I)}, \{A_i^* \cap Y\}_{i \in j(I)}).$$

Proof. Otherwise, for some $\{W_i\}_{i \in j(I)}$ in $M[G^*]$ with $W_i \subseteq Y$, all $i < j(I)$, we have that $\mathcal{Y}$ satisfies that $\{W_i\}_{i \in j(I)}$ is a family of open sets that separates $\{A_i^* \cap Y\}_{i \in j(I)}$.

Notice that $j''\check{\lambda} = \{ \langle j(\check{\mu}), j(p) \rangle : \mu < \lambda \text{ and } p \in Fn(\kappa, 2) \}$ is a $Fn(j''\kappa, 2)$ -name in V , and $j''\check{\lambda}[j''G] = j''\lambda$. Hence, $j''\lambda \subseteq Y$.

We may now turn $j''\lambda$ into a substructure $\mathcal{E}$ of $\mathcal{Y}$. Namely, in $V[G][H]$, let

$$\mathcal{E} = \langle j''\lambda, \langle j(R_n) \cap (j''\lambda)^2 \rangle_{n < \omega}, \langle A_{j(i)}^* \cap j''\lambda \rangle_{i \in I} \rangle.$$

We claim that $\mathcal{E}$ belongs to $V[G]$. First, we may assume that $\dot{R}_n = \bigcup_{\alpha \neq \beta < \lambda} f_{\alpha, \beta}$, where $f_{\alpha, \beta}$ is a function from a maximal antichain $A_{\alpha, \beta}$ of $Fn(\kappa, 2)$ into $\{0, 1\}$, so that for every $p \in A_{\alpha, \beta}$, p forces $\alpha R_n \beta$ iff $f(p) = 1$. Since $Fn(\kappa, 2)$ is ccc, we may enumerate $A_{\alpha, \beta}$ as $\{p_{\alpha, \beta}^n : n < \omega\}$. Notice that $j(\dot{R}_n) = \bigcup_{\alpha \neq \beta < j(\lambda)} f_{\alpha, \beta}^*$, where for $\alpha \neq \beta$ less than λ , $f_{j(\alpha), j(\beta)}^* = j(f_{\alpha, \beta}) : j(A_{\alpha, \beta}) \rightarrow \{0, 1\}$. Thus, for every $\alpha \neq \beta$ with $\alpha, \beta < \lambda$ we have that in $M[G^*]$ the following holds:

$$j(\alpha)j(R_n)j(\beta) \quad \text{iff} \quad j(\alpha)j(\dot{R}_n)[G^*]j(\beta) \quad \text{iff} \quad f_{j(\alpha), j(\beta)}^*(G^* \cap j(A_{\alpha, \beta})) = 1$$

Now notice that $j(A_{\alpha, \beta}) = \{j(p_{\alpha, \beta}^n) : n < \omega\}$ is a maximal antichain of $Fn(j''\kappa, 2)$. Hence,

$$f_{j(\alpha), j(\beta)}^*(G^* \cap j(A_{\alpha, \beta})) = 1 \quad \text{iff} \quad j(f_{\alpha, \beta})(j''G \cap j(A_{\alpha, \beta})) = 1 \quad \text{iff} \quad f_{\alpha, \beta}(G \cap A_{\alpha, \beta}) = 1.$$

Thus $j(R_n) \cap (j''\lambda)^2 = \{\langle j(\alpha), j(\beta) \rangle : f_{\alpha,\beta}(G \cap A_{\alpha,\beta}) = 1\} \in V[G]$. Similarly, we may assume that each $\dot{A}_i$ is a set of functions $\{g_\alpha : \alpha < \lambda\}$, where g_α has domain a maximal antichain B_α of $Fn(\kappa, 2)$ and values in $\{0, 1\}$, such that for every $p \in B_\alpha$, p forces $\alpha \in \dot{A}_i$ iff $g_\alpha(p) = 1$. Then, for every $\alpha < \lambda$ and $i \in I$ we have:

$$j(\alpha) \in A_{j(i)}^* \quad \text{iff} \quad j(\alpha) \in j(\dot{A}_i)[G^*] \quad \text{iff} \quad g_{j(\alpha)}^*(G^* \cap j(B_\alpha)) = 1$$

and since $j(B_\alpha)$ is a maximal antichain of $Fn(j''\kappa, 2)$, we then have that

$$g_{j(\alpha)}^*(G^* \cap j(B_\alpha)) = 1 \quad \text{iff} \quad j(g_\alpha)(j''G \cap j(B_\alpha)) = 1 \quad \text{iff} \quad g_\alpha(G \cap B_\alpha) = 1$$

which shows that $A_{j(i)}^* \cap j''\lambda = \{j(\alpha) : g_\alpha(G \cap B_\alpha) = 1\} \in V[G]$. Hence $\mathcal{E} \in V[G]$.

In $V[G][H]$, by elementarity of j , $\{j^{-1}[W_{j(i)} \cap j''\lambda]\}_{i \in I}$ is a family of open subsets of $\mathcal{X}$ that separates $\{j^{-1}[A_{j(i)}^* \cap j''\lambda]\}_{i \in I}$, and therefore it also separates $\{A_i\}_{i \in I}$. But since $\{A_i\}_{i \in I}$ is not separated in $V[G]$, this contradicts Lemma 1.2 from [6], which shows that “being unseparated” is preserved by adding Cohen reals. This proves the claim. $\square$

Note that since $\mathcal{Y}$ is a substructure of $j(\mathcal{X})$, the family $\langle A_i^* \cap Y \rangle_{i \in j(I)}$ is also discrete and cannot be separated in $j(\mathcal{X})$ either.

Thus, $M[G^*]$ satisfies that there exists $\mathcal{Y} \subseteq j(\mathcal{X})$ of cardinality less than $j(\kappa)$ such that the family $\langle A_i^* \cap Y \rangle_{i \in j(I)}$ is unseparated in $\mathcal{Y}$ and in $j(\mathcal{X})$ and witnesses that $\mathcal{Y}$ is not collectionwise normal. So, by elementarity, $V[G]$ satisfies that there exists $\mathcal{Y} \subseteq \mathcal{X}$ of cardinality less than κ such that the family $\langle A_i \cap Y \rangle_{i \in I}$ is unseparated in $\mathcal{Y}$ and in $\mathcal{X}$ and witnesses that $\mathcal{Y}$ is not collectionwise normal.

We can now give a proof of the following theorem:

Theorem 4.4. *Suppose κ is an ω_1 -strongly compact cardinal and G is $Fn(\kappa^+, 2)$ -generic over V . Then in $V[G]$ every first-countable normal topological space is collectionwise normal. Hence, the NMSC holds.*

Proof. Aiming for a contradiction, suppose that, in $V[G]$, $\mathcal{X}$ is a normal first-countable space that is not collectionwise normal, a fact witnessed by some discrete unseparated family $\langle A_i \rangle_{i \in I}$ of closed sets. Since κ^+ is also ω_1 -strongly compact, by our arguments above there is a subspace $\mathcal{Y}$ of $\mathcal{X}$ of size less than κ^+ that is not collectionwise normal, a fact that is witnessed by the family $\langle A_i \cap Y \rangle_{i \in I}$, which is also discrete and unseparated in $\mathcal{X}$.

Since κ^+ is regular, and $\mathcal{Y}$ has size less than κ^+ , we may now argue as in [6, Corollary 2.6]. Namely, since $\mathcal{X}$ is normal, the family $\langle A_i \rangle_{i \in I}$ is normalized in $\mathcal{X}$, hence $\langle A_i \cap Y \rangle_{i \in I}$ is also normalized in $\mathcal{X}$. Thus, in $V[G]$, the space $\mathcal{X}$ is a normal space in which every point has local character less than $\mathfrak{c} = \kappa$ and in which $\langle A_i \cap Y \rangle_{i \in I}$ is a discrete collection whose union has size less than κ and is normalized but not separated. But this contradicts Theorem 1.4 of [6]. $\square$

Notice that the preceding argument does not require nor assumes Y to be a normal subspace.

5. Notes and questions

As pointed out by the anonymous referee, it seems that ω_1 -strongly compact cardinals have potential for additional applications to General Topology. There are, indeed, recent results in this direction, due to Usuba (see [28], [29]). We hope this paper may inspire further research on the interaction between ω_1 -strongly compact cardinals and General Topology.

The referee has also called our attention to **PCEA**, the so-called Product Category Extension Axiom. **PCEA** states as follows (see [11], [12]).

“For every index set I , there is $J \supseteq I$ and a $\mathfrak{c}$ -complete Boolean homomorphism $h : \mathcal{P}({}^I\omega) \rightarrow \text{RegOpen}({}^J\omega)$ such that $h([p]_I) = [p]_J$ for all basic open sets $[p]_I$.”

Fleissner ([11]) has shown that, similarly to **PMEA**, the consistency of **PCEA** may be obtained by adding strongly compact many random reals, and, furthermore, **NMSC** follows from **PCEA** as well.

Let **PCEA** $_{\sigma}$ be the statement obtained from **PCEA** by relaxing the requirement of completeness of the Boolean homomorphism h from $\mathfrak{c}$ -completeness to σ -completeness (i.e., **PCEA** $_{\sigma}$ is obtained from **PCEA** in a similar fashion as **PMEA** $_{\sigma}$ is obtained from **PMEA**). It seems reasonable that the following questions have positive answer, nevertheless some additional research is needed here.

Question 5.1. Is **PCEA** $_{\sigma}$ enough to imply **NMSC**? Does **PCEA** $_{\sigma}$ hold after adding ω_1 -strongly compact many random reals? Does **PCEA** $_{\sigma}$ hold after adding ω_1 -strongly compact many Cohen reals?

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